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EXPONENTIAL ENERGY DECAY OF SOLUTIONS OF ELASTIC WAVE EQUATIONS WITH THE DIRICHLET CONDITION.

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The purpose of this paper is to prove exponential energy decay of solutions of elastic wave equations with the Dirichlet condition. To prove it we shall use a translation representation of the unitary group which is used to express solutions, and show that there is no point spectrum of the infinitesimal generator of the unitary group. Furthermore using the scattering theory of Lax and Phillips and an argument due to Morawetz, we can get exponential energy decay. In the free boundary condition case by an existence of Rayleigh waves exponential energy decay dose not hold for some solution, however by the behavior of singularities of solutions of the Dirichlet condition, which is almost the same as that of solutions of the wave equation, we can prove exponential energy decay.

1. Preliminary about the free space problem.

We shall consider the Cauchy problem corresponding to the elastic wave equation in an isotropic medium. The displacement $u(t,x) = {}^t(u_1, u_2, u_3)$ satisfies the following equation of linear elasticity in R^3

$$(1.1)\qquad \begin{cases} \partial_t^2 u - (\lambda + \mu)\,\mathrm{grad}\,(\mathrm{div}\, u) - \mu \Delta u = 0, \\ \partial_t^{j-1} u = f_j(x) \ \text{ on } \ t = 0 \quad (j = 1,2), \end{cases}$$

where λ and μ are certain scalar quantities called the "Lamè constants". We assume $\lambda + \mu$ and μ are positive. Let us introduce a norm on $(C_0^\infty(\mathsf{R}^3))^3$:

$$\|f\|_1^2 = (\lambda + \mu)\,\|\mathrm{div}\, f\|_{L^2(\mathsf{R}^3)}^2 + \mu \sum_{j=1}^{3} \|\mathrm{grad}\, f_j\|_{L^2(\mathsf{R}^3)}^2,$$

where $f(x) = {}^t(f_1, f_2, f_3) \in (C_0^\infty(\mathsf{R}^3))^3$. $\mathscr{H}_0$ is defined as the completion of $(C_0^\infty(\mathsf{R}^3))^3$ in the above norm. For simplicity in this paper $(C_0^\infty(\mathsf{R}^3))^3$ is denoted by $C_0^\infty(\mathsf{R}^3)$." Similarly simplified notations are used for S^3, where S is a certain

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function space. The following lemma is proved in Chapter IV of [4] as Lemma 1.1.

LEMMA 1.1. *Suppose* $f = {}^t(f_1, f_2, f_3) \in \mathscr{H}_0$. *Then* $f \in \mathscr{S}'(\mathrm{R}^3) \cap L^2_{\mathrm{loc}}(\mathrm{R}^3)$ and

$$\sum_{j=1}^{3} \int_{|x|<R} |f_j(x)|^2 \, dx \leqq CR^2 \, \|f\|_1^2. \tag{1.2}$$

Next we define a Hilbert space $H_0 = \mathscr{H}_0 \times L^2(\mathrm{R}^3)$ with the energy norm: $\|f\|_E^2 = \{\|f_1\|_1^2 + \|f_2\|_{L^2(\mathrm{R})}^2\}/2$, where $f = \{f_1, f_2\}, f_1 \in \mathscr{H}_0$ and $f_2 \in L^2(\mathrm{R}^3)$. Let $u(t, x)$ be a solution of the Cauchy problem (1.1). Then $u(t, x)$ is represented by the one parameter unitary group $\{U_0(t)\}$ on H_0 with the inifnitesimal generator $A_0 = \begin{pmatrix} 0 & E_3 \\ L(\partial_x) & 0 \end{pmatrix}$, where E_3 is the unit 3×3 matrix and $L(\partial_x)u = (\lambda + \mu)\,\mathrm{grad}\,(\mathrm{div}\, u) + \mu \Delta u$.

We shall use the Radon transform in order to derive an explicit formula for the solutions of (1.1). If $g(x)$ belongs to $\mathscr{S}(\mathrm{R}^3)$, its Radon transform $\tilde{g}(s, \omega)$ is defined by the formula;

$$\tilde{g}(s, \omega) = (2\pi)^{-1} \int_{x\omega = s} g(x) dS = (2\pi)^{-1} \langle g, \delta(x\omega - s) \rangle,$$

where $s \in R$, $\omega \in S^2$. Then the following proprties are valid:

$$\tilde{g}(s, \omega) = \tilde{g}(-s, -\omega), \text{ i.e., } \tilde{g} \text{ ia an even funtion} \tag{1.3}$$

$$\widetilde{\partial g/\partial x_j}(s, \omega) = \omega_j \partial \tilde{g}/\partial s(s, \omega). \tag{1.4}$$

$$\hat{g}(\sigma\omega) = F\tilde{g}(\sigma, \omega), \tag{1.5}$$

where we used the symbol F to denote the Fourier transform with respect to s.

$$g(x) = -(4\pi)^{-1} \int \partial_s^2 \tilde{g}(x\omega, \omega) d\omega, \tag{1.6}$$

$$\|g\|_{L^2(\mathrm{R}^3)}^2 = \|\partial_s \tilde{g}\|_{L^2(\mathrm{R} \times S^2)}^2 / 2. \tag{1.7}$$

The explicit formula of the solution of (1.1) is as follows:

THEOREM 1.2. *If* $f_1, f_2 \in \mathscr{S}(\mathrm{R}^3)$. *Then the unique solution* $u(t, x)$ *of the Cauchy problem* (1.1) *is written as*

$$\begin{aligned} u(t, x) = (4\pi)^{-1} \int \{ &\rho_1^{-1} (\omega \cdot l_1(x\omega - \rho_1 t, \omega))\omega \\ &- \rho_2^{-1} \omega \times (\omega \times l_2(x\omega - \rho_2 t, \omega))\} d\omega, \end{aligned} \tag{1.8}$$

where $\rho_1 = (\lambda + 2\mu)^{1/2}, \rho_2 = \mu^{1/2}$ *and* $l_j(s, \omega) = \partial_s \tilde{f}_2 - \rho_j \partial_s^2 \tilde{f}_1$.

PROOF. Consider the Radon transform of (1.1), then taking the inner and vector product of ω and (1.1), we have

$$(1.9)\qquad \begin{cases} (\partial_t^2 - (\lambda + 2\mu)\,\partial_s^2)((\omega\cdot\tilde{u})\omega) = 0, \\ \partial_t^{j-1}((\omega\cdot\tilde{u})\omega)(0,s,\omega) = (\omega\cdot\tilde{f}_j)\omega \quad (j = 1,2), \end{cases}$$

$$(1.10)\qquad \begin{cases} (\partial_t^2 - \mu\,\partial_s^2)(\omega\times(\omega\times\tilde{u})) = 0, \\ \partial_t^{j-1}(\omega\times(\omega\times\tilde{u}))(0,s,\omega) = \omega\times(\omega\times\tilde{f}_j) \quad (j = 1,2). \end{cases}$$

Let $\tilde{v}(t,s,\omega)$ be the solution of the one dimensional wave equation $(\partial_t^2 - \rho^2\,\partial_s^2)\tilde{v} = 0$ with data $\partial_t^{j-1}\tilde{v}(0,s,\omega) = g_j(s,\omega)(j = 1,2)$. Then since this equation is written as $(\partial_t + \rho\,\partial_s)(\partial_t - \rho\,\partial_s)\tilde{v} = 0$, we can easily show that

$$(1.11)\qquad (\partial_t - \rho\,\partial_s)\tilde{v} = (\tilde{g}_2 - p\,\partial_s\tilde{g}_1)(s - \rho t,\omega).$$

Since $\partial_s\partial_t\tilde{v}(t,x\omega,\omega)$ is an odd function with respect to ω by (1.3), by the inerse theorem (1.6) it follows that $v(t,x) = (4\pi\rho)^{-1}\int(\partial_s\tilde{g}_2 - \rho\,\partial_s^2\tilde{g}_1)(x\omega - \rho t,\omega)d\omega$. Applying this formula to the problem (1.9) and (1.10), we have the desired formula (1.8). The proof is completed.

From the formula (1.8) we can prove the existence of lacuna of the solutions of (1.1).

COROLLARY 1.3. *We assume the support of* $f = (f_1, f_2)\in H_0$, *which is the union of* $\operatorname{supp} f_1$ *and* $\operatorname{supp} f_2$, *is contained in* $\{x\in \mathrm{R}^3: |x| < \mathrm{R}\}$, *then* $(U_0(t)f)(x) = 0$ *for* $\rho_2|t| > |x| + R$.

We shall show a translation representation of $\{U_0(t)\}$.

THEOREM 1.4. *Let us define an operator* T_0 *from* H_0 *to* $L^2(\mathrm{R}\times S^2)$ *by*

$$(1.12)\quad T_0(f)(s,\omega) = \{\rho_1^{1/2}(\omega\cdot l_1(\rho_1 s,\omega))\omega - \rho_2^{1/2}\omega\times(\omega\times l_2(\rho_2 s,\omega))\}/2,$$

where $l_j(s,w) = \partial_s\tilde{f}_2 - P_j\partial_s^2\tilde{f}_1$. *Then we have the following:* i) T_0 *is a unitary operator from* H_0 *to* $L^2(R\times S^2)$. ii) $T_0(U_0(t)f)(s,\omega) = (T_0 f)(s-t,\omega)$ *for all* $f\in H_0$.

PROOF. In order to prove i) we shall introduce an auxiliary operator $T_0'(f)$ defined by $\{(\omega\cdot l_1(s,\omega))\omega - \omega\times(\omega\times l_2(s,\omega))\}/2$. Assume $f\in\mathscr{S}(\mathrm{R}^3)$, then

$$\begin{aligned} |T_0'(f)(s,\omega)|^2 &= \{|\omega\cdot l_1(s,\omega)|^2 + |\omega\times(\omega\times l_2(s,\omega)|^2\}/4 \\ &= \{|\omega\cdot\partial_s\tilde{f}_2|^2 + (\lambda + 2\mu)|\omega\cdot\partial_s^2\tilde{f}_1|^2 \\ &\quad + |\omega\times(\omega\times\partial_s\tilde{f}_2)|^2 + \mu|\omega\times(\omega\times\partial_s^2\tilde{f}_1)|^2\}/4 + g(s,\omega) \end{aligned}$$

where $g(s,\omega)$ is an odd function of s,ω. From (1.4), the above equality is equal to

$$\left\{|\partial_s\tilde{f}_2|^2 + (\lambda+\mu)|\,\partial_2\widetilde{\operatorname{div} f_1}|^2 + \mu\sum_{j=1}^{3}|\partial_s\widetilde{\operatorname{grad} f_{1,j}}|^2\right\}/4 + g(s,\omega),$$

where $f_1 = {}^t(f_{1,1}, f_{1,2}, f_{1,3})$. Making use of the Parseval theorem (1.7), we have

$\|T_0'(f)\|_{L^2(R\times S^2)} = \|f\|_E$, which means that T_0' is an isometric operator from H_0 to $L^2(R \times S^2)$. In order to prove the unitary property of T_0' we shall show that a dense set $X = \{g(s,\omega) \in \mathscr{S}(R \times S^2) : (Fg)(\sigma,\omega) = 0 \text{ near } \sigma = 0\}$ of $L^2(R \times S^2)$ is contained in the range of T_0'. By the definitions of $T_0'(f)$ and $l_j(s,\omega)$, the equivalent condition $T_0'(f) = g$ is

$$(1.13)\quad \begin{cases} g_1(s,\omega) = g(s,\omega) + g(-s,-\omega) = \rho_2\omega \times (\omega \times \partial_s^2 \tilde{f}_1) - \rho_1\omega(\omega\cdot\partial_s^2\tilde{f}_1), \\ g_2(s.\omega) = g(s,\omega) - g(-s,-\omega) = \partial_s\tilde{f}_2, \end{cases}$$

where g_1 and g_2 are defined by the first equalities. Take the Fourier transform of (1.13) with respect to s. Then by (1.5) it follows that

$$(1.14)\quad \begin{cases} (Fg_1)(\sigma,\omega) = \sigma^2(\rho_1\omega(\omega\cdot\hat{f}_1(\sigma\omega)) - \rho_2\omega \times (\omega \times \hat{f}_1(\sigma\omega)), \\ (Fg_2)(\sigma,\omega) = i\sigma\hat{f}_2(\sigma\omega). \end{cases}$$

Let us solve the first equality of (1.14). The right hand side of (1.14) is equal to $\sigma^2((\rho_1 - \rho_2)\omega^t\omega + \rho_2)\hat{f}_1(\sigma\omega)$, and the inverse matrix of $(\rho_1 - \rho_2)\omega^t\omega + \rho_2 E_3$ is $((\rho_2 - \rho_1)/\rho_1\rho_2)\omega^t\omega + \rho_2^{-1}E_3$. Thus (1.14) is equivalent to

$$\begin{cases} \hat{f}_1(\xi) = |\xi|^{-2}((\rho_2 - \rho_1)\xi^t\xi/(\rho_1\rho_2|\xi|^2) + \rho_2^{-1}E_3)(Fg_1)(|\xi|,\xi/|\xi|), \\ \hat{f}_2(\xi) = -i|\xi|^{-1}(Fg_2)(|\xi|,\xi/|\xi|), \end{cases}$$

where the right hand sides of the aboves belong to $\mathscr{S}(R^3)$ if $g \in X$. A second auxiliary operator T_0'' on $L^2(R \times S^2)$ to itself is defined as follows;

$$T_0''(k)(s,\omega) = \rho_1^{1/2}\omega(\omega\cdot k)(\rho_1 s,\omega) - \rho_2^{1/2}\omega \times (\omega \times k)(\rho_2 s,\omega).$$

It is clear that $T_0(f) = T_0''(T_0'(f))$ and T_0'' is an isometric oprator on $L^2(R \times S^2)$. The relation $T_0''(k) = h$ is equivalent to $(\omega\cdot h)(s\cdot\omega) = \rho_1^{1/2}(\omega\cdot k)(\rho_1 s,\omega)$ and $(\omega \times h)(s,\omega) = \rho_2^{1/2}(\omega \times h)(\rho_2 s,\omega)$. Thus if we put $k(s,\omega) = \rho_1^{-1/2}\omega(\omega\cdot h)(\rho_1^{-1}s,\omega) - \rho_2^{-1/2}\omega \times (\omega \times h)(\rho_2^{-1}s,\omega)$, then $T_0''(k) = h$. So T_0'' is unitary.

In order to prove the statement ii) we may prove $T_0'(U_0(t)f)(s,\omega) = \{\omega(\omega\cdot l_1)(s - \rho_1 t,\omega) - \omega \times (\omega \times l_2)(s - \rho_2 t,\omega)\}/2$. This is an easy consequence from the definition $l_j(s,\omega)$ and (1.11). The proof is completed.

Making use of the translation representation T_0 of $\{U_0(t)\}$, we can express the solution $u(x,t)$ of (1.1) as follows:

$$(1.15)\quad \begin{aligned} u(t,x) = (2\pi)^{-1}\int\{&\rho_1^{-3/2}\omega(\omega\cdot k)(\rho_1^{-1}x\omega - t,\omega) \\ &-\rho_2^{-3/2}\omega \times (\omega \times k)(\rho_2^{-1}x\omega - t,\omega)\}d\omega, \end{aligned}$$

where $k(s,\omega) = T_0(f)(s,\omega)$ for $f \in \mathscr{S}(R^3)$.

2. Preliminary about the obstacle problem.

We shall consider a mixed boundary value problem for the elastic wave equation with the Dirichlet condition in the exterior domain. Let $\mathcal{O}$ be a compact obstacle in R^3 with the smooth boundary such that $\Omega = \mathrm{R}^3 \setminus \mathcal{O}$ is connected. The problem to be considered is as follows:

$$
\begin{cases} \partial_t^2 \mathrm{u} - (\lambda + \mu)\,\mathrm{grad}\,(\mathrm{div}\, u) - \mu \Delta u = 0 \ \text{in} \mathrm{R} \times \Omega, \\ \partial_t^{j-i} \mathrm{u} = \mathrm{f}_j(\mathrm{x}) \ \text{ on } \ t = 0 \ (j = 1, 2), \ u = 0 \ \text{ on } \ \mathrm{R} \times \partial\Omega. \end{cases} \tag{2.1}
$$

First by the argument of Section 1.2 in [9] we have the following

PROPOSITION 2.1. *The stationary problem of* (2.1)

$$
\begin{cases} (\lambda + \mu)\,\mathrm{grad}\,(\mathrm{div}\, u) + \mu \Delta u = f(x) \ \text{ in}\, \Omega, \\ u = g(x) \ \text{ on } \ \partial\Omega, \end{cases}
$$

is a coercive elliptic boundary value problem.

We introduce the following notation: Ω_R denotes the set of $\{x \in \Omega \colon |x| < R\}$ and $E(u(t), R)$ stands for the energy conaind in the dmain Ω_R of a solution $u(t, x)$ of the elastic wave equation (2.1) at time t, that is

$$
E(u(t), R) = \int_{\Omega_R} \{(\lambda + 2\mu)\,|\mathrm{div}\, u(t, x)|^2 + \mu \sum_{j=1}^{3} |\mathrm{grad}\, u_j(t, x)|^2 + |u_t(t, x)|^2\} dx/2,
$$

where $u = {}^t(u_1, u_2, u_3)$. We begin by proving the following inequality, that is classical for the wave equation, i.e., $\lambda + \mu = 0$.

THEOREM 2.2. *Let $u(t, x)$ be a smooth solution of* (2.1). *Then the following energy inequality holds:*

$$
E(u(T), R - \rho\,|T|) \leqq E(u(0), R) \leqq E(u(T), R + \rho|T|), \tag{2.2}
$$

where $\rho = (3\lambda + 4\mu)^{1/2}$.

PROOF. For $T > 0$, we shall prove the right inequality of (2.2). Put

$$
X_0(t, x) = |u_t|^2 + (\lambda + \mu)\,|\mathrm{div}\, u|^2 + \mu \sum_{k=1}^{3} |\mathrm{grad}\, u_k|^2,
$$

$$
X_j(t, x) = -\mu \sum_{k=1}^{3} \left(\frac{\partial u_k}{\partial t} \frac{\partial \bar{u}_k}{\partial x_j} + \frac{\partial \bar{u}_k}{\partial t} \frac{\partial u_k}{\partial x_j} \right) - (\lambda + \mu) \left(\frac{\partial u_j}{\partial t} \mathrm{div}\, \bar{u} + \frac{\partial \bar{u}_j}{\partial t} \mathrm{div}\, u \right).
$$

Then the following relation holds:

$$
\frac{\partial X_0}{\partial t} + \sum_{j=1}^{3} \frac{\partial X_j}{\partial x_j} = (u_{tt} - L(\partial_x)u) \cdot \bar{u}_t + u_t \cdot (\bar{u}_{tt} - L(\partial_x)\bar{u}) = 0.
$$

For some $\alpha > 0$ denote $\{(t, x) \in (0, T) \times \Omega \colon |x| < R + \alpha t\}$ by D, then from the divergence theorem it follows that $\int_{\partial D} (X_0 n_t + X \cdot n)\, dS = 0$, where (n_t, n) is the unit outer normal vector of ∂D and $X = (X_1, X_2, X_3)$. Since $n_t = 0$ and $X = 0$ on

$\partial D \cap (R \times \partial\Omega)$, we see that

$$E(u(T), R + \alpha T) = E(u(0), R) + \int_\Gamma (X_0 n_t + X \cdot n) dS,$$

where $\Gamma = \partial D \cap \{(t, x) : |x| = R + \alpha t\}$. In order to obtain the desired inequality (2.2) we shall look for the smallest positive α such that

$$X_0 - x \cdot X/(\alpha|x|) > 0, \tag{2.3}$$

because on Γ the unit outer normal vector is $(-\alpha, x/|x|)/(1 + \alpha^2)$.

Put $\mu' = \mu\alpha^{-1}, \lambda' = \lambda\alpha^{-1}$, $Y_1 = {}^t(x_1 u_{1,t}, x_2 u_{2,t}, x_3 u_{3,t})/|x|$, $Y_2 = {}^t(x_2 u_{1,t}, x_3 u_{1,t}, x_1 u_{2,t}, x_3 u_{2,t}, x_1 u_{3,t}, x_2 u_{3,t})$, $Y_3 = {}^t(u_{1,1}, u_{2,2}, u_{3,3})$ and $Y_4 = {}^t(u_{1,2}, u_{1,3}, u_{2,1}, u_{2,3}, u_{3,1}, u_{3,2})$, where we use simplified notations $v_t = \partial v/\partial t$, $v_k = \partial v/\partial x_k$. From $\sum_{k=1}^{3} |u_{k,t}|^2 = \sum_{k,j=1}^{3} |x_j u_{k,j}/|x||^2$, (2.3) is denoted by the following quadratic form: $X_0 - x \cdot X/(\alpha|x|) = (AY) \cdot Y$, where $Y = {}^t({}^tY_1, {}^tY_2, {}^tY_3, {}^tY_4)$ and

$$A = \begin{pmatrix} E_3 & 0 & \mu' E_3 + (\lambda' + \mu')I & 0 \\ 0 & E_6 & 0 & \mu' E_6 \\ \mu' E_3 + (\lambda' + \mu')I & 0 & \mu E_3 + (\lambda + \mu)I & 0 \\ 0 & \mu' E_6 & 0 & \mu E_6 \end{pmatrix}.$$

Here I is the 3×3 matrix whose all components are 1. Let us look for an eigenvalue a of A. The equivalent condition $AY = aY$ is

$$\begin{cases} Y_2 + \mu' Y_4 = aY_2, & Y_1 + (\mu' E_3 + (\lambda' + \mu')I) Y_3 = aY_1, \\ \mu' Y_2 + \mu Y_4 = aY_4, & (\mu' E_3 + (\lambda' + \mu')I' Y_1 + (\mu E_3 + (\lambda + \mu)I Y_3 = aY_3. \end{cases} \tag{2.4}$$

By the first part of (2.4) we have that $Y_2 = Y_4 = 0$, or a is a root of $f(a) = \mu'^2 + (\mu - a)(a - 1) = 0$. By $f((\mu + 1)/2 > 0$, a condition that the equation $f(a) = 0$ has only non-negative roots is $f(0) \leqq 0$, that is $\mu^{1/2} \leqq \alpha$. Since $\det(\mu' E_3 + (\lambda' + \mu')I) = \mu'^2(3\lambda' + 4\mu') > 0$ and $f(1) = 0$, if $a = 1$ in (2.4), then $Y = 0$. Thus an equivalent condition of the second part of (2.4) is

$$\begin{cases} Y_1 = (\mu' E_3 + (\lambda' + \mu')I) Y_3/(a - 1), \\ \{(\mu' E_3 + (\lambda' + \mu')I)^2 + (a - 1)(\mu E_3 + (\lambda + \mu)I) - a(a - 1)\} Y_3 = 0. \end{cases} \tag{2.5}$$

The last equality of (2.5) is $[\{\mu'^2 + (\mu - a)(a - 1)\} + \{(a - 1)(\lambda + \mu) + 2\mu'(\lambda' + \mu') + 3(\lambda' + \mu')^2\}I] Y_3$, which is denoted by $(p + qI)Y_3 = 0$. Thus if Y_1, Y_3 and a satisfy (2.5), then $Y_1 = Y_3 = 0$, or $\det(pE_3 + qI) = p^2(p + 3q) = 0$. We have already proved that the roots of $p = f(a) = 0$ are non-negative if $\alpha \geqq \mu^{1/2}$. Put $g(a) = p + 3q = -\{a^2 - (3\lambda + 4\mu + 1)a - \mu'^2 - 6\mu'(\lambda' + \mu') - 9(\lambda' + \mu')^2 + 3\lambda + 4\mu\}$: then we have $g((3\lambda + 4\mu + 1)/2) = ((3\lambda + 4\mu - 1)/2)^2 + ((3\lambda + 4\mu)/2)^2 > 0$, $g(0) = ((3\lambda + 4\mu)/\alpha)^2 - (3\lambda + 4\mu)$. An equivalent condition that all roots of

$g(a) = 0$ are non-negative is $g(0) \leqq 0$, i.e., $\alpha \geqq (3\lambda + 4\mu)^{1/2}$. The proof is completed.

From Theorem 2.2 we have the following

COROLLARY 2.3. *If initially the total energy of u is finite, u has the same energy for all time.*

In order to express solutions of (2.1) by using an one parameter unitary group, we introduce function spaces. The Hilbert space H_D is the completed space of $C_0^\infty(\Omega)$ by the following norm

$$\|g\|_D^2 = (\lambda + \mu)\int_\Omega |\operatorname{div} g|^2\, dx + \mu \sum_{j=1}^3 \int_\Omega |\operatorname{grad} g_j|^2\, dx,$$

where $g = {}^t(g_1, g_2, g_3)$. The space H consists of all vectors $f = \{f_1, f_2\}$ whose first component f_1 belongs to H_D and whose second component f_2 is square integrable over Ω, and the norm of H is $\|f\|_E^2 = \|f_1\|_D^2 + \|f_2\|_{L^2(\Omega)}^2$. The domain of a shew self adjoint operator A which is similarly defined on H to A_0 is the set of all data $f = \{f_1, f_2\}$ such that $f_1 \in H_D$ and $L(\partial_x)f_1$, defined in the sense of distributions, is square integrable over Ω and that $f_2 \in L^2(\Omega)$ belongs to H_D (see Theorem 1.2 in Chapter V of [4]). The one parameter unitary group on H with the infinitesimal generator A is denoted by $\{U(t)\}$.

From Lemma 1.1 and an energy inequality of a coercive elliptic boundary value problem, we have

LEMMA 2.4. *If $u(x)$ belongs to H_D and $L(\partial_x)u$ is square integrable, then all second derivatives of u are square integrable, and*

$$\sum_{|\alpha|=2} \|\partial_x^\alpha u\|_{L^2(\Omega)} \leqq C(\|u\|_D + \|Lu\|_{L^2(\Omega)}). \tag{2.6}$$

PROOF. Let $\alpha(x)$ be a $C_0^\infty(R^3)$ function such that $\alpha = 1$ in a neighbourhood of $\mathrm{R}^3 \setminus \Omega$. Taking the Fourier transform of $(1 - \alpha)u$ and using the inequality (1.2) and the elliptic estimate for a coercive elliptic boundary value problem: $L(\partial_x)(\alpha u) = f \in \mathrm{L}^2(\Omega_R)$, $\alpha u = 0$ on $\partial\Omega_R$, where $\Omega_R = \Omega \cap \{x : |x| < R\}$ and $\operatorname{supp} \alpha \subset \{x : |x| < R\}$, we have the desired (2.6). The proof is completed.

The first component of an element of $D(A)$ satisfies the conditions of Lemma 2.4. So we have

THEOREM 2.5. i) *Theorem 2.2 and its corollary 2.3 hold for all solutions u* of the form $[U(t)f]_1$, where *f is in H.*

ii) *Let F be the set of data f such that $\|Af\|_E + \|f\|_E \leqq C$, then F is precompact in the local energy norm $\|f\|_E^{\Omega'}$, that is the energy of f on Ω', for any bounded subset Ω' of Ω.*

Making use of Theorem 2.5, we have the following

THEOREM 2.6 (see Lemma 2.2, 2.3 and 2.4 in Chapter V of [4]). *If A has no point spectrum, then the local energy decay*

$$\lim_{t\to\infty} \|U(t)f\|_E^{\Omega'} = 0 \tag{2.8}$$

holds for all f in H and every bounded subdomain Ω' of Ω.

In order to prove the assumption of Theorem 2.6 we shall define a operator $J_j (j = 1, 2)$ for $C_0^\infty(\mathsf{R} \times S^2)$ given by

$$(J_j k)(x) = {}^t\{\int P_j(\omega)k(\rho_j^{-1}x\omega, \omega)d\omega, -\int P_j(\omega)(\partial_s k)(\rho_j^{-1}x\omega, \omega)d\omega\}/(2\pi\rho_j^{3/2}), \tag{2.9}$$

where $P_1(\omega)$ is the 3×3 matrix given by $\omega^t\omega$ and $P_2(\omega)a = (E_3 - P_1(\omega))a = -\omega \times (\omega \times a)$. Since for any $\phi = (\phi_1, \phi_2) \in C_0^\infty(\mathsf{R}^3)$

$$\langle J_j k, \phi\rangle = \int k(s, \omega)P_j(\omega)(\tilde{\phi}_1 + \rho_j \partial_s \tilde{\phi}_2)(\rho_j s, \omega)ds\, d\omega/\rho_j^{1/2}, \tag{2.10}$$

we can extend the operator J_j for $k \in \mathscr{D}'(\mathsf{R} \times S^2)$ as the weak limit of the right hand side of (2.10). The following properties of J_j are easily proved from (2.9): $J_j(-\partial_s) = \begin{pmatrix} 0 & E_3 \\ \rho_j^2 \Delta & 0 \end{pmatrix} J_j = A_j J_j$, where A_j is defined by the last equality, $(J_1 + J_2)T_0(g) = g$ for $g \in H_0$, $J_1(\omega \times (\omega \times k)) = J_2((\omega \cdot k)\omega) = 0$, the rotations of the first three components and the last three components of $J_j k$ are zero, and the divergences of the first three components and the last three components of $J_2 k$ are zero. We also need to define similar operators $\mathscr{J}_j$ $(j = 1, 2)$ which are defined by (2.9) as $P_2(\omega) = E_3$ when $j = 2$, and as $P_1(\omega) = 1$ and for scalar functions $k(s, \omega)$ when $j = 1$. The operators $\mathscr{J}_j$ satisfy that $\begin{pmatrix} 0 & E \\ \rho_j^2 \Delta & 0 \end{pmatrix} \mathscr{J}_j = -\mathscr{J}_j \partial_s$, where $E = 1$ if $j = 1$ and $E = E_3$ if $j = 2$, and the following

THEOREM 2.7 (see Theorem 3.2 in Chapter IV of [4]). *Let l be a distribution which is zero for $|s| > r$; then $\mathscr{J}_j l$ is zero for $|x| > r/\rho_j$ if and only if l satisfies the orthogonal conditions $\langle s^\beta Y_m(\omega), l\rangle = 0$ for all $\beta \leqq m$ and all spherical harmonics $Y_m(\omega)$ of order m.*

Making use of the operators and Theorem 2.7, we can prove the following

THEOREM 2.8. *The generator A has no point spectrum.*

PROOF. We assume that there is $f = \{f_1, f_2\} \in D(A)$ such that $(A - i\sigma)f = 0$. First we assume $\sigma = 0$, i.e., $L(\partial_x)f_1 = 0$, $f_2 = 0$. Take an inner product $L(\partial_x)f_1$ and $g \in C_0^\infty(\Omega)$ in $L^2(\Omega)$; then by the integration by parts we see that $(f_1, g)_D = 0$ for all $g \in C_0^\infty(\Omega)$, where $(\,,)_D$ is the inner product of H_D. Since $C_0^\infty(\Omega)$ is a dense set of H_D, 0 is not in the spctrum of A.

Next we shall consider the case $\sigma \neq 0$. Let $\Phi(x)$ be a scalar C^∞ function which vanishes near $\partial\Omega$ and is equal to one for $|x| > \rho$. Then by the ellipticity of $L(\partial_x)$ $g = \Phi f$ satisfies the equation $(A_0 - i\sigma)g = h = {}^t(h_1, h_2) \in C_0^\infty(\mathrm{R}^3)$. Since f belongs to $D(A)$, the translation representer $k(s, \omega)$ of g satisfies the following conditions: $\partial_s k \in L^2(\mathrm{R} \times S^2)$ and $-(\partial_s + i\sigma)k = T_0(h)$. These conditions imply that the support of $k(s, \omega)$ is compact, becasue the support of $T_0(f)$ is compact. Taking the divergence of the first three components and the last three components of the both side of $(A_1 - i\sigma)J_1((\omega \cdot k)\omega) = J_1((\omega \cdot T_0(h))\omega) = h - J_2(T_0(h))$ and making use of the properties on J_j, we have

$$\left\{\begin{pmatrix} 0 & 1 \\ \rho^2 \Delta & 0 \end{pmatrix} - i\sigma\right\} \mathscr{G}_1(\rho_1^{-1}\, \omega \cdot \partial_s k) = {}^t\{\operatorname{div} h_1, \operatorname{div} h_2\}. \tag{2.10}$$

Since the left hand side of (2.10) is equal to $-\mathscr{G}_1((\partial_s + i\sigma)(\rho_1^{-1}\omega \cdot \partial_s k))$, and the supports of $(\partial_s + i\sigma)(\omega \cdot \partial_s k)$ and $\operatorname{div} h_j$ have compact suports, we can apply Theorem 2.7 as $l = (\partial_s + i\sigma)(\omega \cdot \partial_s k)$, which implies that for all spherical harmonics $Y_m(\omega)$ of order m and $\beta \leqq m$

$$i\sigma \langle s^\beta Y_m(\omega), \omega \cdot \partial_s k\rangle - \beta\langle s^{\beta-1} Y_m(\omega), \omega \cdot \partial_s k\rangle = 0.$$

By the induction with respect to β we see that for all $\beta \leqq m$ and $Y_m(\omega)$ $\langle s\beta^K Y_m(\omega), \omega \cdot \partial_s k\rangle = 0$. It follows from Theorem 2.7 that $\mathscr{G}_1(\omega \cdot k)$ has the compact support. By taking the divergences of the first three components and the last three components of $\mathscr{G}_1(\omega \cdot k)$, we can conclude that $(J_1 k)(x)$ has the compact support. Similarly taking rotations of the first three components and the last three components of $(A_2 - i\sigma)(J_2(\omega \times (\omega \times k)))$, we see that $J_2(k)(x)$ has the compact support. Thus $g = (J_1 + J_2)k$ has the compact support. From this fact and the analyticity of f_1 in Ω the function f vanishes on Ω. The proof is completed.

3. Theorems on the propagation of singularities.

In this section we shall study the propagation of singularities of solutions to the boundary value problem of elastic wave equations with the Dirichlet condition:

$$\partial_t^2 u - (\lambda + \mu)\operatorname{grad}(\operatorname{div} u) - \mu\Delta u = f \quad \text{in } \Omega \times \mathrm{R}, \tag{3.1}$$

$$u = g \quad \text{on } \partial\Omega \times \mathrm{R}, \tag{3.2}$$

where Ω is an arbitrary domain with the smooth boundary in R^3 and f and g are smooth functions. In the free boundary condition case, i.e., the boundary condition is $\sum_i n_i(x)\sigma_{ij}(u) = 0$, where the stress tensor $\sigma_{ij}(u)$ is $\lambda(\operatorname{div} u)\delta_{ij} + \mu(\partial u_j/\partial x_i + \partial u_i/\partial x_j)$ and $n(x)$ is the unit outer normal vector at $x \in \partial\Omega$, the same problem is discussed in Chapter I of [9]. Since in the case of the Dirichlet boundary condition the proofs of theorems, which will be mentioned below, are almost same as these of theorems appeared in Chapter I of [9], we state these on a propagation of singularities without proofs. In order to state the theorems we need the notion of the set $WF_b(u)$ in $(T^*(\Omega \times \mathrm{R})\setminus 0) \cup (T^*(\partial\Omega \times \mathrm{R})\setminus 0)$ for a solution $u(x, t)$ of (3.1) and (3.2), which is defined in Definition 1.7 of [6]. It is not easy to completely understand the meaning of $WF_b(u)$. That is a generalization of singularities of $u(t, x)$ in $\bar{\Omega} \times \mathrm{R}$ in the following sense: Let π be the projection from $T^*(\Omega \times \mathrm{R}) \cup T^*(\partial\Omega \times \mathrm{R})$ to $\bar{\Omega} \times \mathrm{R}$. Then (t_0, x_0) does not belong to $\pi(WF_b(u))$ if and only if there exists an open neighbourhood of V_0 of (t_0, x_0) such that $u(t, x) \in C^\infty(\bar{\Omega} \cap V_0)$. Thus if we consider the projected rays to $\bar{\Omega} \times \mathrm{R}$ of the rays appeared in the following theorems, our theorems are regarded as these on the propagation of singularities along the projected ray in $\bar{\Omega} \times \mathrm{R}$ to solutions of (3.1) and (3.2).

Since (3.1) is rotation free, we may assume that in a neighbourhood of $0 \in \partial\Omega$ Ω is defined by $x_3 > g(x')$ with $(\operatorname{grad} g)(0) = (\nabla g)(0) = 0$, where $x' = (x_1, x_2)$. By the coordinate transform κ in $T^*(\mathrm{R}^3 \times \mathrm{R})$: $y' = x'$, $y_3 = x_3 - g(x')$, $t = t$, $\eta' = \xi' + (\nabla g)(x)\xi_3, \eta_3 = \xi_3$, $\tau = \tau$, the determinant of the principal symbol of (3.1) becomes to $(\tau^2 - \mu|\bar{\eta} + Gn_3|^2)^2(\tau^2 - (\lambda + 2\mu)|\bar{\eta} + G\eta_3|^2) =$
$-\mu(\lambda + 2\mu)|G|^4\{(\eta_3 - a)^2 + r_2\}^2\{(\eta_3 - a)^2 + r_1\}$, where $G = (-\nabla g, 1)$, $a(y', \eta') = (\eta' \cdot \nabla g(y'))/|G|^2, \bar{\eta} = (\eta', 0)$ and

$$r_k(y', \eta', \tau) = \{(\rho_k^2|\eta'|^2 - \tau^2)|G|^2 - \rho_k^2(\eta' \cdot \nabla g)^2\}/(\rho_k^2|G|^4) \tag{3.3}$$

with $\rho_1 = (\lambda + 2\mu)^{1/2}$ and $\rho_2 = \mu^{1/2}$. Here we remark that κ is identical if $x = 0$ and $(0, t_0, \xi, \tau)$ belongs to $T^*(\mathrm{R} \times \partial\Omega)$ if and only if $\xi_3 = 0$. All points in $T^*_{(0,t_0)}(\mathrm{R} \times \Omega)$ are classified in the following five classes: i) $r_2(0, \xi'_0, \tau_0) > 0$, ii) $(r_1 r_2)(0, \xi'_0, \tau_0) < 0$, iii) $r_1(0, \xi'_0, \tau_0) < 0$, iv) $r_2(0, \xi'_0, \tau_0) = 0$, v) $r_1(0, \xi'_0, \tau_0) = 0$, because $(r_1 - r_2)(0, \xi, \tau) = (\lambda + \mu)\tau^2/(\mu(\lambda + 2\mu)) \geqq 0$. First we shall consider the case i). By the similar argument of proving Theorem 1.6 of [9] we have the following

THEOREM 3.1. *If* $(0, \xi'_0, \tau_0)$ *is an elliptic point*, i.e., $r_2(0, \xi'_0, \tau_0) > 0$, *then for any* $t_0 \in \mathrm{R}$ *the point of* $(0, t_0, \xi'_0, 0, \tau_0) \in T^*(\partial\Omega \times \mathrm{R})\setminus 0$ *does not belong to* $WF_b(u)$.

Let us consider the null bicharacteristic of $\tau^2 - \rho_k^2|\xi|^2$ passing through $(0, t_0, \xi'_0, \xi_3^\varepsilon, \tau_0)$, where $\xi_3^\varepsilon = \varepsilon(\tau_0^2/\rho_k^2 - |\xi'_0|^2)^{1/2}$ with $\varepsilon^2 = 1$. We denote by γ_k^ε the ray given by $\{(-\rho_k^2\xi_0^\varepsilon(t - t_0)/\tau_0, t, \xi_0^\varepsilon, \tau_0) \in T^*(\Omega \times R)\}$, where $\xi_0^\varepsilon = (\xi'_0, \xi_3^\varepsilon)$. The equivalent condition that $(0, \xi'_0, \tau_0)$ satisfies ii) is $\rho_1^2|\xi'_0|^2 > \tau_0^2 > \rho_2^2|\xi'_0|^2$. In this case by the similar argument of proving Theorem 1.8 of [9] we can prove the following

THEOREM 3.2. *We assume* $(0, \xi'_0, \tau_0)$ *satisfies the condition* ii). *Then* γ_1^ε $(\varepsilon = +, -)$ *does not exist and if* $WF_b(u) \cap \gamma_2^\varepsilon$ *is empty, then the point* $(0, t_0, \xi'_0, 0, \tau_0)$ *does not belong to* $WF_b(u)$ *and* $WF_b(u) \cap \gamma_2^{-\varepsilon}$ *is also empty.*

In the case iii), that is $\rho_2^2|\xi_0'|^2 > \tau_2^2$, there exist γ_1^ε and γ_2^ε for $\varepsilon^2 = 1$. By the similar argument of proving Theorem 1.9 of [9] we have the following

THEOREM 3.3. *In the case* iii) *if* $WF_b(u) \cap (\gamma_1^\varepsilon \cup \gamma_2^\varepsilon)$ *is empty*, $(0, t_0, \xi_0', \tau_0)$ *does not belong to* $WF_b(u)$ *and* $WF_b(u) \cap (\gamma_1^{-\varepsilon} \cup \gamma_2^{-\varepsilon})$ *is also empty.*

If $r_k(0, \xi_0', \tau_0) = 0$, the null bicharacteristic of $\tau^2 - \rho_k^2|\xi|^2$ passing through $\rho_0 = (0, t_0, \xi_0', 0, \tau_0)$ is tangent to $T^*(\partial\Omega \times \mathrm{R})$. In this case we need a notion of a generalized bicharacteristic γ_i of $\tau^2 - \rho_i|\xi|^2$, which is denoted in Definition 3.1 of [7]. In $T^*(\Omega \times \mathrm{R})$ γ_k is a null bicharacteristic of $\tau^2 - \rho_k|\xi|^2$ and near ρ_0 satisfying ii) or iii) that is a broken null bicharacteristic defined by $\gamma_k^+ \cup \gamma_k^-$. If $\mathcal{O} = \mathrm{R}^3 \setminus \Omega$ is convex near 0, then for any $(0, \xi_0', \tau_0)$ belong iv) or v) γ_k is uniquely defined as the null bicharacteristic of $\tau^2 - \rho_k|\xi|^2$ passing through ρ_0. Therefore if $\mathcal{O}$ is a convex set, the following theorems are easily understood as theorems on the propagation of singularities along the broken characteristic curves of $\tau^2 - \rho_k|\xi|^2$ in $\bar{\Omega} \times \mathrm{R}$ to the solution $u(x,t)$. On the other hand if $\mathcal{O}$ is concave near 0, then for $(0, \xi_0', \tau_0)$ belonging to iv) or v) γ_k is the projected ray to $T^*(\partial\Omega \times \mathrm{R})$ of the ray $\kappa^{-1}(y'(t), 0, t, a(y_1(y), \eta'(t), \tau(t))$, where the ray $\{(y'(t), t, \eta'(t), \tau(t)) : |t| < \sigma\}$ is the null bicharacteristic of $-\rho_k^2|G|^2 r_k$ passing through $(0, t_0, \xi_0', \tau_0)$ parametrized by time t. If a null bicharacteristic is a tangential ray to $T^*(\partial\Omega \times \mathrm{R})$ of order ∞, generally γ_k is not unique. Thus let $\Gamma_i^+ (\Gamma_i^-)$ be a union of all half generalized bicharacteristics of $\tau^2 - \rho_k^2|\xi|^2$ starting at $(0, t_0, \xi_0', \tau_0)$ whose t component is greater (less) than t_0. Then by the similar argument of proving Theorem 1.24 in [9] we have the following

THEOREM 3.4. *We assume the case* iv). *Then* γ_1 *does not exist and if* $WF_b(u) \cap \Gamma_2^\varepsilon$ *is empty, then* $(0, t, \xi_0', 0, \tau_0)$ *does not belong to* $WF_b(u)$ *and* $WF_b(u) \cap \Gamma_2^{-\varepsilon}$ *is also empty, where* $\varepsilon = +$ *or* $-$.

In the case v) by the similar argument of proving Theorem 1.26 of [9] we have the following

THEOREM 3.5. *In the case* v) *we assume* $WF_b(u) \cap (\gamma_2^\varepsilon \cup \Gamma_1^{\varepsilon'})$ *is empty. Then* $(0, t, \xi_0', 0, \tau_0)$ *does not belong to* $WF_b(u)$ *and* $WF_b(u) \cap (\gamma_2^{-\varepsilon} \cup \Gamma_1^{-\varepsilon'})$ *is also empty, where* $\varepsilon, \varepsilon'$ *are* $+$ *or* $-$.

4. Exponential energy decay.

In this section we shall show exponential energy decay of solutions of elastic wave equations with the Dirichlet condition for non-trapping obstacles. In order to define non-trapping obstacles we use generalized bicharacteristics stated in Section 3. The projection into Ω of the generalized bicharacteristics is called generalized geodestics. We say that Ω is non-trapping if for any sufficiently large R there exists T_R such that no generalized geodesic of length T_R *lies completely* *within* $\Omega_R = \{x \in \Omega: |x| < R\}$. By theorems in Section 3 and the remark on $Wf_b(u)$ and generalized bicharacteristics convex obstacles are non-trapping. Since we can prove that star-shaped obstacles are also non-trapping, the following theorem is a generalized one of [2].

THEOREM 4.1. *We assume that Ω is non-trapping. Then for any R there exists positive contants $\alpha = \alpha(\Omega, R)$ and $C = C(\Omega, R)$ such that if the support of $f \in H$ is contained in $\Omega_R = \Omega \cap \{x: |x| < R\}$, then*

$$E(U(t)f\colon \Omega_R) \leqq Ce^{-\alpha t}E(f), \tag{4}$$

where $E(f)$ is the total energy of f in Ω, i.e., $E(f) = \|f\|_E$ *and $E(g\colon \Omega_R)$ is the local energy in Ω_R.*

In order to prove the above theorem we use an argument due to Morawetz in [8], where she considered the wave equation. Instead of Bertrami's form, which is used in the proof of Lemma 2 in [8], if we use the following lemma, we can use Theorem 1 of [8] for elastic wave equations.

LEMMA 4.2. *Let K be the subset* $(0, T) \times \{x: |x| < \mathrm{R}\}$ *of* $\mathrm{R}_+ \times R^3$. *We suppose that* $(\partial_t^2 - L(\partial_x))u = 0$ *for* $(t, x) \in \mathrm{R}_+ \times \mathrm{R}^3 \setminus K$, $(\partial_t^j u)(0, x) = 0$ *for* $|x| > R + \rho_2 T$ *and* supp $\partial_t^j u(T, x)$ *is compact, where* $j = 0,1$ *and* $\rho_2 = \mu^{1/2}$. *Then u vanishes in* $\rho_2(t - T) > R + |x|$.

PROOF. The lemma is true for the wave equation $\partial_t - v^2\Delta$ by Beltrami's form (see p. 575 of [1]). Since div u and rot u satisfy the wave equation with the speed ρ_2 and ρ_1, respectively, div u = rot u = 0 in $\rho_2(\mathrm{t} - \mathrm{T}) > \mathrm{R} + |\mathrm{x}|$. It follows that $\partial_t^2 u$ vanishes there. However from Corollary 1.3 and the assumption that supp $\partial_t^j u(T, x) \subset \{x: |x| < R_0\}$ for some R_0, we see that u vanishes in $\rho_2(\mathrm{t} - \mathrm{T}) > \mathrm{R}_0 + |\mathrm{x}|$. This implies that for fixed x we have $u(x, T_0) = (\partial_t u)(x, T_0) = 0$ *if* T_0 is sufficiently large. Thus by the equality

$$u(x, t) = u(x, T_0) + (\partial_t u)(x, T_0)(t - T_0) + \int_{T_0}^{t} ds \int_{T_0}^{s} (\partial_t^2 u)(x, \tau) d\tau$$

it follows that $u(t, x) = 0$ if $\rho_2(t - T) > R + |x|$. The proof is completed.

In order to show (4.1) by Theorem 1 of [8] we only prove that there exists a function $p(t)$ such that $\lim_{t \to \infty} p(t) = 0$ and

$$E(U(t)f\colon \Omega_R) \leqq p(t)E(f). \tag{4.2}$$

Let us give a proof of (4.2), by using the idea of [5]. Let $\alpha(x)$ be an element of $C_0^\infty(\{x: |x| < R\})$ such that $\alpha = 1$ for $|x| < R/2$, where we assume $\{x: |x| < R/2\} \supset \partial\Omega$. Since Ω is non-trapping, by the theorems of Section 3 there exists T such that

$$U(t)f \in C^\infty(\{x, t) \in \bar{\Omega} \times [T, \infty); |x| < R + \rho_2(t - T)\}) \tag{4.3}$$

for all $f \in X = \{f \in H: \operatorname{supp} f \subset \bar{\Omega}_R\}$. First we shall show that the function $p_1(t) = \sup\{E(U(t)(\alpha U(T)f)): \Omega_R): f \in X_0\}$, where $X_0 = X \cap \{f: E(f) = 1\}$, converges to 0 as t trends to ∞. By the closed graph theorem it follows that the mapping $\kappa(f) = \alpha U(T)f$ from X to $C^\infty(\bar{\Omega}_R)$ is continuous. Now we assume that $p_1(t)$ does not converge to 0 as t tends to ∞. Then there exist sequences $\{t_n\}$ of R and $\{f_n\}$ of X such that t_n tends to ∞ and for any t_n and f_n

$$E(U(t_n)(\alpha U(T)f_n): \Omega_R) > \delta > 0. \tag{4.4}$$

By the continuity of κ and Ascoli-Arzela's theorem we may assume that there exists $g \in H$ such that $\alpha U(T)f_n$ converges to g in H. Then by (4.4) it follows that

$\lim_{t_n \to \infty} E(U(t_n)g: \Omega_R) \neq 0$, which contradicts (2.8).

Next we shall consider $U(t)(1-\alpha)U(T)f$. First we prove the following

$$U(t)\alpha U(T)f \in C^\infty(\bar{\Omega} \times \bar{R}_+). \tag{4.5}$$

By (4.3) and Lemma 2.2 it follows that $g = U(t)\alpha U(T)f = U(t+T)f - U(t)(1-\alpha)U(T)f$ is a C^∞ function in $\{(x,t) \in \bar{\Omega} \times \mathsf{R}: t \geqq 0, |x| < R/2 - (3\lambda + 4\mu)^{1/2}t\}$. If we assume that $(\bar{x}, \bar{t}, \bar{\xi}, \bar{\tau}) \in WF_b(g)$, then by theorems of Section 3 there exists a ray $\rho(t)$ consisting of generalized bicharacteristics of $\tau^2 - \rho_1^2|\xi|^2$ or $\tau^2 - \rho_2^2|\xi|^2$ such that $\rho(\bar{t}) = (\bar{x}, \bar{t}, \bar{\xi}, \bar{\tau})$ and $\rho(t) \in WF_b(g)$ for all $t \leqq \bar{t}$. Let $\rho(0) = (x_0, 0, \xi_0, \tau_0)$; then $R/2 \leqq |x_0| < R$. Since for $\Phi \in C_0^\infty(U_0)$ with $\Phi = 1$ near x_0, where U_0 is a small neighbourhood of x_0, $U(t)\alpha U(T)f$ is equal to $U(t)\Phi\alpha U(T)f$ near $(x_0, 0)$, which is in $C^\infty(\bar{\Omega} \times \mathsf{R})$ because $\Phi\alpha U(T)f \in D(A^\infty)$, we have a contradiction. Next we shall show the following

$$w(x,t) = U_0(t)(1-\alpha)U(T)f \in C^\infty(\{x: |x| \leqq R/2\} \times \bar{\mathsf{R}}_+\}). \tag{4.6}$$

From Lemma 2.2 $w(x,t)$ is equal to $U(t)(1-\alpha)U(T)f$ in $\bar{\Omega} \times \{t: |t| < \delta\}$ for small δ. By this fact, (4.3) and (4.5) it follows that $w(x,t) \in C^\infty\{(x,t) \in \mathsf{R}^3 \times \mathsf{R}: |x| < R + \rho_2 t, 0 \leqq t < \delta\}$. Let $\bar{\rho} = (\bar{x}, \bar{t}, \bar{\xi}, \bar{\tau})$ belongs to $WF(w)$, where $|\bar{x}| \leqq R/2, t \geqq 0$: then there exists the null bicharactristic $\rho(t)$ passing through $\bar{\rho}$ of $\tau^2 - \rho_k^2|\xi|^2 (k = 1, 2)$ such that $\rho(t) \in WF(w)$ for $t \leqq \bar{t}$. Since the absolute value of the x component $x(0)$ of $\rho(0)$ is greater than R, the absolute value of the x component $x(t)$ of $\rho(t)$ is a decreasing function of t. For small $t > 0$ $\rho(t) \in WF_b(U(t)(1-\alpha)U(T)f) = WF_b(U(t+T)f)$ by (4.5). This implies that $\rho(t) \in WF_b(U(t+T)f)$ near $t = -T$. Since $|x(t)|$ is decreasing, $|x(-T)| > R$. This is a contradiction to $f \in X$.

Let us consider the mapping $\kappa(f) = w(x,t)$ from X to $C^\infty\{(x,t): |x| \leqq R/2, t \in [0, T_0]\}$, where T_0 is an arbitrary positive number; then from the closed graph theorem we have

$$\sup\{|\partial_t^j \partial_x^\alpha w(x,t)|: t \in [0, T_0], |x| \leqq R/2\} \leqq C_{j,\alpha} E(f). \tag{4.7}$$

By Huygen's principle and Lemma 2.2 it is not difficult to construct $\tilde{v}(x,t) = {}^t\{\tilde{v}_1, \tilde{v}_2\} \in H^{2m}(\Omega \times \mathrm{R}_+) \cap \mathscr{E}'(\bar{\Omega}_R \times (\delta, T_1))$ such that $L(\partial_x)^k(\tilde{v}_j - w_j) = 0$ on $\partial\Omega$, where $j = 1, 2$, $k = 0, 1, \ldots, m$, $0 < \delta < T_1$, $w(x,t) = {}^t\{w_1, w_2\}$, and

$$\|\tilde{v}_1\|_{H^{2m}} + \|\tilde{v}_2\|_{H^{2m}} \leqq C \sum_{j+|\alpha| \leqq M} \sup_{(x,t)\in D} |\partial_t^j \partial_x^\alpha w(x,t)| \tag{4.8}$$

for some positive constants C and M, where $D = \{(x,t)\colon\ t \in [0, T_1],\ |x| \leqq R/2\}$. Then $w - \tilde{v}$ satisfies the following

$$(\partial_t - A)(w - \tilde{v}) = \tilde{f}(x,t), \quad (w - \tilde{v})(x,0) = (1 - \alpha)U(T)f, \tag{4.9}$$

where $\tilde{f} = -{}^t\{\partial_t\tilde{v}_1 - \tilde{v}_2, \partial_t\tilde{v}_2 - L(\partial_x)\tilde{v}_1\}$. Here we remark that $(w - \tilde{v})(t, \cdot) \in D(A^{2m})$ and $\tilde{f}(t, \cdot) \in D(A^{2m-2})$ for all t. If f belongs to $D(A)$, then by the representation theorem to solutions of (4.9) we have $(w - \tilde{v})(x,t) = U(t)(1 - \alpha)\mathrm{U}(T)\mathrm{f} + \int_0^t U(\mathrm{t} - \mathrm{s})\tilde{f}(\mathrm{s}, \cdot)ds$. It follows that if t is sufficiently large, $U(t)(1 - \alpha)U(T)f = U(t)g$, where $g(x) = -\int_0^{T_0} U(-s)\tilde{f}(s, \cdot)ds$, which belongs to $D(A^{2m-2}) \cap \mathscr{E}'(\Omega_{R_0})$, where R_0 only depends on R. Since $E(U(t)(1 - \alpha)U(T)f\colon\ \Omega_R) \leqq E(U(t)g\colon\ \Omega_R)$, we shall show that $p_2(t) = \sup\{E(U(t)\mathrm{g}\colon\ \Omega_R)\colon\ f \in X_0 \cap D(A)\}$ converges to 0 as t tends to ∞. By the way of proving the property on $p_1(t)$, we may show that for all $f \in X_0 \cap D(A)$ $E_{m'}(g) = \|g_1\|_{H^{2m'+1}} + \|g_2\|_{H^{2m'}} \leqq C$, where m' is sufficiently large. By the similar way of proving (2.6) we have

$$\|g_1\|_{H^{2m'+1}} + \|g_2\|_{H^{2m'}} \leqq C\{E(A^{2m'-1}g) + E(A^{2m'}g)\}. \tag{4.10}$$

From (4.7), (4.8) and (4.10) it follows that for all $f \in X_0 \cap D(A)$ $E_{m'}(g)$ is bounded by some positive constant C that is independent of f. Since $X \cap D(A)$ is a dense set of X, the proof is completed as $p(t) = p_1(t) + p_2(t)$.

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