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ON THE RELATIVE ADJUNCTION MAPPING

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Introduction.

Let $\varphi\colon X \to Y$ be a proper algebraic map with connected fibres from a connected quasi-projective n-dimensional manifold, $n \geqq 2$, onto a complex quasi-projective variety Y. Let L be an algebraic line bundle on X that is very ample relatively to φ. In this article we use Reider's technique [4] in a local setting to generalize results about the adjunction bundle $K_X \otimes L^{n-1}$ that are standard when Y is a point (see [9] for statement in this case and references to the literature).

For simplicity we will state only the dim $Y > 0$ results and only the algebraic version in this introduction, but actually prove both algebraic and analytic versions of a number of results in the paper.

THEOREM I ((4.1), (5.1)). *The natural morphism* $\varphi^*\varphi_*(K_X \otimes L^{n-1}) \to K_X \otimes L^{n-1}$ *is onto unless* φ *is a* $\mathbf{P}^{n-1}$ *bundle over a smooth curve and* $L_F \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$ *for a fibre* F *of* φ.

Assuming the above map is onto it is easy to construct a normal quasi-projective space X' with a surjective morphism $\varphi'\colon X' \to Y$, an algebraic morphism with connected fibres $\phi\colon X \to X'$, and a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\varphi"'] & & X' \arrow[dl, "\varphi'"] \\ & Y & \end{tikzcd}$$

such that $K_X \otimes L^{n-1} \approx \phi^*\mathscr{L}$ for a line bundle $\mathscr{L}$ on X' which is ample and spanned by global sections relatively to φ'.

THEOREM II (5.3). *If* $\dim X > \dim X'$ *then either:*
i) $n \geqq 3$, $\dim Y = 2$ *and* $\varphi\colon X \to Y$ *is a* $\mathbf{P}^{n-2}$ *bundle over a surface with* $L_F \cong \mathcal{O}_F(1)$ *for a fibre* F; *or*
ii) $\dim Y = 1$ *and either* $\varphi\colon X \to Y$ *has a* $n-1$ *dimensional quadric* Q *for a gen-*

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eral fibre with L_Q the pull-back of $\mathcal{O}_{\mathbf{P}^n}(1)$ under the embedding $Q \subset \mathbf{P}^n$ or X is a linear $\mathbf{P}^{n-2}$ bundle over a surface X' that fibres over the curve Y.

THEOREM III (5.3). *If $\dim X = \dim X'$, X' is smooth and ϕ expresses X as X' with a finite set B blown up. There is a line bundle L' on X' that is ample relatively to φ' and such that $L \approx \phi^* L' \otimes [\phi^{-1}(B)]^{-1}$ and $K_X \otimes L^{n-1} \approx \phi^*(K_{X'} \otimes L'^{n-1})$. Further $K_{X'} \otimes L'^{n-1}$ is very ample relatively to φ'.*

The proof of the above results depends on relative forms of Reider's results for proper morphisms $\varphi\colon X \to Y$, where X is a smooth surface, and line bundles L on X. These results are proved in (1.3), (2.1) and (3.1).

We give local criteria for certain bundles to be k-spanned. The k-spannedness is a notion of higher order emdedding introduced and extensively studied in [3] and [2]. It should be noted that 0-spanned is equivalent to be spanned and 1-spanned is equivalent to be very ample. The proofs of these local results are simple reductions to the compact case when X and Y are quasi-projective and L, φ are algebraic. We also show the analytic case, when Y is a surface and φ a modification, based on an elaborate reduction to results of Elkik and Artin [1].

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§0. Preliminaries.

We work over the complex numbers field $\mathbf{C}$. Through the paper we shall denote by $\pi\colon S \to Y$ a desingularization of either a *germ* Y of an isolated surface singularity $\{y\}$ or a *normal Stein* (respectively *affine*) surface Y. Then there exist a holomorphic surjection $\not{p}\colon \mathcal{S} \to \mathcal{Y}$ of a smooth projective surface $\mathcal{S}$ onto a projective surface $\mathcal{Y}$ and a commutative diagram (see §1 for details)

$$\begin{array}{ccc} S & \xrightarrow{\pi} & Y \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow{\not{p}} & \mathcal{Y} \end{array}$$

where the vertical arrows are holomorphic embeddings of open sets in the complex analytic topology.

Let $\mathcal{L}$ be a line bundle on S (resp. on $\mathcal{S}$). We say that $\mathcal{L}$ is *numerically effective*, *nef* for short, if $\mathcal{L} \cdot C \geqq 0$ for every effective compact divisor C on S (resp. curve C on $\mathcal{S}$). We say that a nef line bundle $\mathcal{L}$ on $\mathcal{S}$ is *big* if $\mathcal{L} \cdot \mathcal{L} > 0$. We shall denote by K_S and $K_{\mathcal{S}}$ the *canonical sheaves* of the holomorphic 2-forms on S and $\mathcal{S}$ respectively. We simply write $\pi_{(i)}(\mathcal{L})$ or $\not{p}_{(i)}(\mathcal{L})$ to mean the *Leray's sheaves* $R^i\pi_*(\mathcal{L})$ or $R^i\not{p}_*(\mathcal{L})$, $i > 0$.

(0.1) We fix some more notation.
$\sim$ (resp. $\approx$), the numerical (respectively linear) equivalence of divisors;
$\chi(\mathscr{L}) = \sum(-1)^i h^i(\mathscr{L})$, the Euler characteristic of a line bundle $\mathscr{L}$, where $h^i(\mathscr{L})$ stands for the complex dimension of $H^i(\cdot, \mathscr{L})$;
$|\mathscr{L}|$, the complete linear system associated to $\mathscr{L}$ and $\Gamma(\mathscr{L})$, the space of its global sections. We say that $\mathscr{L}$ is *spanned* if it is spanned by $\Gamma(\mathscr{L})$;
K_X, the canonical bundle of the holomorphic n-forms of a n-dimensional analytic (or algebraic) manifold X.

As usual we don't distinguish between locally free sheaves and vector bundles nor between line bundles and Cartier divisors. Hence we shall freely switch from the multiplicative to the additive notation and viceversa.

We recall the definition of k-spannedness and the basic construction for it given in [2].

(0.2) *k-spannedness.* Let $\mathscr{L}$ be a line bundle on S (resp. on $\mathscr{S}$). We say that $\mathscr{L}$ is *k-spanned* for $k \geqq 0$, if for any admissible 0-cycle $\mathscr{Z}$ of degree $k+1$ on S (resp. on $\mathscr{S}$) the map

$$\Gamma(\mathscr{L}) \to \Gamma(\mathscr{L} \otimes \mathcal{O}_{\mathscr{Z}})$$

is onto. Here *admissible* means that the ideal sheaf $\mathscr{J}_{\mathscr{Z}}$ defining $\mathscr{Z}$ is isomorphic to $\mathcal{O}_{S,z}$ (resp. to $\mathcal{O}_{\mathscr{S},z}$) for $z \notin \mathrm{Supp}(\mathscr{Z})$ and $\mathscr{J}_{\mathscr{Z}}$ is generated by $(u_i, v_i^{k_i})$ at $z_i \in \mathrm{Supp}(\mathscr{Z})$ where $\sum_i k_i = k+1$ and (u_i, v_i) are local coordinates at z_i. Note $\deg \mathscr{Z} = \mathrm{length}(\mathcal{O}/\mathscr{J}_{\mathscr{Z}})$. Note also that 0-spanned is equivalent to $\mathscr{L}$ being spanned by $\Gamma(\mathscr{L})$ and 1-spanned is equivalent to very ample.

(0.3) THEOREM ([2], (2.1)). *With the notation as above, let $\mathscr{L}$ be a nef and big line bundle on $\mathscr{S}$ and let $\mathscr{L} \cdot \mathscr{L} \geqq 4k+5$. Then either $K_{\mathscr{S}} + \mathscr{L}$ is k-spanned or there exists an effective divisor D on $\mathscr{S}$ such that $\mathscr{L} - 2D$ is $\mathbb{Q}$-effective, D contains some admissible 0-cycle of degree $k'+1 \leqq k+1$ where the k-spannedness fails and*

$$\mathscr{L} \cdot D - k' - 1 \leqq D \cdot D < \mathscr{L} \cdot D/2 < k' + 1.$$

Let us recall the following definition.

(0.4) We say that (S, L) is a *conic bundle* if there exists a proper holomorphic surjection π: $S \to C$ where C is a smooth curve, the fibers of π are connected and $K_S \otimes L = \pi^* H$ where H is ample and L is both relatively ample with respect to π and nef and big.

§1. The local k-spannedness Theorem.

We need the following simple consequence of Elkik's approximation theorem ([1], pp. 49–50 and p. 55).

(1.1) THEOREM. *Let π: $S \to Y$ be a desingularization of a germ Y of an isolated surface singularity $\{y\}$. Let E be a vector bundle on S. Then there exists a holomorphic surjection $\not{p}$: $\mathcal{S} \to \mathcal{Y}$ of a smooth projective surface $\mathcal{S}$ onto a projective surface $\mathcal{Y}$ with an isolated singularity $\mathcal{y}$ such that π: $S \to Y$ is the pull-back of $\not{p}$: $\mathcal{S} \to \mathcal{Y}$ under a biholomorphism φ of Y into $\mathcal{Y}$. Further there is a holomorphic vector bundle $\mathcal{E}$ on $\mathcal{S}$ which gets pulled back to E under the map $S \to \mathcal{S}$ induced as above.*

PROOF. Let (Y, E_Y) be the pair consisting of Y and the coherent sheaf $E_Y = \pi_* E$. By Elkik's approximation theorem quoted above, there exists a pair (Y', E') where E' is a coherent algebraic sheaf on an algebraic (affine) surface Y' and E' is locally free outside of an isolated singularity $\mathcal{y}$. Further there exists a biholomorphism φ' of Y into Y' with $\varphi'(y) = \mathcal{y}$ and $E_Y = \varphi'^* E'$. Now, choose a projective compactification Y'' of Y' and let Y''' be the desingularization of it away from $\mathcal{y}$. After possibly blowing up Y''' away from $\mathcal{y}$ we obtain a normal projective surface $\mathcal{Y}$ such that the pull-back, modulo torsion, of E' to $\mathcal{Y}$ is a coherent sheaf $\mathcal{E}'$ locally free away from $\mathcal{y}$; furthermore the morphism φ' gives rise to a biholomorphism φ of Y into $\mathcal{Y}$. Let $\not{p}$: $\mathcal{S} \to \mathcal{Y}$ be a desingularization of $\mathcal{Y}$ at the singular point $\mathcal{y} = \varphi(y)$, constructed by replacing a smoll neighborhood V' of $\mathcal{y}$ with $\pi^{-1}(V)$ where V is a neighborhood of y in Y biholomorphic to V'. Then clearly $S = Y \times_{\mathcal{Y}} \mathcal{S}$ and φ induces a biholomorphism of S into $\mathcal{S}$. Since $\not{p}^* \mathcal{E}'$ and E agree on some open set of the form $\not{p}^{-1}(\mathcal{U}) \setminus \not{p}^{-1}(\mathcal{y})$, where $\mathcal{U}$ is a neighborhood of $\mathcal{y}$, $\not{p}^* \mathcal{E}'$ and E patch together to give a locally free vector bundle $\mathcal{E}$ on $\mathcal{S}$ agreeing with E over $\not{p}^{-1}(\mathcal{U})$. This completes the proof.

Note that the same argument as above and the Kawamata-Viehweg vanishing theorem lead to the Grauert-Riemenschneider vanishing theorem in the following special case we need.

(1.2) THEOREM. *Let π: $S \to Y$ be a desingularization of a germ Y of an isolated surface singularity. Let L be a nef line bundle on S. Then $\pi_{(1)}(K_S \otimes L) = 0$.*

PROOF. The notation are as in Theorem (1.1). Let $\mathcal{L}$ be the extension of L on $\mathcal{S}$ given by (1.1). Choose a very ample line bundle $\mathcal{H}$ on $\mathcal{Y}$ and set

$$\mathcal{M} = \mathcal{L} \otimes \not{p}^* \mathcal{H}^N, \qquad N \gg 0.$$

Note that $\mathcal{M}$ is spanned outside of $\not{p}^{-1}(\mathcal{y})$, $\mathcal{y}$ the singular point of $\mathcal{Y}$, for $N \gg 0$. Indeed,

$$\Gamma(\mathcal{S}, \mathcal{M}) \cong \Gamma(\mathcal{Y}, \not{p}_* \mathcal{M}) \cong \Gamma(\not{p}_* \mathcal{L} \otimes \mathcal{H}^N)$$

and $\not{p}_* \mathcal{L} \otimes \mathcal{H}^N$ is spanned for $N \gg 0$ by Serre's Theorem A. Further, for any irreducible component C of $\not{p}^{-1}(\mathcal{y})$, $\mathcal{M} \cdot C = \mathcal{L} \cdot C = L \cdot C \geqq 0$ since $\not{p}^* \mathcal{H}$ is trivial on C and L is nef. Thus $\mathcal{M}$ is nef and big for $N \gg 0$,

$$H^2(\mathcal{Y}, p_*(K_{\mathcal{S}} \otimes \mathcal{M})) \cong H^2(\mathcal{Y}, p_*(K_{\mathcal{S}} \otimes \mathcal{L}) \otimes \mathcal{H}^N) = (0)$$

by Serre's Theorem B and $H^1(\mathcal{S}, K_{\mathcal{S}} \otimes \mathcal{M}) = (0)$ by the Kawamata-Viehweg vanishing theorem. Thus from the Leray's spectral sequence for p and $K_{\mathcal{S}} \otimes \mathcal{M}$ we infer that $H^0(\mathcal{Y}, p_{(1)}(K_{\mathcal{S}} \otimes \mathcal{M})) = (0)$ and hence

$$p_{(1)}(K_{\mathcal{S}} \otimes \mathcal{M}) \cong \pi_{(1)}(K_S \otimes L) = 0$$

since $p_{(1)}(K_{\mathcal{S}} \otimes \mathcal{M})$ is supported at the singular point of $\mathcal{Y}$.

(1.3) Corollary (The local k-spannedness Theorem). *Let π: $S \to Y$ be a desingularization of a germ Y of an isolated singularity $\{y\}$ and let L be a nef line bundle on S. Given an admissible 0-cycle $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of degree $k+1$ with* $\operatorname{Supp}(\mathcal{Z}) \subset \pi^{-1}(y)$, *then $H^1(S, K_S \otimes L \otimes \mathcal{J}_{\mathcal{Z}}) = (0)$ where $\mathcal{J}_{\mathcal{Z}}$ is the defining ideal of $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ and the map*

$$\Gamma(S, K_S \otimes L) \to \Gamma(\mathcal{Z}, K_S \otimes L \otimes \mathcal{O}_{\mathcal{Z}})$$

is onto, unless there exists a compact effective divisor D with $\operatorname{Supp}(D) \subset \pi^{-1}(y)$ *such that D contains some non-trivial sub 0-cycle of $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of degree $k'+1 \leqq k+1$ and*

$$L \cdot D - k' - 1 \leqq D \cdot D < 0. \tag{1.3.1}$$

Proof. Let $\mathcal{L}$ be the extension of L on $\mathcal{S}$ guaranteed by (1.1) and set $\mathcal{M} = \mathcal{L} \otimes p^*\mathcal{H}^N$, $N > 0$, as in the proof of (1.2). Then applying Theorem (0.3) to the pair $(\mathcal{S}, \mathcal{M})$ we see that the map

$$\imath\colon\ \Gamma(\mathcal{S}, K_{\mathcal{S}} \otimes \mathcal{M}) \to \Gamma(\mathcal{Z}, K_{\mathcal{S}} \otimes \mathcal{M} \otimes \mathcal{O}_{\mathcal{Z}})$$

is onto, unless there exists a compact effective divisor D on $\mathcal{S}$ containing some non-trivial sub 0-cycle of $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of degree $k'+1 \leqq k+1$ and

$$\mathcal{M} \cdot D - k' - 1 \leqq D \cdot D < \mathcal{M} \cdot D/2 < k' + 1.$$

Since $\mathcal{M} \cdot D/2 > k'+1$ if $D \cdot p^*\mathcal{H} > 0$ and $N \gg 0$, we conclude that $D \cdot p^*\mathcal{H} = 0$. Thus $D \subset \pi^{-1}(y)$ on S, $\mathcal{M} \cdot D = L \cdot D$ and

$$L \cdot D - k' - 1 \leqq D \cdot D < 0.$$

Now if $\imath$ is surjective then clearly the map

$$\Gamma(S, K_S \otimes L) \to \Gamma(\mathcal{Z}, K_S \otimes L \otimes \mathcal{O}_{\mathcal{Z}})$$

is also surjective and hence $H^1(S, K_S \otimes L \otimes \mathcal{J}_{\mathcal{Z}}) = (0)$ since $H^1(S, K_S \otimes L) = H^1(Y, \pi_*(K_S \otimes L)) = (0)$ in view of Theorem (1.2).

§2. Piecing together local results.

In this section we prove the following "global" k-spannedness criterion. Let us set

$$\mathcal{D}_k(L) = \{\text{effective compact divisors } D \text{ on } S, L\cdot D - k - 1 \leqq D\cdot D < 0\}.$$

(2.1) THEOREM. *Let $\pi\colon S \to Y$ be a desingularization of a normal Stein (respectively affine) surface. Let L be a nef line bundle on S. Then given any admissible 0-cycle $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ of degree $k+1$ with $\operatorname{Supp}(\mathcal{Z}) \subset S \setminus \mathcal{D}_k(L)$ one has $H^1(S, K_S \otimes L \otimes \mathcal{J}_{\mathcal{Z}}) = (0)$ for the defining ideal sheaf $\mathcal{J}_{\mathcal{Z}}$ of $(\mathcal{Z}, \mathcal{O}_{\mathcal{Z}})$ and the map*

$$\Gamma(S, K_S \otimes L) \to \Gamma(S, K_S \otimes L \otimes \mathcal{O}_{\mathcal{Z}})$$

is onto.

PROOF. Let $\operatorname{Sing}(Y)$ denote the singular locus of Y and let $Z = \pi(\mathcal{Z}) \cup \operatorname{Sing}(Y)$. Let D_1, D_2 be two Cartier divisors on Y such that $D_1 \cap D_2$ is a finite set containing Z. By using the local k-spannedness Theorem (1.3) it thus follows that if we show the surjectivity of the map

$$\rho\colon \Gamma(S, K_S \otimes L) \to \Gamma(\mathcal{C}, K_S \otimes L \otimes \mathcal{O}_{\mathcal{C}}),$$

where $\mathcal{C}$ is the O-cycle of S defined as $\mathcal{C} = (k+1)\pi^{-1}(D_1) \cap (k+1)\pi^{-1}(D_2)$, we will be done. Indeed $\operatorname{Supp}(\mathcal{Z}) \subset \pi^{-1}(D_1) \cap \pi^{-1}(D_2)$, therefore $\mathcal{Z} \subset (k+1)(\pi^{-1}(D_1) \cap \pi^{-1}(D_2))$ since $\deg \mathcal{Z} \leq k+1$. This implies that $\mathcal{J}_{\mathcal{Z}} \supset \mathcal{J}_{\mathcal{C}}$, $\mathcal{J}_{\mathcal{C}}$ the defining ideal of $\mathcal{C}$. Hence there is a surjective map

$$\Gamma(\mathcal{C}, K_S \otimes L \otimes \mathcal{O}_{\mathcal{C}}) \to \Gamma(\mathcal{Z}, K_S \otimes L \otimes \mathcal{O}_{\mathcal{Z}})$$

so we get the result after proving that ρ is onto. To see this, by using the hypercohomology spectral sequence of the Koszul complex

$$0 \to \pi^*\mathcal{O}_Y(-(k+1)(D_1 + D_2)) \otimes K_S \otimes L \to \dots$$
$$\to \pi^*[\mathcal{O}_Y(-(k+1)D_1) \oplus \mathcal{O}_Y(-(k+1)D_2)] \otimes K_S \otimes L \to$$
$$\to K_S \otimes L \to K_S \otimes L \otimes \mathcal{O}_{\mathcal{C}} \to 0$$

we are reduced to showing that, for $i > 0$,

$$H^i(S, \pi^*\mathcal{O}_Y(-(k+1)(D_1 + D_2)) \otimes K_S \otimes L) = (0). \tag{2.1.1}$$

Note that the same proof as in Theorem (1.2) gives us $\pi_{(i)}(K_S \otimes L) = 0$ for $i > 0$. Therefore, by the projection formula

$$\pi_{(i)}(\pi^*\mathcal{O}_Y(-(k+1)(D_1 + D_2)) \otimes K_S \otimes L) =$$
$$\pi_{(i)}(K_S \otimes L) \otimes \mathcal{O}_Y(-(k+1)(D_1 + D_2)) = 0$$

and hence (2.1.1) follows by the Leray's spectral sequence for π and the fact that

Y is either a Stein or an affine surface. In the same way, $\pi_{(i)}(K_S \otimes L) = 0$ for $i > 0$ leads to $H^1(S, K_S \otimes L) = (0)$, whence $H^1(S, K_S \otimes L \otimes \mathcal{J}_{\mathscr{Z}}) = (0)$.

The following consequence of the Theorem above generalizes a result of Sakai [5], §7.

(2.2) COROLLARY. *Let π: $S \to Y$ be a minimal desingularization of a Stein (respectively affine) surface Y. Then*
(2.2.1) *K_S is k-spanned outside of $\mathscr{D}_k(\mathcal{O}_S) = \{$effective compact divisors D, $-k-1 \leqq D \cdot D < 0\}$;*
(2.2.2) *K_S^t is k-spanned outside of the set of -2 rational curves for $t \geqq k+2$;*
(2.2.3) *if $v = \min_D \{D \cdot K_S\}$ where D is an effective compact divisor and* Supp(D) *is not made up with only -2 rational curves, K_S^t is k-spanned outside of the set of -2 rational curves for $(t-1)v \geqq k+1$.*

PROOF. First, note that K_S is nef since π is the minimal desingularization. Then Theorem (2.1) applies to say that for any admissible 0-cycle $\mathscr{Z}$ of degree $k+1$, with Supp$(\mathscr{Z}) \subset S \setminus \mathscr{D}_k(\mathcal{O}_S)$, the map

$$\Gamma(S, K_S) \to \Gamma(\mathscr{Z}, K_S \otimes \mathcal{O}_{\mathscr{Z}})$$

is onto, that is K_S is k-spanned outside of $\mathscr{D}_k(\mathcal{O}_S)$ and (2.2.1) is proved.

Similarly we know that for any admissible 0-cycle $\mathscr{Z}$ of degree $k+1$ with Supp$(\mathscr{Z}) \subset S \setminus \mathscr{D}_k((t-1)K_S)$ the map

$$\Gamma(S, K_S^t) \to \Gamma(\mathscr{Z}, K_S^t \otimes \mathcal{O}_{\mathscr{Z}})$$

is onto and hence K_S^t is k-spanned outside of $\mathscr{D}_k((t-1)K_S)$. Now, since we are looking for the k-spannedness outside of -2 rational curves, we can assume $K_S \cdot D > 0$. Therefore $\mathscr{D}_k((t-1)K_S) = \varnothing$ whenever $t \geqq k+2$, which proves (2.2.2). The same argument gives us (2.2.3).

Let us point out the following consequence of Theorem (2.1) in the case when the line bundle L is assumed to be k-spanned.

(2.3) COROLLARY. *Let π: $S \to Y$ be a desingularization of a Stein (respectively affine) surface Y and let L be a k-spanned line bundle on S. Then*
(2.3.1) *$K_S \otimes L$* is *$(k-1)$-spanned;*
(2.3.2) *$K_S \otimes L$ is k-spanned if π is the minimal desingularization.*

PROOF. It runs parallel to that of Corollary (2.2). Note that, since L is k-spanned, $L \cdot D \geqq k$ for any effective compact divisor D on S. So (2.3.1) is clear.

As to (2.3.2), we see that $K_S \otimes L$ is k-spanned unless there exists an effective compact divisor D such that $L \cdot D = k$, $D \cdot D = -1$. Now, $h^0(D, L_D) \geqq k+1$ since L, and hence L_D, is k-spanned. Then D is a smooth $\mathbf{P}^1$, contradicting the minimality of π.

§3. The case of algebraic morphisms onto a curve.

In this section we carry out the analogues of the main results of sections 1 and 2 in the case of a smooth surface with a given algebraic morphism onto a smooth *affine* curve.

The following is the analogue of the local k-spannedness Theorem (1.3).

(3.1) THEOREM. *Let $p\colon S \to C$ be a proper algebraic morphism from a smooth surface S onto a smooth affine curve C. Let L be a spanned line bundle on S such that the restriction of L to the general fibre of p has positive degree. Given a finite set of points $c_1, \ldots, c_t$ on C and an admissible 0-cycle $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$ of degree $k+1$ with* $\operatorname{Supp}(\mathscr{Z}) \subset \cup_i p^{-1}(c_i)$, *$i = 1, \ldots, t$, then the map*

$$\Gamma(K_S \otimes L) \to \Gamma(K_S \otimes L \otimes \mathcal{O}_{\mathscr{Z}})$$

is onto and $H^1(S, K_S \otimes L \otimes \mathscr{J}_{\mathscr{Z}}) = (0)$ where $\mathscr{J}_{\mathscr{Z}}$ is the defining ideal of $\mathscr{Z}$ in S, unless there exists a compact effective divisor D on S with $\operatorname{Supp}(D) \subset p^{-1}(c_1) \cup \ldots \cup p^{-1}(c_t)$ *such that D contains a non-trivial sub 0-cycle of $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$ of length $k'+1 \leqq k+1$ and*

$$L \cdot D - k' - 1 \leqq D \cdot D \leqq 0.$$

PROOF. First, note that there exists a commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} \\ C & \longrightarrow & R \end{array}$$

where V, R are both projective and smooth, r is a proper algebraic morphism with connected fibres and $S \to V$, $C \to R$ are embeddings. Further there exists a line bundle $\mathscr{L}$ on V such that

$$\mathscr{L}_S \cong L.$$

To construct the above diagram, take projective compactifications S', C' of S, C and let S'', R be the desingularizations (away from S and C) of S', C' respectively. Let L' be the extension of L to S' and let L'' be the pull-back of L' to S''. After possibly blowing up S'' away from S we obtain a smooth surface V and a line bundle $\mathscr{L}$ on V, the pull-back modulo torsion of L'' on V, with the requested properties.

Now, let H be an ample line bundle on R and set

$$\mathscr{M} = \mathscr{L} \otimes r^* H^N, \quad N \gg 0.$$

Clearly we can assume that the support of H is contained in $R \setminus C$, so we have $\mathscr{M} \cong L$ on S. Furthermore we claim that $\mathscr{M}$ *is nef on* V. Indeed

$$\Gamma(V, \mathscr{M}) \cong \Gamma(R, r_* \mathscr{M}) \cong \Gamma(r_* \mathscr{L} \otimes H^N)$$

and $r_* \mathscr{L} \otimes H^N$ is spanned for $N \gg 0$ by Serre's Theorem A. Then $\Gamma(V, \mathscr{M})$ spans the restriction $\mathscr{M}_S$ because L is spanned on S and $\mathscr{M}$ agrees with L on S. Therefore the base locus $\mathrm{Bs}\,|\mathscr{M}|$ of $\mathscr{M}$ is contained in $V \setminus S$. By possibly replacing $\mathscr{M}$ with $\mathscr{M} \setminus \{$divisorial components of $\mathrm{Bs}\,|\mathscr{M}|\}$, we can assume $\mathrm{Bs}\,|\mathscr{M}|$ to be finite and hence $\mathscr{M}$ to be nef.

Note also that $\mathscr{M} \cdot \mathscr{M} > 0$ for $N \gg 0$ since L restricted to the general fibre of p has positive degree. Then by Theorem (3.1) of [2] applied to the pair $(V, \mathscr{M})$ (see also (0.2), (0.3) in this paper) we know that the restriction

$$\bar{\rho}\colon\ \Gamma(K_V \otimes \mathscr{M}) \to \Gamma(K_V \otimes \mathscr{M} \otimes \mathcal{O}_{\mathfrak{X}})$$

is onto, unless there exists an effective divisor Δ on V containing some sub 0-cycle of $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ of degree $k' + 1 \leqq k + 1$ and such that

$$\mathscr{M} \cdot \Delta - k' - 1 \leqq \Delta \cdot \Delta < \mathscr{M} \cdot \Delta/2 < k' + 1.$$

After replacing integer N above with $M = N + m$ for some $m > 2k' + 2$ we infer that

$$\mathscr{M} \cdot \Delta/2 = (\mathscr{L} \otimes r^* H^M) \cdot \Delta/2 > k' + 1$$

whenever $r^* H \cdot \Delta \geqq 0$, a contradiction. Thus we conclude that the support of Δ is contained in the union of fibres of r. Let D be the restriction of Δ to S; then $\mathrm{Supp}(D) \subset \cup_i p^{-1}(c_i)$ and it is not empty. Therefore, since $\mathscr{M} \cdot D = L \cdot D$ on S,

$$L \cdot D - k' - 1 \leqq D \cdot D < L \cdot D/2 < k' + 1.$$

Now from the commutative diagram

$$\begin{array}{ccc}
\Gamma(K_V \otimes \mathscr{M}) & \xrightarrow{\ \bar{\rho}\ } & \Gamma(K_V \otimes \mathscr{M} \otimes \mathcal{O}_{\mathfrak{X}}) \\
\downarrow & & \downarrow\wr \\
\Gamma(K_S \otimes L) & \xrightarrow{\ \rho\ } & \Gamma(K_S \otimes L \otimes \mathcal{O}_{\mathfrak{X}})
\end{array}$$

we see that ρ is surjective whenever $\bar{\rho}$ is also. Finally we claim that

$$r_{(1)}(K_V \otimes \mathscr{M}) = 0.$$

It thus follows that $H^1(S, K_S \otimes L \otimes \mathscr{J}_{\mathfrak{X}}) = (0)$. Indeed $r_{(1)}(K_V \otimes \mathscr{M}) = 0$ implies $p_{(1)}(K_S \otimes L) = 0$; hence by Leray's spectral sequence $H^1(S, K_S \otimes L) \cong H^1(C, p_*(K_S \otimes L)) = (0)$, so that the surjectivity of ρ gives the result.

To prove $r_{(1)}(K_V \otimes \mathscr{M}) = 0$ note that, for $N \gg 0$,

$$H^1(R, r_*(K_V \otimes \mathscr{M})) \cong H^1(R, r_*(K_V \otimes \mathscr{L}) \otimes H^N) = (0)$$

by the projection formula and Serre's Theorem B. By using again Leray's spectral sequence one sees that

$$H^0(R, r_{(1)}(K_V \otimes \mathscr{M})) \cong H^1(V, K_V \otimes \mathscr{M})$$

and the right-hand group is zero by the Kawamata-Viehweg vanishing. It thus follows that $r_{(1)}(K_V \otimes \mathscr{M}) = 0$: otherwise by the projection formula and Serre's Theorem A it would be

$$H^0(R, r_{(1)}(K_V \otimes \mathscr{M})) \cong H^0(R, r_{(1)}(K_V \otimes \mathscr{L}) \otimes H^N) \neq (0),$$

a contradiction. This completes the proof.

Note that the analogue of Theorem (2.1) is now an obvious consequence of the Theorem above. Here $\mathscr{D}'_k(L)$ denotes the set of the effective compact divisors D on S such that $L \cdot D - k - 1 \leqq D \cdot D \leqq 0$.

(3.2) THEOREM. *Let p, S, C and L be as in* (3.1). *Then given any admissible 0-cycle $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$ of degree $k+1$ with* $\mathrm{Supp}(\mathscr{Z}) \subset S \setminus \mathscr{D}'_k(L)$ *one has $H^1(S, K_S \otimes L \otimes \mathscr{J}_{\mathscr{Z}}) = (0)$ for the defining ideal sheaf $\mathscr{J}_{\mathscr{Z}}$ of $(\mathscr{Z}, \mathcal{O}_{\mathscr{Z}})$ and the map*

$$\Gamma(S, K_S \otimes L) \to \Gamma(\mathscr{Z}, K_S \otimes L \otimes \mathcal{O}_{\mathscr{Z}})$$

is onto.

In the case when L is k-spanned we have another easy consequence of Theorem (3.1) (compare with (2.3)).

(3.3) COROLLARY. *Let p: $S \to C$ be a proper algebraic morphism from a smooth surface S onto a smooth affine curve C. Let L be a k-spanned line bundle on S. Then*
(3.3.1) *$K_S \otimes L$ is $(k-2)$-spanned;*
(3.3.2) *$K_S \otimes L$ is $(k-1)$-spanned unless p is a $\mathbf{P}^1$ bundle and $L_f \cong \mathcal{O}_{\mathbf{P}^1}(k)$ for every fibre f of p;*
(3.3.3) *If p is a minimal fibration* (i.e. *no -1 rational curves are contained in the fibres*), *then $K_S \otimes L$ is k-spanned unless either p is a $\mathbf{P}^1$ bundle and $L_f \cong \mathcal{O}_{\mathbf{P}^1}(k)$ for any fibre f of p, or $L \cdot D = k+1$, $D \cdot D = 0$ and D is a degree $k+1$ curve in $\mathbf{P}^k$.*

PROOF. Since L is k-spanned, $L \cdot D \geqq k$ for any effective compact divisor D on S. So (3.3.1) is clear. Further, $K_S \otimes L$ is $(k-1)$-spanned unless there exists an effective compact divisor D such that

$$L \cdot D = k, \ D \cdot D = 0.$$

Now, $H^0(D, L_D) \geqq k+1$ since L and hence L_D is k-spanned. Then D embeds in a $\mathbf{P}^k$, under the morphism given by $\Gamma(L_D)$, as a smooth $\mathbf{P}^1$ and it is a fibre of p. It thus follows that p is a $\mathbf{P}^1$ bundle since $\mathscr{N}_{D|S} \cong \mathcal{O}_D$, so that D deforms. This gives (3.3.2).

Finally $K_S \otimes L$ is k-spanned unless there exists an effective compact divisor D such that either

$$L \cdot D = k+1 \text{ and } D \cdot D = 0$$

or

$$L \cdot D = k \text{ and either } D \cdot D = 0 \text{ or } -1.$$

The case $L \cdot D = k$ and $D \cdot D = -1$ is excluded by the minimality assumption and we are done.

§4. The relative adjunction mapping, part I.

In this section we use the results of §§1, 2 to generalize, in both the analytic and algebraic cases, results about the adjoint bundle $K_X \otimes L^{n-1}$ that are standard in the absolute case. Here we deal with morphisms onto surfaces.

(4.1) THEOREM. *Let X be a n-dimensional connected analytic (respectively algebraic) manifold and let π: $X \to Y$ be a holomorphic (respectively algebraic) proper morphism from X onto a normal analytic (respectively algebraic) surface Y. Let L be a holomorphic (respectively algebraic) line bundle on X that is very ample relative to π and such that the natural morphism $\pi^*\pi_*L \to L$ is onto. Then $\pi^*\pi_*(K_X \otimes L^{n-1}) \to K_X \otimes L^{n-1}$ is onto. Further, either:*

(4.1.1) *there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & X' \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd}$$

where X is the blowing up of a discrete set B on an analytic (respectively algebraic) manifold X', $L \approx \phi^*L' \otimes \mathcal{O}_X(\phi^{-1}(B))$ for a line bundle L' on X' that is ample relatively to π', i.e. *$K_X \otimes L^{n-1} \approx \phi^*(K_{X'} \otimes L'^{n-1})$ and $K_{X'} \otimes L'^{n-1}$ is very ample relatively to π'; or*

(4.1.2) *$n \geqq 3$ and π is a $\mathbf{P}^{n-2}$ bundle with $L_F \cong \mathcal{O}_{\mathbf{P}^{n-2}}(1)$ for any fibre F of π.*

PROOF. It suffices to prove the statement under the assumption that Y is a normal Stein (respectively affine) surface and L is very ample. Then note that $\pi^*\pi_*(K_X \otimes L^{n-1}) \to K_X \otimes L^{n-1}$ onto simply means $K_X \otimes L^{n-1}$ to be spanned.

First, let us assume $n = 2$. Thus clearly $K_X \otimes L$ is spanned by Theorem (2.1). Further $K_X \otimes L$ is 1-spanned unless there exists an effective compact divisor D, passing through some 0-cycle of degree $\leqq 2$ where the 1-spannedness fails, such that

$$L \cdot D - 2 \leqq D \cdot D < 0.$$

Therefore $L \cdot D = 1$, $D \cdot D = -1$ and hence D is a smooth rational -1 curve. Note that $(K_X \otimes L)_D \approx \mathcal{O}_D$ for such curves. Note also that these curves are disjoint; since otherwise $D_1 \cdot D_2 > 0$, $D_i^2 = -1$, $i = 1, 2$, would imply $(D_1 + D_2)^2 \geqq 0$, contradicting the fact that they are both contracted by π. Let X' denote X with all such smooth rational -1 curves D with $L \cdot D = 1$ contracted and let ϕ: $X \to X'$ be the contraction. Let L' be the line bundle associated to the divisor $\phi(A)$ for a general $A \in |L|$. Then $K_X \otimes L \approx \phi^*(K_{X'} \otimes L')$, $K_{X'} \otimes L'$ is 1-spanned and we are as in (4.1.1).

Now let $n \geqq 3$. Through any point $x \in X$ we can choose a smooth surface S obtained as transversal intersection of $n-2$ general elements of $|L|$. Let s be the section of $\mathscr{E} = L \oplus \ldots \oplus L$ ($n-2$ copies) associated to the sections defining the $n-2$ general elements of $|L|$. Let

$$0 \to K_X \otimes L \to K_X \otimes L \otimes \mathscr{E} \to K_X \otimes L \otimes \wedge^2 \mathscr{E} \to \ldots$$
$$\ldots \to K_X \otimes L \otimes \wedge^{n-2} \mathscr{E} \to (K_X \otimes L \otimes \wedge^{n-2} \mathscr{E})_S \to 0$$

be the tensor product of $K_X \otimes L$ and the hypercohomology spectral sequence associated to s. Since $\wedge^{n-2} \mathscr{E} \approx L^{n-2}$ we see that

$$K_X \otimes L \otimes \wedge^{n-2} \mathscr{E} \approx K_X \otimes L^{n-1} \text{ and } (K_X \otimes L \otimes \wedge^{n-2} \mathscr{E})_S \approx K_S \otimes L_S.$$

By Theorem (2.1), $K_S \otimes L_S$ is spanned by global sections and thus by the hypercohomology spectral sequence, $K_X \otimes L^{n-1}$ is spanned by global sections at x if $h^i(K_X \otimes L \otimes \wedge^{n-2-i} \mathscr{E}) = 0$ for $0 < i \leqq n-2$. Since $\wedge^{n-2-i} \mathscr{E}$ is a direct sum of copies of L^{n-2-i} we are reduced to showing that $h^i(K_X \otimes L^{n-1-i}) = 0$ for $0 < i \leqq n-2$. By using Leray's spectral sequence and the fact that S is a Stein (respectively affine) surface, this follows from the relative Kodaira vanishing theorem which states that $R^i\pi_*(K_X \otimes L^j) = 0$ for $i > 0, j > 0$.

To prove the second part of the statement choose a finite number of global sections which span $K_X \otimes L^{n-1}$ and look at the morphism σ: $X \to \mathrm{P_C}$ associated to $\Gamma(K_X \otimes L^{n-1})$. Let (π, σ): $X \to Y \times \mathrm{P_C}$ be the holomorphic (respectively algebraic) *proper* map induced by π and σ and let $s \circ \phi$ be the Remmert-Stein factorization. Look at the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "{(\pi,\sigma)}"] \arrow[dr, "\phi"] \arrow[ddr, "\pi"'] & & Y \times \mathrm{P_C} \arrow[ddl, "q"] \\
& X' \arrow[ur, "s"'] \arrow[d, "\pi'"] & \\
& Y &
\end{tikzcd}$$

Let F be a general fibre of π. Note that the restriction of ϕ to F is the morphism associated to $\Gamma(F, K_X \otimes L^{n-1})$. To see this it suffices to show that $H^1(K_X \otimes L^{n-1} \otimes J_F) = (0)$ where J_F is the ideal sheaf of F. By the Leray spectral sequence and the fact that Y is Stein (affine) we are reduced to showing that $H^1(K_U \otimes L_U^{n-1} \otimes J_F) = (0)$ where $U = \pi^{-1}(V)$ for some Stein (affine) neighborhood V of $\pi(F)$. Since F is a general fibre, we can choose V so that there are $w = \dim Y = \dim X - \dim F$ holomorphic (algebraic) functions $\{g_1, \ldots, g_w\}$ on V that generate the maximal ideal sheaf $\mathfrak{m}$ of $\pi(F)$ on V. We have the usual Koszul complex; use the tensor product of $K_U \otimes L_U^{n-1}$ with the Koszul complex obtained by wedging exterior products of $\mathcal{O}_U^w$ by $g_1 \oplus \dots \oplus g_w$:

$$0 \to K_U \otimes L_U^{n-1} \otimes \mathcal{O}_U \to K_U \otimes L_U^{n-1} \otimes \mathcal{O}_U^w \to K_U \otimes L_U^{n-1} \otimes \wedge^2 \mathcal{O}_U^w \to \dots \to$$
$$\to K_U \otimes L_U^{n-1} \otimes J_F \to 0.$$

Now compute the cohomology using the Grauert-Riemenschneider vanishing theorem and the hypercohomology spectral sequence.

Thus well known properties of the adjunction mapping say us that either $(K_X \otimes L^{n-1})_F \approx K_F \otimes L_F^{\dim F+1}$ is trivial or $\dim F = \dim \phi(F)$. In the former case $F \cong \mathbf{P}^{n-2}$ and $L_F \cong \mathcal{O}_{\mathbf{P}^{n-2}}(1)$ (see f.e. [7], 0.3); it thus follows that $\pi\colon X \to Y$ is a $\mathbf{P}^{n-2}$ bundle in view of [8], (3.3), so we find class (4.1.2). In the latter case $\dim \phi(F) = n - 2$ where $\phi(F)$ is nothing but the general fibre of π' and therefore $\dim X' = n$.

From now on, let us assume $n = 3$; the proof in the general case is an easy modification of the arguments below. Let E be any connected divisor which contracts to a single point under ϕ and let S be a smooth general element in the complete linear system $|L|$. Then by the above the intersection $E \cap S$ is a -1 rational curve ℓ such that $E \cdot L \cdot L = L \cdot \ell = L_S \cdot \ell = 1$. It thus follows that E is reduced and irreducible: further $E \cong \mathbf{P}^2$ and $L_E \cong \mathcal{O}_{\mathbf{P}^2}(1)$. Now $(K_X \otimes L^2)_E \approx \mathcal{O}_E$ since clearly $K_X \otimes L^2 \approx \phi^* M$ for some (ample) line bundle M on X'. Therefore the adjunction formula gives $\mathcal{N}_{E|X} \cong \mathcal{O}_{\mathbf{P}^2}(-1)$ which means that $\phi(E)$ is a smooth point on X'.

Next, let C be an irreducible component of some 1-dimensional fibre of ϕ. Let $\{c_1, \dots c_s\}$ be the set of points cut out on C by a general element $A \in |L|$. Again the result we proved for $n = 2$ applies to say that $K_S \otimes L_S$ is very ample outside of -1 rational curves. Then we see that there is a line ℓ_S passing through all the points, $c_1, \dots, c_s$ and which contracts to the point $\phi(C)$. Now *we claim* that there exists a compact divisor A_S of X containing ℓ_S and such that $\phi(A_S) = \phi(C)$. Then, from the fact that there are at most a countable number of divisors which contracts to points, it thus follows that there exists a divisor A, among the A_S's, which contains the curve C. Hence the previous argument applies again to show that $A \cong \mathbf{P}^2$.

The claim above follows from standard arguments of deformation theory and it is proved in [6], (0.5.4). We give here a sketch of the proof for reader's convenience. Write $\ell = \ell_S$ and note that there is an exact sequence

$$0 \to \mathcal{O}_\ell(-1) \to \mathcal{N}_{\ell|X} \to \mathcal{O}_\ell(1) \to 0$$

where either $\mathcal{N}_{\ell|X} \cong \mathcal{O}_\ell \oplus \mathcal{O}_\ell$ or $\mathcal{N}_{\ell|X} \cong \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1)$. If $\mathcal{N}_{\ell|X} \cong \mathcal{O}_\ell \oplus \mathcal{O}_\ell$ one sees that deformations of ℓ fill out a neighborhood of ℓ in X and hence $\dim \phi(X) < 3$ since $\dim \phi(\ell) = 0$ implies $\dim \phi(\ell_t) = 0$ for small deformations ℓ_t of ℓ, a contradiction. Therefore $\mathcal{N}_{\ell|X} \cong \mathcal{O}_\ell(1) \oplus \mathcal{O}_\ell(-1)$; this implies that deformations of ℓ lie on a divisor Δ of X and $\dim \phi(\Delta) < 2$. It thus follows that Δ is smooth in a neighborhood of ℓ and $\mathcal{N}_{\ell|\Delta} \cong \mathcal{O}_\ell(1)$. Then, since $\ell \cdot \ell_t = 1$, any small deformation ℓ_t of ℓ meets ℓ and hence $\pi(\ell_t) = \pi(\ell)$. Therefore there exists an irreducible component A_ℓ of Δ which contains ℓ and lies over a point of Y, so that A_ℓ is the requested compact divisor.

Thus we have proved that X' is smooth and ϕ: $X \to X'$ is the blowing up of X' at a finite set B of points. Further the arguments above show that $K_X \otimes L^2$ is 1-spanned outside of $\phi^{-1}(B)$. Then $K_X \otimes L^2 \cong \phi^*(K_{X'} \otimes L'^2)$ where $L' \cong (\phi_* L)^{**}$ and $K_{X'} \otimes L'^2$ is very ample; so we are as in (4.1.1).

§5. The relative adjunction mapping, part II.

In this section we use the results of previous sections to generalize, in the algebraic case only, results about the adjoint bundle $K_X \otimes L^{n-1}$ that are standard in the absolute case.

(5.1) THEOREM. *Let X be a n-dimensional connected quasi-projective manifold and let p: $X \to Y$ be an algebraic proper morphism from X onto a normal quasi-projective variety Y of dimension $\geqq 1$. Let L be a line bundle on X that is very ample relative to p. Then the natural map $p^*p_*(K_X \otimes L^{n-1}) \to K_X \otimes L^{n-1}$ is onto unless* $\dim Y = 1$ *and p is a* P^{n-1} *bundle on Y with $L_F \cong \mathcal{O}_F(1)$ for any fibre F of p.*

PROOF. It sufficies to assume that Y is affine with L very ample and to look at the spannedness of $K_X \otimes L^{n-1}$.

The case when $\dim Y = 2$ has been worked out in (4.1). Note also that, by slicing X with general elements of $|L|$ and by looking over the proof of (4.1), one sees that $K_X \otimes L^{n-1}$ is spanned by global sections whenever $\dim Y \geqq 3$.

Thus we reduce to the case when $\dim Y = 1$. First, let us assume $n = 2$. Then by Theorem (3.1), $K_X \otimes L$ is spanned unless there exists an effective compact divisor D passing through some point where the spannedness fails and sucht that

$$L \cdot D - 1 \leqq D \cdot D \leqq 0.$$

Hence $L \cdot D = 1, D \cdot D = 0$. Therefore D is a smooth P^1 and it is a fibre of p. It thus follows that p is a P^1 bundle since $\mathcal{N}_{D|S} \cong \mathcal{O}_D$, so that D deforms.

Then from now on we can assume $n \geqq 3$ (and $\dim Y = 1$). Fix a general fibre F of p. Let S be the smooth surface on X obtained as transversal intersection of $n-2$ general members of $|L|$ and let p_S: $S \to Y$ be the restriction of p to S. Denote by $f = F \cap S$ a general fibre of p_S. From the above we know that $K_S \otimes L_S^2$ is spanned and $(K_S \otimes L_S^2)_f \approx (K_X \otimes L^n)_f$ is trivial on f. Again, the same arguments as in the proof of (4.1) show that $K_X \otimes L^n$ is spanned by global sections. Let Ψ be the morphism associated to $\Gamma(K_F \otimes L_F^n)$. Then Ψ is the restriction to F of the morphism associated to $\Gamma(K_X \otimes L^n)$. General properties of the adjunction mapping say us that either $\dim \Psi(F) = 0$ or $\dim \Psi(F) = \dim F = n - 1$. In the former case $K_F \otimes L_F^{\dim F+1}$ is trivial so that $F \cong \mathbf{P}^{n-1}$ and $L_f \cong \mathcal{O}_{\mathbf{P}^{n-1}}(1)$; it thus follows that $p: X \to Y$ is a $\mathbf{P}^{n-1}$ bundle in view of [8], (3.3). The latter case is excluded since $(K_X \otimes L^n)_f \approx (K_F \otimes L_F^n)_f$ is trivial and f varies with S to cover the whole F. So we are done.

To prove the structure Theorem (5.3) below, the following Proposition is useful.

(5.2) PROPOSITION. *Let p: $S \to C$ be a proper algebraic morphism from a quasi-projective smooth surface S to a smooth quasi-projective curve C. Let L be a line bundle on S that is ample and spanned relative to p and assume the natural map $p^*p_*(K_S \otimes L) \to K_S \otimes L$ to be onto. Then $K_S \otimes L$ is very ample relative to p unless either p: $S \to C$ is a conic bundle and $L_f \cong \mathcal{O}_f(2)$ for any fibre f of p or there exists a commutative diagram*

$$\begin{tikzcd} S \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & S' \arrow[dl, "p'"] \\ & C & \end{tikzcd}$$

where φ is the blowing up of a discrete set B on a smooth surface S', $L \approx \varphi^*L' \otimes \mathcal{O}_S(\varphi^{-1}(B))$ for an ample line bundle L' on S', i.e. $K_S \otimes L \approx \varphi^*(K_{S'} \otimes L')$ and $K_{S'} \otimes L'$ is very ample relative to p'.

PROOF. It suffices to prove this when C is affine. By Theorem (3.1) we see that $K_S \otimes L$ is 1-spanned unless there exists an effective compact divisor D, passing through some 0-cycle of degree $\leqq 2$ where the 1-spannedness fails, such that

$$L \cdot D - 2 \leqq D \cdot D \leqq 0.$$

Hence either

$$L \cdot D = 1, \ D \cdot D = 0 \text{ or } -1,$$

or

$$L \cdot D = 2, \ D \cdot D = 0.$$

Case $L \cdot D = 1$, $D \cdot D = 0$ is excluded since $K_S \otimes L$ is assumed to be spanned. If $L \cdot D = 2, D \cdot D = 0$, then $|L_D|$ embeds D as a degree 2 curve and D is a fibre of p. It thus follows that p is a conic bundle since $\mathcal{N}_{D|S} \cong \mathcal{O}_D$, so that D deforms. Finally, if $L \cdot D = 1$, $D \cdot D = -1$ then D is a smooth $\mathbf{P}^1$ and it is contained in a fibre of p; indeed $p(D)$ is a point since D is compact. Note that $(K_S \otimes L)_D \cong \mathcal{O}_D$. Note also that such $\mathbf{P}^1$'s are contained in disjoint fibres: otherwise $D_1 \cdot D_2 = 1$, $D_i^2 = -1$, $i = 1,2$, would imply $(D_1 + D_2)^2 = 0$, $L \cdot (D_1 + D_2) = 2$ and $D_1 + D_2$ would be a reducible connected fibre of a conic bundle as in the case above.

Let S' denote S with all such smooth rational -1 curves D with $L \cdot D = 1$ contracted and let φ: $S \to S'$ be the contraction. Let L' be the line bundle associated to the divisor $\varphi(A)$ for a general $A \in |L|$. Then $K_S \otimes L \approx \varphi^*(K_{S'} \otimes L')$ is 1-spanned.

(5.3) THEOREM. *Let $p: X \to Y$ and L be as in (5.1) and let assume the natural map $p^*p_*(K_X \otimes L^{n-1}) \to K_X \otimes L^{n-1}$ to be onto. Then either:*
(5.3.1) $n \geqq 3$, $\dim Y = 2$ *and $p: X \to Y$ is a $\mathbf{P}^{n-2}$ bundle over Y with $L_F \cong \mathcal{O}_F(1)$ for any fibre F of p;*
(5.3.2) $\dim Y = 1$ *and either $p: X \to Y$ has $n-1$ dimensional quadric Q for general fibre with L_Q the pull-back of $\mathcal{O}_{\mathbf{P}^n}(1)$ under the embedding $Q \subset \mathbf{P}^n$, or X is a linear $\mathbf{P}^{n-2}$ bundle over X' a surface that fibres over the curve Y;*
(5.3.3) *there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & Y & \end{tikzcd}$$

where X' is smooth, ϕ expresses X as the blowing up of X' at a finite set B, $L \approx \phi^*L' \otimes \mathcal{O}_X(\phi^{-1}(B))$ for some relatively ample line bundle L' on X', i.e. *$K_X \otimes L^{n-1} \approx \phi^*(K_{X'} \otimes L'^{n-1})$, and $K_{X'} \otimes L'^{n-1}$ is very ample relatively to p'.*

PROOF. It suffices to show the case when Y is affine and $K_X \otimes L^{n-1}$ is spanned. Choose a finite number of global sections which span $K_X \otimes L^{n-1}$ and look at the morphism σ: $X \to \mathbf{P}_{\mathbf{C}}$ associated to $\Gamma(K_X \otimes L^{n-1})$. Let (p, σ): $X \to Y \times \mathbf{P}_{\mathbf{C}}$ be the algebraic *proper* morphism induced by p and σ and let $s \circ \phi$ be the Remmert-Stein factorization of (p, σ). Look at the commutative diagram

$$\begin{tikzcd} X \arrow[rr, "(p,\sigma)"] \arrow[dr, "\phi"] \arrow[ddr, "p"'] & & Y \times \mathbf{P}_{\mathbf{C}} \arrow[ddl, "q"] \\ & X' \arrow[ur, "s"] \arrow[d, "p'"] & \\ & Y & \end{tikzcd}$$

First, assume $\dim X > \dim X'$ and let F be a general fibre of p. Then $(K_X \otimes L^{n-1})_F \approx K_F \otimes L_F^{n-1}$ and the restriction of ϕ to F is the morphism associated to $\Gamma(K_F \otimes L_F^{n-1})$. Note also that $\dim X > \dim X'$ if and only if $\dim F > \dim \phi(F)$. Let Δ be a general fibre of the induced map ϕ_F: $F \to \phi(F)$. Then $K_\Delta \sim -L_\Delta^{n-1}$, hence by a classical result due to Kobayshi-Ochiai (see e.g. [7], (0.3)), we have $n - 1 \leqq \dim \Delta + 1$, so that

$$n - 1 \leqq \dim \Delta + 1 \leqq n - \dim Y + 1.$$

Therefore $\dim Y \leqq 2$. If $\dim Y = 2$, one has $\dim \Delta = n - 2$, that is $\Delta = F$, $\dim \phi(F) = 0$ and $K_F \sim -L_F^{\dim F + 1}$. Hence $F \cong \mathbf{P}^{n-2}$ and $L_F \cong \mathcal{O}_{\mathbf{P}^{n-2}}(1)$; by the smoothness of X and the argument in [8], §3 it thus follows that p: $X \to Y$ is a $\mathbf{P}^{n-2}$ bundle, so we find class (5.3.1). If $\dim Y = 1$, then either $\dim \Delta = n - 1$ or $\dim \Delta = n - 2$. In the former case $\Delta = F$, $\dim \phi(F) = 0$ and $K_F \sim -L_F^{\dim F}$; hence $F \cong Q$, Q a quadric in $\mathbf{P}^n$, with $L_F \cong \mathcal{O}_Q(1)$, so we are in class (5.3.2). In the latter case one has $\dim \phi(F) = 1$ and $K_\Delta \sim -L_\Delta^{\dim \Delta + 1}$; therefore $\Delta \cong \mathbf{P}^{n-2}$ and $L_\Delta \cong \mathcal{O}_{\mathbf{P}^{n-2}}(1)$. From this we conclude that X to X' is a scroll over a surface, that is $K_X + (n-1)L \approx \phi^*\mathscr{L}$ for some ample line bundle $\mathscr{L}$ on X'. By the smoothness of X it follows that X to X' is a linear $\mathbf{P}^{n-2}$ bundle over the surface X' that fibres over the curve Y, so we are again in class (5.3.2).

Thus we can assume $\dim X = \dim X'$. Let S be a smooth surface obtained as transversal intersection of $n-2$ general members of $|L|$. If the general fibre F of p is of positive dimension, denote by $f = S \cap F$ the general fibre of the restriction p_S: $S \to Y$ of p to S. Note that p_S: $S \to Y$ cannot be a conic bundle. Otherwise $(K_S \otimes L_S)_f$ would be trivial, then, since the restriction of ϕ to S is the map associated to $\Gamma(K_S \otimes L_S)$, f would be contracted to a point under ϕ and therefore $\dim F > \dim \phi(F)$, that is $\dim X > \dim X'$, a contradiction. Now, by using (5.2) we see that X' is smooth, ϕ is the blowing up of X' at a finite set of points and we are in class (5.3.3). If $\dim F = 0$ an analysis of the positive dimensional fibres of ϕ, as done in the proof of Theorem (4.1), shows that we fall again in class (5.3.3).

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