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PARABOLIC PSEUDO-DIFFERENTIAL INITIAL-BOUNDARY VALUE PROBLEMS

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Introduction.

Let $\bar{\Omega}$ be a compact and connected, n-dimensional C^∞ manifold with boundary Γ and interior Ω for some $n \geqq 2$. Suppose that E and F_k are complex vector bundles over $\bar{\Omega}$ and Γ, respectively, and let E^t denote the trivial extension of E to a bundle over $\bar{\Omega} \times \bar{\mathbf{R}}_+$, i.e., E is lifted to E^t by means of the projection $\bar{\Omega} \times \bar{\mathbf{R}}_+ \to \bar{\Omega}$, where $\bar{\mathbf{R}}_+ = \{t \in \mathbf{R} : t \geqq 0\}$. In this paper we are interested in studying parabolic initial-boundary value problems of the form

$$(\partial_t^m + \partial_t^{m-1}(P_\Omega^{(1)} + G^{(1)}) + \ldots + (P_\Omega^{(m)} + G^{(m)}))u = f \quad \text{in } \Omega \times \mathbf{R}_+,$$

$$\sum_{j=0}^{m-1} \partial_t^j T_k^{(m-j)} u = g_k \quad \text{on } \Gamma \times \mathbf{R}_+, k = 0,\ldots,md - 1 \text{ (see 1.2)},$$

$$\partial_t^j u|_{t=0} = h_j \quad \text{in } \Omega, j = 0,\ldots,m-1,$$

for sections u of E^t. Here $\partial_t = \partial/\partial t$, the operator $P_\Omega^{(j)}$ from E to E is defined by a classical pseudo-differential operator $P^{(j)}$ of order jd with the transmission property at Γ, $G^{(j)}$ is a singular Green operator of order jd from E to E, and $T_k^{(m-j)}$ is a trace operator of order $k - jd \geqq 0$ from E to F_k. The number d, the parabolic weight, will be an even positive integer.

It is known that the existence of a solution u requires some kind of compatibility of the data f, g_k, h_j. From several related discussions we refer to Agranovič–Višik [2], Donaldson [9], Iliev [18], Lions–Magenes [20] for the point of view using the solvability requirement of a suitable subproblem, to Bove–Franchi–Obrecht [6], Piriou [22] for the case of homogeneous initial values, and to Grubb [12], Grubb–Solonnikov [14], Ladyženskaja–Solonnikov–Ural'ceva [19] Lions–Magenes [20], Rempel–Schulze [27], Višik–Èskin [32] for the first order case ($m = 1$); see also Bogatova–Glushko [3], Čan Zuĭ Ho–Èskin [7], Schulze [28], and confer for example Agranovič [1], Chazarain–Piriou [8] for related hyperbolic problems. Our main purpose is here to find such compatibility conditions under which the problem is well-posed. So

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this work is an extension of our previous papers [24], [25], [26], which deal with scalar differential operators and the quarter space $\mathbf{R}^n_+ \times \mathbf{R}_+$.

In solving the problem we use the Laplace transformation method in the case of homogeneous initial values. By means of an adapted Paley–Wiener–Schwartz theorem this leads to a polynomially parameter-dependent elliptic boundary problem in Ω. A general theory for such parameter-dependent pseudo-differential boundary problems has recently been developed by Gerd Grubb in her monograph [12]. Since we want to apply this theory to our special case, it is convenient to formulate the problem following [12].

We shall work within the framework of standard Sobolev spaces (for functions or sections of vector bundles). These and other preliminaries are declared in Section 1. In Section 2 we consider polynomially parameter-elliptic pseudo-differential boundary problems and state the crucial results needed from Grubb [12]. The transition between parabolic problems with homogeneous initial values and parameter-elliptic boundary problems depends on an appropriate Laplace transform calculus, which is given in Section 3. In Section 4 we first derive the compatibility conditions which the data f, g_k, h_j have to satisfy, if the general problem has a solution. The main result is then that these conditions will also be sufficient for the well-posedness of the problem.

1. Notations.

1.1. Let $|y|$ denote the Euclidean norm of $y \in \mathbf{R}^k$ or $y = \operatorname{Re} y + i \operatorname{Im} y \in \mathbf{C}^k$ with $k \geqq 1$, and set $\langle y \rangle = (1 + |y|^2)^{1/2}$ for $y \in \mathbf{R}^k$. We write $x = (x_1, \ldots, x_{k-1}, x_k) = (x', x_k)$ for the generic point of $\mathbf{R}^k_x = \mathbf{R}^k = \mathbf{R}^{k-1} \times \mathbf{R}$ (with $k \geqq 2$) and D_j for the differential operator $-i\,\partial/\partial x_j$. The norm in a normed (complex) vector space H is denoted by $\|\cdot\|_H$.

1.2. We assume that the base manifold $\bar{\Omega}$ is smoothly imbedded into a compact and connected, n-dimensional Riemannian C^∞ manifold Σ without boundary. Let x and x' denote points in Σ and Γ, respectively, and choose a normal coordinate x_n near Γ such that Γ has (by indentification) a neighborhood in Σ in the form of the collar $\Sigma'_2 = \Gamma \times]-2,2[= \{x = (x', x_n) : x' \in \Gamma,\ x_n \in\ ]-2,2[\}$. Moreover, Σ and Γ are supposed to have positive C^∞ densities dx and dx', respectively, satisfying $dx = dx'\,dx_n$ in Σ'_2. See, for example, Hörmander [16], [17], Rempel–Schulze [27], Treves [31].

It is assumed that there is a Hermitean complex C^∞ vector bundle $\tilde{E}$ over Σ with fiber dimension $N \geqq 1$ such that $E = \tilde{E}|_{\bar{\Omega}}$. For the sake of simplicity it is also supposed that similarly $F_k = \tilde{F}_k|_\Gamma$ for some Hermitean complex C^∞ vector bundle $\tilde{F}_k$ over Σ with fiber dimension $M_k \geqq 0$ (it is notationally convenient to include here the case $M_k = 0$). The bundle $\tilde{E}$ over Σ'_2 can be identified with its lifting by the projection $\Sigma'_2 \to \Gamma$ (see Hörmander [17, Section 20.1]), and hence the normal derivatives $D_n^j u = (-i\partial/\partial x_n)^j u$ of sections u of $\tilde{E}$ near Γ are well-defined. An analogous identification is done for the bundle $\tilde{F}_k$. When a special choice of manifold and bundle structures is allowed, it is supposed to be done as in Grubb [12, Appendix], to which we refer for further details. Thus especially:

(i) Γ has a coordinate system $\kappa_i' : \Gamma_i \to X_i'$, $i = 1, \ldots, i_1$, and the collar $\Sigma_2' = \Gamma \times]-2,2[$ is covered by the coordinate patches $\Sigma_i = \Gamma_i \times]-2,2[$ for the charts $\kappa_i : \Sigma_i \to X_i = X_i' \times]-2,2[$ with $\kappa_i(x', x_n) = (\kappa_i'(x'), x_n), i = 1, \ldots, i_1$. For other charts $\kappa_i : \Sigma_i \to X_i$ with $i_1 < i \leqq i_\Sigma$ (for some finite i_Σ) the closures of the coordinate patches Σ_i do not intersect Γ, $\Sigma_i \subset \Omega$ and $X_i \subset \mathbb{R}_+^n$ for $i_1 < i \leqq i_2$, and $\Sigma_i \subset \Sigma \backslash \bar{\Omega}$ and $X_i \subset \mathbb{R}_-^n$ for $i > i_2$, where $\mathbb{R}_\pm^n = \{(x', x_n) \in \mathbb{R}^n : x_n \gtrless 0\}$.

(ii) The bundles $\tilde{E}$ and $\tilde{F}_k$ are trivial over every Σ_i with local trivializations

$$\psi_i : \tilde{E}|_{\Sigma_i} \to X_i \times \mathbb{C}^N,$$

$$\psi_i' : E|_{\Gamma_i} \to X_i' \times \mathbb{C}^N \quad \text{for} \quad i = 1, \ldots, i_1,$$

and, respectively,

$$\zeta_{k,i} : \tilde{F}_k|_{\Sigma_i} \to X_i \times \mathbb{C}^{M_k},$$

$$\zeta_{k,i}' : F_k|_{\Gamma_i} \to X_i' \times \mathbb{C}^{M_k} \quad \text{for} \quad i = 1, \ldots, i_1$$

such that, for example, if $x \in \Sigma_i$ then $\psi_i(w) = (\kappa_i(x), \psi_{i,x} w)$ for $w \in \tilde{E}_x$ with the fiber isomorphism $\psi_{i,x} : \tilde{E}_x \to \mathbb{C}^N$.

(iii) The charts $\kappa_i : \Sigma_i \to X_i$ are so chosen that the structures of cotangent bundles can also be described suitably.

1.3. Let $C^\infty(X; H)$ denote the space of C^∞ functions from an open subset X of $\mathbb{R}^k$ (or some C^∞ manifold) to a Hilbert space H, and $C_0^\infty(X; H)$ the space of $C^\infty(X; H)$ functions with compact support in X. The symbol $H = \mathbb{C}$ will not be indicated. Let $r_X : u \mapsto u|_X$ be the restriction operator to X and e_X the corresponding extension by zero outside X. In particular, the restriction to $x_k > 0$ is generally denoted by r^+, and the extension by zero to $x_k < 0$ by e^+. We then set

$$C^\infty(\bar{X}; H) = r_X C^\infty(\mathbb{R}^k; H), \qquad C_c^\infty(\bar{X}; H) = r_X C_0^\infty(\mathbb{R}^k; H),$$

and

$$C^\infty(X \cap \bar{\mathbb{R}}_+^k; H) = r^+ C^\infty(X; H), \qquad C_c^\infty(X \cap \bar{\mathbb{R}}_+^k; H) = r^+ C_0^\infty(X; H).$$

Analogously we let, for example, $C^\infty(\Sigma, \tilde{E})$ denote the space of C^∞ sections of the bundle $\tilde{E}$ over Σ. Note that $C^\infty(\Sigma, \tilde{E}) = C_0^\infty(\Sigma, \tilde{E})$ because Σ is supposed to be compact.

We shall use the standard Sobolev spaces H^s for $s \geqq 0$, allowing however their norms to depend on a complex parameter z (see [12], [15], [17], [20], [21], [23], [27], [30], [33]). To fix these spaces it therefore suffices to define the basic space

$H^s_z(\mathsf{R}^k)$. Note that in all cases the parameter-independent spaces H^s are special cases of H^s_z for $z = 0$, and that $H^s_z = H^{s,\mu}$ in the notation of Grubb [12] with $\mu = |z|^{1/d}$.

We set

$$H^s_z(\mathsf{R}^k) = \{u \in \mathscr{S}'(\mathsf{R}^k) : \langle(\xi, \mu)\rangle^s \mathscr{F} u \in L^2(\mathsf{R}^k_\xi)\}$$

with the norm

$$\|u\|_{H^s_z(\mathsf{R}^k)} = \|\langle(\xi, \mu)\rangle^s \mathscr{F} u\|_{L^2(\mathsf{R}^k_\xi)},$$

where $\mathscr{S}'(\mathsf{R}^k)$ is the Schwartz space of tempered distributions in R^k, the dual space of $\mathscr{S}(\mathsf{R}^k)$. For the Fourier transformation $\mathscr{F} = \mathscr{F}_{x \to \xi}$ we use the definition

$$(\mathscr{F} u)(\xi) = \int e^{-i(x_1\xi_1 + \ldots + x_k\xi_k)} u(x)\, dx.$$

Further, if X is an open subset in R^k, then $H^s_z(\bar{X}) = r_X H^s_z(\mathsf{R}^k)$, equipped with the infimum norm

$$\|u\|_{H^s_z(\bar{X})} = \inf\{\|v\|_{H^s_z(\mathsf{R}^k)} : u = r_X v\}$$

(in "smooth" cases the closure notation only points out the way of definition). The definitions of the spaces $H^s_z(\mathsf{R}^k; H)$ and $H^s_z(\bar{X}; H)$ of functions valued in a (separable) Hilbert space H are analogous. The closure of $C^\infty_0(\mathsf{R}_+; H)$ in $H^s(\bar{\mathsf{R}}_+; H)$ is the space $H^s_0(\mathsf{R}_+; H)$. In order to obtain a scale of interpolation spaces (for $s \geqq 0$), replace the space $H^s_0(\mathsf{R}_+; H)$, for $s = k + 1/2$ with $k \in \mathsf{N}$, by the intermediate space $[H^{k+1}_0(\mathsf{R}_+; H), H^k_0(\mathsf{R}_+; H]_{1/2}$ provided with the norm

$$(\|u\|^2_{H^s(\bar{\mathsf{R}}_+; H)} + \|t^{-1/2}\partial^k_t u\|^2_{L^2(\mathsf{R}_+; H)})^{1/2},$$

and denote by $H^s_{(0)}(\mathsf{R}_+; H)$ the Hilbert spaces so obtained (see [20], [24]). We shall also need for $\rho > 0$ the space

$$H^s(\bar{\mathsf{R}}_+, \rho; H) = \{u \in \mathscr{D}'(\mathsf{R}_+; H) : e^{-\rho t} u \in H^s(\bar{\mathsf{R}}_+; H)\}$$

with the norm

$$\|u\|_{H^s(\bar{\mathsf{R}}_+, \rho; H)} = \|e^{-\rho t} u\|_{H^s(\bar{\mathsf{R}}_+; H)};$$

the space $H^s_{(0)}(\mathsf{R}_+, \rho; H)$ is defined analogously.

The Sobolev space of order s for sections of $\tilde{E}$ over Σ is denoted by $H^s_z(\Sigma, \tilde{E})$. For any choice of a (finite) system of local trivializations $\psi_1, \ldots, \psi_{i_\Sigma}$ of $\tilde{E}$ and a corresponding partition of unity $\phi_1, \ldots, \phi_{i_\Sigma}$ (see 1.2), the definition

$$\|u\|^2_{H^s_z(\Sigma, \tilde{E})} = \sum_{i=1}^{i_\Sigma} \|\psi_i \circ (\phi_i u)\|^2_{H^s_z(\mathsf{R}^n; \mathsf{C}^N)}$$

gives a norm of $H_z^s(\Sigma, \tilde{E})$, where

$$\psi_i \circ v = (\psi_i^{-1})^* v \colon y \mapsto \psi_{i,\kappa_i^{-1}(y)} v(\kappa_i^{-1}(y))$$

(see [12], [17], [27]). Such norms are equivalent and $H_z^s(\Sigma, \tilde{E})$ is a Hilbert space with respect to each of them; indeed, $H_z^s(\Sigma, \tilde{E})$ is the completion of $C^\infty(\Sigma, \tilde{E})$. Analogously, a Hilbert norm for the space $H_z^s(\Gamma, F_k)$ of sections of F_k over Γ is given by

$$\|u\|^2_{H_z^s(\Gamma, F_k)} = \sum_{i=1}^{i_1} \|\zeta'_{k,i} \circ (\phi'_i u)\|^2_{H_z^s(\mathbb{R}^{n-1}; \mathbb{C}^{M_k})},$$

where $\phi'_i = r_\Gamma \phi_i$. The space $H_z^s(\bar{\Omega}, E) = r_\Omega H_z^s(\Sigma, \tilde{E})$ is also a Hilbert space with respect to the corresponding infimum norm. Recall that the trace operator $\gamma^j \colon u \mapsto r_\Gamma D_n^j u$ ($x_n = 0$ in local coordinates) is continuous from $H_z^s(\bar{\Omega}, E)$ to $H_z^{s-j-1/2}(\Gamma, E) = H_z^{s-j-1/2}(\Gamma, E|_\Gamma)$ for $s > j + 1/2$. The trace operator with respect to the variable t will be denoted by $\gamma_t \colon v \mapsto v|_{t=0}$.

2. Polynomially parameter-elliptic operators and parabolicity.

The considerations in this section are adapted to our polynomially parameter-dependent situation from the general theory of Grubb [12]. For further details we therefore refer to this monograph; see also Boutet de Monvel [4], [5], Hörmander [16], [17], Rempel–Schulze [27].

2.1 We shall consider an operator (system)

$$A(z) = \begin{bmatrix} P_\Omega(z) + G(z) \\ T(z) \end{bmatrix} \colon C^\infty(\bar{\Omega}, E) \to \begin{matrix} C^\infty(\bar{\Omega}, E) \\ \times \\ C^\infty(\Gamma, F) \end{matrix}, \tag{2.1}$$

where F stands for the direct sum $F_1 \oplus \ldots \oplus F_{md-1}$, the operator $P_\Omega(z)$ is defined by $P_\Omega(z) = r_\Omega P(z) e_\Omega$ with the *polynomially parameter-dependent pseudo-differential operator*

$$P(z) = z^m + z^{m-1} P^{(1)} + \ldots + P^{(m)} \colon C^\infty(\Sigma, \tilde{E}) \to C^\infty(\Sigma, \tilde{E}),$$

$G(z)$ is the *singular Green operator*

$$G(z) = z^{m-1} G^{(1)} + \ldots + G^{(m)} \colon C^\infty(\bar{\Omega}, E) \to C^\infty(\bar{\Omega}, E),$$

and $T(z) = (T_k(z))_{0 \leq k < md}$ is a *trace operator* with

$$T_k(z) = z^{m-1} T_k^{(1)} + \ldots + T_k^{(m)} \colon C^\infty(\bar{\Omega}, E) \to C^\infty(\Gamma, F_k).$$

Here we assume that $P^{(j)}$ is a pseudo-differential operator of order jd from $\tilde{E}$ to $\tilde{E}$ with the *transmission property* at Γ, $G^{(j)}$ is a singular Green operator of order jd

and class $r \leqq jd$ from E to E, and $T_k^{(j)}$ is a trace operator of order $r = k - (m-j)d \geqq 0$ and class $r+1$ from E to F_k and $T_k^{(j)} = 0$ for $k < (m-j)d$. Such an operator $A(z)$ is called a *(parameter-dependent) Green operator*, and it is of order md, class $\leqq md$, and in general of regularity d (for $P(z)$) and $v \geqq 1/2$ (for $G(z)$ and $T(z)$) in the sense of [12]. Recall that this means that for any system of local trivializations (see 1.2) $\psi_i : \tilde{E}|_{\Sigma_i} \to X_i \times \mathbf{C}^N$, $\zeta'_{k,i} : F_k|_{\Gamma_i} \to X'_i \times \mathbf{C}^{M_k}$ the operators

$$P^{(i,j)}(z) : C_0^\infty(X_i)^N \to C^\infty(X_j)^N : u \mapsto \psi_j \circ (P(z)(\psi_i^{-1} \circ u)),$$

$$G^{(i,j)}(z) : C_c^\infty(X_i \cap \bar{\mathbf{R}}^n_+)^N \to C^\infty(X_j \cap \bar{\mathbf{R}}^n_+)^N : u \mapsto \psi_j \circ (G(z)(\psi_i^{-1} \circ u)),$$

and

$$T_k^{(i,j)}(z) : C_c^\infty(X_i \cap \bar{\mathbf{R}}^n_+)^N \to C^\infty(X'_j)^{M_k} : u \mapsto \zeta'_{k,j} \circ (T_k(z)(\psi_i^{-1} \circ u))$$

form a Green operator in the above sense (see also 2.2).

2.2. The *parameter-ellipticity* of the Green operator (2.1) is defined by means of the *interior* and *boundary symbols*. For this, take a system of local trivializations as in 1.2, and let $\mathscr{P}^{(j)}$, $\mathscr{G}^{(j)}$, and $\mathscr{T}_k^{(j)}$ denote generically the corresponding local representations of $P^{(j)}$, $G^{(j)}$, and $T_k^{(j)}$, respectively. It is now assumed that (see [5], [12], also for notation):

(i) $\mathscr{P}^{(j)}$ is a pseudo-differential operator with a symbol

$$p_j(x, \xi) \in S^{jd}_{\mathrm{tr}}(X_i, \mathbf{R}^n) \oplus L(\mathbf{C}^N; \mathbf{C}^N),$$

i.e., $p_j(x, \xi)$ is an $N \times N$-matrix formed, polyhomogeneous pseudo-differential symbol of order jd with the transmission property at $x_n = 0$.

(ii) $\mathscr{G}^{(j)}$ is a singular Green operator with a symbol

$$g_j(x', \xi', \xi_n, \eta_n) \in S^{jd-1}(X'_i, \mathbf{R}^{n-1}, H^+ \hat{\oplus} H^-_{r-1}) \oplus L(\mathbf{C}^N; \mathbf{C}^N),$$

i.e. $g_j(x', \xi, \eta_n)$ is an $N \times N$-matrix formed, polyhomogeneous singular Green symbol of order jd and class $r \leqq jd$.

(iii) $\mathscr{T}_k^{(j)} = 0$ for $k < (m-j)d$, and $\mathscr{T}_k^{(j)}$ with $r = k - (m-j)d \geqq 0$ is a trace operator with a symbol

$$t_k^{(j)}(x', \xi', \xi_n) \in S^r(X'_i, \mathbf{R}^{n-1}, H_r^-) \oplus L(\mathbf{C}^N; \mathbf{C}^{M_k}),$$

i.e., $t_k^{(j)}(x', \xi)$ is an $M_k \times N$-matrix formed, polyhomogeneous trace symbol of order r and class $r+1$.

Thus it follows with $z = \mu^d e^{i\theta}$ that the parameter-dependent symbol

$$p_\theta(x, \xi, \mu) = p(x, \xi, z) = z^m + z^{m-1} p_1(x, \xi) + \ldots + p_m(x, \xi)$$

is of regularity d, and the symbols

$$g_\theta(x', \xi, \eta_n, \mu) = g(x', \xi, \eta_n, z) = z^{m-1} g_1(x', \xi, \eta_n) + \ldots + g_m(x', \xi, \eta_n)$$

and

$$t_{k,\theta}(x',\xi,\mu) = t_k(x',\xi,z) = z^{m-1}t_k^{(1)}(x',\xi) + \ldots + t_k^{(m)}(x',\xi)$$

are of regularity $\nu \geqq 1/2$ (see [12, Proposition 2.3.14]), so that we have for all $\theta \in [0,2\pi[$ (see [12, Sections 2.1–2.3])

$$p_\theta(x,\xi,\mu) \in S_{\mathrm{tr}}^{md,d}(X_i, \mathrm{R}^n, \bar{\mathrm{R}}_+) \oplus L(\mathrm{C}^N;\mathrm{C}^N),$$

$$g_\theta(x',\xi',\xi_n,\eta_n,\mu) \in S^{md-1,\nu}(X_i', \mathrm{R}^{n-1}, H^+ \hat{\oplus} H_{r-1}^-, \bar{\mathrm{R}}_+) \oplus L(\mathrm{C}^N;\mathrm{C}^N),$$

$$t_{k,\theta}(x',\xi',\xi_n,\mu) \in S^{k,\nu}(X_i', \mathrm{R}^{n-1}, H_k^-, \bar{\mathrm{R}}_+) \oplus L(\mathrm{C}^N;\mathrm{C}^{M_k}).$$

The principal symbol

$$p^0(x,\xi,z) = z^m + z^{m-1}p_1^0(x,\xi) + \ldots + p_m^0(x,\xi)$$

is quasi-homogeneous in (ξ, z) of degree md in the sense that

$$p^0(x,\tau\xi,\tau^d\xi) = \tau^{md}p^0(x,\xi,z) \quad \text{for } \tau \geqq 1, |\xi| \geqq 1,$$

and with it there is associated the (*strictly quasi-homogeneous*) *interior symbol*

$$p^h(x,\xi,z) = |\xi|^{md}p^0(x,\xi/|\xi|, z/|\xi|^d), \quad \xi \neq 0.$$

Similarly one associates the strictly quasi-homogeneous symbols $g^h(x',\xi,\eta_n,z)$ and $t^h(x',\xi,z)$ for $\xi' \neq 0$ (see [12, Section 2.8]) with the principal symbols

$$g^0(x',\xi,\eta_n,z) = z^{m-1}g_1^0(x',\xi,\eta_n) + \ldots + g_m^0(x',\xi,\eta_n)$$

and

$$t^0(x',\xi,z) = \begin{bmatrix} t_0^0(x',\xi,z) \\ \vdots \\ t_{md-1}^0(x',\xi,z) \end{bmatrix}, \; t_k^0(x',\xi,z) = z^{m-1}t_k^{(1)0}(x',\xi) + \ldots + t_k^{(m)0}(x',\xi).$$

Then the (*strictly quasi-homogeneous*) *boundary symbol*

$$a^h(x',\xi',\xi_n,z) = \begin{bmatrix} p^h(x',0,\xi',\xi_n,z) + g^h(x',\xi',\xi_n,\eta_n,z) \\ t^h(x',\xi',\xi_n,z) \end{bmatrix}$$

defines, through the operator definitions with respect to x_n (see [12, Section 2.4]), the (*strictly quasi-homogeneous*) *boundary symbol operator*

$$a^h(x',\xi,D_n,z) = \begin{bmatrix} p^h(x',0,\xi',D_n,z)_{\mathrm{R}_+} + g^h(x',\xi',D_n,z) \\ t^h(x',\xi',D_n,z) \end{bmatrix}$$

for $x' \in X_i'$, $\xi' \neq 0$, and

$$a^h(x',\xi',D_n,z) : \mathscr{S}(\bar{\mathrm{R}}_+)^N \to \mathscr{S}(\bar{\mathrm{R}}_+)^N \times \mathrm{C}^M$$

extends by continuity to a continuous operator

$$a^h(x',\xi',D_n,z):H^{md}(\bar{\mathrm{R}}_+)^N \to L^2(\mathrm{R}_+)^N \times \mathrm{C}^M,$$

where $\mathscr{S}(\bar{\mathrm{R}}_+) = r^+\mathscr{S}(\mathrm{R})$ and $M = \sum_k M_k$.

With these concepts and notations it is now possible to give the definitions of parameter-ellipticity and parabolicity.

2.3. DEFINITION. *Let $\theta \in [0,2\pi[$ be given. The parameter-dependent Green operator $A(z)$, given by* (2.1), *is (polynomially) parameter-elliptic on the ray $z = \rho\, e^{i\theta}$ with $\rho \geqq 0$, if the local representations satisfy the conditions* (i)–(iii):

(i) *The interior symbol $p^h(x,\xi,z):\mathrm{C}^N \to \mathrm{C}^N$ is invertible for all $x \in X_i$ and all* $(\xi,z) \neq (0,0)$.

(ii) *The boundary symbol operator*

$$a^h(x',\xi',D_n,z):H^{md}(\bar{\mathrm{R}}_+)^N \to L^2(\mathrm{R}_+)^N \times \mathrm{C}^M$$

is bijective for all $\xi' \neq 0$, all z, and for all $x' \in X_i'$.

(iii) *The boundary symbol operator $a^h(x',\xi',D_n,z)$ converges in the first symbol seminorms (see Grubb* [12, *Sections* 2.1, 2.3, 2.8]) *for $\xi' \to 0$ to a bijective operator*

$$a^h(x',0,D_n,z):H^{md}(\bar{\mathrm{R}}_+)^N \to L^2(\mathrm{R}_+)^N \times \mathrm{C}^M$$

for all $z \neq 0$ and all $x' \in X_i'$.

Note that condition (i) is equivalent to the ellipticity of the symbol $p(x,\xi,z)$ because of positive regularity (see Grubb [12, Proposition 2.1.11]); positive regularity is also used for (iii). Conditions (ii) and (iii) contain in an appropriate sense generalizations for the Šapiro–Lopatinskiĭ condition and the normality of trace operators; for analysis of (ii) and (iii) see Grubb [12], [13].

2.4. DEFINITION. *The operator*

$$A(\partial_t) = (P_\Omega(\partial_t) + G(\partial_t), T(\partial_t)) = \begin{bmatrix} P_\Omega(\partial_t) + G(\partial_t) \\ T(\partial_t) \end{bmatrix},$$

where $P_\Omega(\partial_t) = P(\partial_t)_\Omega = r_\Omega P(\partial_t) e_\Omega$,

$$P(\partial_t) = \partial_t^m + \partial_t^{m-1} P^{(1)} + \ldots + P^{(m)},$$

$$G(\partial_t) = \partial_t^{m-1} G^{(1)} + \ldots + G^{(m)},$$

$$T(\partial_t) = \partial_t^{m-1} T^{(1)} + \ldots + T^{(m)},$$

is parabolic, if the parameter-dependent Green operator $A(z) = (P_\Omega(z) + G(z), T(z))$ is parameter-elliptic on every ray $z = \rho e^{i\theta}$ with $-\pi/2 \leqq \theta \leqq \pi/2$ (i.e., for all z with $\operatorname{Re} z \geqq 0$).

2.5 REMARK. The transmission property and the parabolicity imply that d is even: if $\det(p^0(x',0,0,\xi_n,z))=0$, then also $\det(p^0(x',0,0,-\xi_n,(-1)^d z))=0$.

Reasoning as in Grubb [12, Section 1.5], we see that the limit operator $a^h(x',0,D_n,z)$ in the convergence requirement (iii) of the parameter-ellipticity is of the form

$$\begin{bmatrix} s_m(x')D_n^{md} + z\,s_{m-1}(x')D_n^{(m-1)d} + \ldots + z^m \\ s_0^{(m)}(x')\,\gamma^0 \\ \vdots \\ s_{d-1}^{(m)}(x')\,\gamma^{d-1} \\ \vdots \\ s_{(m-1)d}^{(m)}(x')\gamma^{(m-1)d} + z\,s_{(m-1)d}^{(m-1)}(x')\gamma^{(m-2)d} + \ldots + z^{m-1}s_{(m-1)d}^{(1)}(x')\gamma^0 \\ \vdots \\ s_{md-1}^{(m)}(x')\gamma^{md-1} + z\,s_{md-1}^{(m-1)}(x')\gamma^{(m-1)d-1} + \ldots + z^{m-1}s_{md-1}^{(1)}(x')\gamma^{d-1} \end{bmatrix},$$

where the $s_j(x')$ are $N\times N$-matrices and the $s_k^{(j)}(x')$ are $M_k\times N$-matrices. The differential operator can be reduced in a usual way to a system of first order differential operators with the coefficient matrix S, which is an $md\times md$-matrix of $N\times N$-blocks. It is known that

$$\det(\xi_n I - S) = \det(s_m(x')\xi_n^{md} + z\,s_{m-1}(x')\xi_n^{(m-1)d} + \ldots + z^m);$$

apply for example the generalized Gauss algorithm (see Gantmacher [10]). The roots of this polynomial now appear in pairs, and hence we can conclude from the bijectivity of the operator

$$a^h(x',0,D_n,z)\colon H^{md}(\bar{\mathrm{R}}_+)^N \to L^2(\mathrm{R}_+)^N \times \mathrm{C}^{M_0}\times\ldots\times\mathrm{C}^{M_{md-1}}$$

that $M=\sum_k M_k = \frac{1}{2}\,d\,m\,N.$

Note also that the forms of the operators $G^{(j)}$ and $T^{(j)}$ are partly determined by condition (iii) (cf. Grubb [12, Section 1.5]). In particular, the operator $T_k^{(j)}$ with $r=k-(m-j)d\geqq 0$ is of the form

$$T_k^{(j)} = s_k^{(j)}\gamma^r + \sum_{0\leqq l<r} S_{k,l}^{(j)}\gamma^l + T_k'^{(j)},$$

where $T_k'^{(j)}$ is of the class 0 and the coefficient $s_k^{(j)}$ is a homomorphism from $E|_\Gamma$ to F_k. Furthermore, the formulation of the second compatibility condition (II) on the data will lead us to assume that the operators $S_{k,l}^{(j)}$ are differential operators on Γ (see 4.5).

2.6. In the rest of this section we suppose that the operator $A(\partial_t) = (P_\Omega(\partial_t) + G(\partial_t), T(\partial_t))$ is parabolic. We shall need two basic results for the corresponding Green operator $A(z) = (P_\Omega(z) + G(z), T(z))$. The first one gives the continuity.

2.7. THEOREM. *For any $s \geqq 0$, the Green operator $A(z)$ from $C^\infty(\bar{\Omega}, E)$ to $C^\infty(\bar{\Omega}, E) \times C^\infty(\Gamma, F)$ extends by continuity to a continuous operator*

$$A(z) = \begin{bmatrix} P_\Omega(z) + G(z) \\ T(z) \end{bmatrix} : H_z^{s+md}(\bar{\Omega}, E) \to \begin{matrix} H_z^s(\bar{\Omega}, E) \\ \times \\ \prod_{0 \leq k < md} H_z^{s+md-k-1/2}(\Gamma, F_k) \end{matrix}$$

with the norm bounded uniformly for all z, $\operatorname{Re} z \geqq 0$.

This result is essentially Corollary 2.5.6 in Grubb [12]. The uniformity of the norm in z can be seen from the proofs of the local results preceding that corollary or obtained easily by means of these local results and the fact that the operators are polynomials in z.

2.8. In order to be able to apply the results of Grubb [12, Chapter 3] to the Green operator $A(z) = (P_\Omega(z) + G(z), T(z))$, we modify the trace operator $T(z)$ such that it is of order md, too. To this end, let $-\Delta_\Gamma$ denote the positive Laplace–Beltrami operator on Γ. As in Grubb [12, Section 3.3] we now define an elliptic pseudo-differential operator $\Lambda(z)$ from F to F by the diagonal matrix $\operatorname{diag}(\Lambda_{md-k}(z) I_{F_k})_{0 \leq k < md}$ with $\Lambda_k(z) = ((-\Delta_\Gamma)^{md} + |z|^{2m})^{k/(2md)}$, where I_{F_k} is the identity on F_k. Then it is easy to see that the operator

$$A'(z) = \begin{bmatrix} I & 0 \\ 0 & \Lambda(z) \end{bmatrix} A(z) = \begin{bmatrix} P_\Omega(z) + G(z) \\ T'(z) \end{bmatrix}$$

with $T'(z) = \Lambda(z)\, T(z)$ is a parameter-elliptic Green operator of order md (in both entries), and $T(z)$ and $T'(z)$ are of the same regularity. Thus if follows from Grubb [12, Chapter 3] (see especially Theorem 3.4.1) that the operator $A'(z)$ has a parametrix

$$B'(z) = [Q'_\Omega(z) + H'(z) \quad K'(z)] : \begin{matrix} C^\infty(\bar{\Omega}, E) \\ \times \\ C^\infty(\Gamma, F) \end{matrix} \to C^\infty(\bar{\Omega}, E)$$

of order $-md$ such that the pseudo-differential operator $Q'(z)$ is of regularity d, and the singular Green operator $H'(z)$ and the *Poisson operator* $K'(z)$ (see [12, Section 2.4]) are of regularity $\nu' = \min\{\nu, d\}$. Moreover, for z with $\operatorname{Re} z \geqq 0$ and $|z|$ sufficiently large, the inverse $A'(z)^{-1}$ of the operator $A'(z)$ exists and is equal to a parametrix $B'(z)$. Then, by continuity, the operator $A'(z)^{-1}$ extends to a continuous operator from $H_z^s(\bar{\Omega}, E) \times H_z^{s-1/2}(\Gamma, F)$ to $H_z^{s+md}(\bar{\Omega}, E)$ for any $s \geqq 0$ such that the estimate

$$\|A'(z)^{-1}(f, g)\|_{H_z^{s+md}(\bar{\Omega},E)} \leqq C_s(\|f\|_{H_z^s(\bar{\Omega},E)} + \|g\|_{H_z^{s-1/2}(\Gamma,F)}) \tag{2.2}$$

holds uniformly with respect to z. Since $A(z)$ depends polynomially on z and hence $A'(z)$ depends analytically on z, we also know by Grubb [12] that $A'(z)^{-1}$ is analytic in z (in the operator norm).

Now we return to the original operator $A(z)$ and see that

$$B(z) = B'(z)\begin{bmatrix} I & 0 \\ 0 & \Lambda(z) \end{bmatrix} = [Q_\Omega(z) + H(z) \quad K(z)]$$

is a parametrix of $A(z)$, and that the inverse $A(z)^{-1}$ exists and is equal to a parametrix $B(z)$ for all z with $\operatorname{Re} z \geqq 0$ and $|z|$ sufficiently large. Here $Q(z) = Q'(z), H(z) = H'(z)$, and $K(z) = K'(z)\Lambda(z) = [K_0(z) \ldots K_{md-1}(z)]$, where $K_k(z)\colon C^\infty(\Gamma, F_k) \to C^\infty(\bar{\Omega}, E)$ is a Poisson operator of order $-k$ and regularity ν'. The inverse $A(z)^{-1}$ also satisfies an estimate corresponding to (2.2). Thus we are led to the following result which will be crucial.

2.9. THEOREM. *For any $s \geqq 0$ there is $\rho > 0$ such that for every z with $\operatorname{Re} z \geqq \rho$ the inverse $A(z)^{-1}$ of the Green operator $A(z)$ (extended as in Theorem 2.7) exists, depends analytically on z in the norm of operators from $H^s(\bar{\Omega}, E) \times \prod_k H^{s+md-k-1/2}(\Gamma, F_k)$ to $H^{s+md}(\bar{\Omega}, E)$, and satisfies the estimate*

$$\|A(z)^{-1}(f, g)\|_{H_z^{s+md}(\bar{\Omega},E)}$$
$$\leqq C_s\Big(\|f\|_{H_z^s(\bar{\Omega},E)} + \|g\|_{\prod_k H_z^{s+md-k-1/2}(\Gamma, F_k)}\Big)$$

uniformly in z.

3. Adapted Laplace transform calculus.

In this section we recall and give some results concerning the Laplace transformation $\mathscr{L}$ defined by

$$(\mathscr{L}v)(z) = \int_0^\infty e^{-zt}\, v(t)\, dt;$$

see Agranovič–Višik [2], Donaldson [9], Schwartz [29], and also [24], [26].

3.1. The basic fact is that, for any $s \geqq 0$ and $\rho > 0$, the Laplace transformation $\mathscr{L}$ is an isomorphism from the space $H_{(0)}^s(\mathsf{R}_+, \rho; H)$ (see 1.3) to the space $\mathscr{H}^s(\mathsf{C}_\rho; H)$ of such analytic functions $\hat{u}$ from $\mathsf{C}_\rho = \{z \in \mathsf{C} : \operatorname{Re} z > \rho\}$ to the Hilbert space H that

$$\|\hat{u}\|^2_{\mathscr{H}^s(\mathsf{C}_\rho;H)} = \sup_{\sigma>\rho} \int_{-\infty}^{\infty} |\sigma + i\tau|^{2s} \|\hat{u}(\sigma + i\tau\|^2_H \, d\tau < \infty.$$

The space $\mathscr{H}^s(\mathsf{C}_\rho; H)$ is complete and its norm satisfies the equality

$$\|\hat{u}\|^2_{\mathscr{H}^s(\mathsf{C}_\rho;H)} = \int |\rho + i\tau|^{2s} \|\hat{u}(\rho + i\tau)\|^2_H \, d\tau, \tag{3.1}$$

where, by definition, $\hat{u}(\rho + i\tau) = \mathscr{F}_{t\to\tau} e^+(e^{-\rho t}\mathscr{L}^{-1}\hat{u})$. Let now Ξ stand for Ω or Γ, and B stand for E or F_k, respectively. Then we see that for any $s \geqq 0$ and $\rho > 0$ the Laplace transformation $\mathscr{L}$ is an isomorphism from the space

$$H^{(s)}_{[0]}(\Xi \times \mathsf{R}_+, \rho, B^t) = H^0(\bar{\mathsf{R}}_+, \rho; H^s(\bar{\Xi}, B)) \cap H^{s/d}_{(0)}(\mathsf{R}_+, \rho; H^0(\bar{\Xi}, B))$$

to the space

$$\mathscr{H}^{(s)}(\mathsf{C}_\rho, \bar{\Xi}, B) = \mathscr{H}^0(\mathsf{C}_\rho; H^s(\bar{\Xi}, B)) \cap \mathscr{H}^{s/d}(\mathsf{C}_\rho; H^0(\bar{\Xi}, B)).$$

We suppose here, as well as in what follows, that if two normed spaces H_1 and H_2 are algebraically and topologically included in some third normed space, the norm in the space $H_1 \cap H_2$ is given by

$$\|u\|_{H_1 \cap H_2} = (\|u\|^2_{H_1} + \|u\|^2_{H_2})^{1/2}.$$

It follows from (3.1) that

$$\|\hat{u}\|^2_{\mathscr{H}^{(s)}(\mathsf{C}_\rho,\Xi,B)} = \int \|\hat{u}(\rho + i\tau)\|^2_{H^s_{\rho+i\tau}(\Xi,B)} \, d\tau \tag{3.2}$$

for $\hat{u} \in \mathscr{H}^{(s)}(\mathsf{C}_\rho, \bar{\Xi}, B)$, where $\hat{u}(\rho + i\tau)$ is defined as in (3.1).

A fundamental property of the Laplace transformation $\mathscr{L}$ is the relation

$$\mathscr{L}\,\partial_t^j u = z^j \mathscr{L} u, \tag{3.3}$$

now for $u \in H^{(s+jd)}_{[0]}(\Xi \times \mathsf{R}_+, \rho, B^t)$; note that the multiplication by z^j gives a continuous mapping

$$\hat{u} \mapsto z^j \hat{u} : \mathscr{H}^{(s+jd)}(\mathsf{C}_\rho, \bar{\Xi}, B) \to \mathscr{H}^{(s)}(\mathsf{C}_\rho, \bar{\Xi}, B) \tag{3.4}$$

(for the proof fix a system of local trivializations and deduce directly or by means of [24, Proposition 5.7]). The equation (3.3) is the first step in proving the next commutation property of $\mathscr{L}$.

3.2. THEOREM. *If $s \geqq 0$ and $\rho > 0$, the parabolic operator $(P_\Omega(\partial_t) + G(\partial_t), T(\partial_t))$ satisfies the relation*

$$\mathscr{L}(P_\Omega(\partial_t) + G(\partial_t), T(\partial_t))u = (P_\Omega(z) + G(z), T(z))\mathscr{L}u \tag{3.5}$$

for all $u \in H^{(s+md)}_{[0]}(\bar{\Omega} \times \mathsf{R}_+, \rho, E^t)$.

PROOF. We restrict ourselves to an outline; note that a part of the statement is that both sides of (3.5) are well-defined. Let us define

$$H^{(s)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t) = H^0(\bar{\mathsf{R}}_+, \rho; H^s(\bar{\Omega}, E)) \cap H^{s/d}(\bar{\mathsf{R}}_+, \rho; H^0(\bar{\Omega}, E))$$

and analogously the space $H^{(s)}(\Gamma \times \bar{\mathsf{R}}_+, \rho, F_k^t)$. Since the continuity of the operators $P_\Omega^{(j)}$, $G^{(j)}$, and $T_k^{(j)}$ is known, it is routine to show that $P_\Omega(\partial_t)$, $G(\partial_t)$ and $T_k(\partial_t)$ are (more precisely, they define in a natural way) continuous operators from the space $H^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t)$ to $H^{(s)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t)$ and correspondingly to $H^{(s+md-k-1/2)}(\Gamma \times \bar{\mathsf{R}}_+, \rho, F_k^t)$. By interpolation, by using the continuity of the extension by zero of $e^{-\rho t}u$ for $t < 0$ (see [24]), and recalling the properties of intermediate derivatives (cf. Lions–Magenes [20, Chapter 1]), one then proves that $P_\Omega(\partial_t)$, $G(\partial_t)$, and $T_k(\partial_t)$ are, moreover, continuous operators from $H_{[0]}^{(s+md)}(\bar{\Omega} \times \mathsf{R}_+, \rho, E^t)$ to $H_{[0]}^{(s)}(\bar{\Omega} \times \mathsf{R}_+, \rho, E^t)$ and correspondingly to $H_{[0]}^{(s+md-k-1/2)}(\Gamma \times \mathsf{R}_+, \rho, F_k^t)$. By using local trivializations and (3.4), we obtain similarly the continuity of the operators

$$P_\Omega(z), G(z) : \mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \bar{\Omega}, E) \to \mathscr{H}^{(s)}(\mathsf{C}_\rho, \bar{\Omega}, E)$$

and

$$T_k(z) : \mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \bar{\Omega}, E) \to \mathscr{H}^{(s+md-k-1/2)}(\mathsf{C}_\rho, \Gamma, F_k).$$

A standard argument now shows that the operators $P_\Omega^{(j)}$, $G^{(j)}$, and $T_k^{(j)}$ commute with $\mathscr{L}$, from which the assertion follows by means of (3.3).

3.3. THEOREM. *Let $s \geqq 0$, and let $\rho > 0$ be chosen according to Theorem 2.9 for the parameter-elliptic operator $(P_\Omega(z) + G(z), T(z))$ corresponding to the parabolic operator $(P_\Omega(\partial_t) + G(\partial_t), T(\partial_t))$. Then the estimate*

$$\|\hat{u}\|_{\mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \bar{\Omega}, E)} \leqq C\Big(\|(P_\Omega(z) + G(z))\,\hat{u}\|_{\mathscr{H}^{(s)}(\mathsf{C}_\rho, \bar{\Omega}, E)} + \|T(z)\hat{u}\|_{\prod_k \mathscr{H}^{(s+md-k-1/2)}(\mathsf{C}_\rho, \Gamma, F_k)}\Big)$$

holds for all $\hat{u} \in \mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \bar{\Omega}, E)$.

PROOF. For $\hat{u} \in \mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \bar{\Omega}, E)$ we have $(P_\Omega(z) + G(z))\,\hat{u} \in \mathscr{H}^{(s)}(\mathsf{C}_\rho, \bar{\Omega}\, E)$ and $T_k(z)\hat{u} \in \mathscr{H}^{(s+md-k-1/2)}(\mathsf{C}_\rho, \Gamma, F_k)$ (see the proof of the preceding theorem). Now we consider the equality (see (3.2))

$$\|\hat{u}\|^2_{\mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \bar{\Omega}, E)} = \int \|\hat{u}(\rho + i\tau)\|^2_{H^{s+md}_{\rho+i\tau}(\bar{\Omega}, E)} d\tau,$$

where $\hat{u}(\rho + i\tau) = \mathscr{F}_{t \to \tau} e^+(e^{-\rho t}\mathscr{L}^{-1}\hat{u})$. From Theorem 2.9 it follows that the inequality

$$\|\hat{u}\|^2_{H^{s+md}_{\rho+i\tau}(\bar{\Omega},E)} \leqq C\Big(\|(P_\Omega(\rho+i\tau)+G(\rho+i\tau))\,\hat{u}(\rho+i\tau)\|^2_{H^s_{\rho+i\tau}(\bar{\Omega},E)} + \|T\|(\rho+i\tau)\,\hat{u}(\rho+i\tau)\|^2_{\prod_k H^{s+md-k-1/2}_{\rho+i\tau}(\Gamma,F_k)}\Big) \tag{3.6}$$

holds for almost all τ. Noting that (for suitable v)

$$(\rho+i\tau)^j \mathscr{F}_{t\to\tau}(e^{-\rho t}v) = \mathscr{F}_{t\to\tau}(e^{-\rho t}\partial_t^j v),$$

and using (3.3) and Theorem 3.2, we get

$$(P_\Omega(\rho+i\tau)+G(\rho+i\tau)\,\hat{u}(\rho+i\tau) = \mathscr{F}_{t\to\tau}e^+(e^{-\rho t}\mathscr{L}^{-1}(P_\Omega(z)+G(z))\,\hat{u})$$

and

$$T(\rho+i\tau)\,\hat{u}(\rho+i\tau) = \mathscr{F}_{t\to\tau}e^+(e^{-\rho t}\mathscr{L}^{-1}\,T(z)\hat{u}).$$

To complete the proof it therefore suffices to integrate (3.6) with respect to τ.

3.4. THEOREM. *Assume that the hypotheses in Theorem* 3.3 *are satisfied. If* $\hat{f}\in\mathscr{H}^{(s)}(\mathbf{C}_\rho,\bar{\Omega},E)$ *and* $\hat{g}\in\prod_k\mathscr{H}^{(s+md-k-1/2)}(\mathbf{C}_\rho,\Gamma,F_k)$ *are given, there exists a unique* $\hat{u}\in\mathscr{H}^{(s+md)}(\mathbf{C}_\rho,\bar{\Omega},E)$ *such that* $(P_\Omega(z)+G(z),T(z))\hat{u}=(\hat{f},\hat{g})$.

PROOF. The uniqueness part is obvious from Theorem 3.3. In order to verify the existence, consider the analytic mappings $z\mapsto\hat{f}(z):\mathbf{C}_\rho\to H^s(\bar{\Omega},E)$ and $z\mapsto\hat{g}(z):\mathbf{C}_\rho\to\prod_k H^{s+md-k-1/2}(\Gamma,F_k)$. It follows from Theorem 2.9 that the only candidate for a solution is given by

$$\hat{u}(z) = \begin{bmatrix} P_\Omega(z)+G(z) \\ T(z)\end{bmatrix}^{-1}\begin{bmatrix}\hat{f}(z)\\ \hat{g}(z)\end{bmatrix}$$

in such a way that the mapping $z\mapsto\hat{u}(z):\mathbf{C}_\rho\to H^{s+md}(\bar{\Omega},E)$ is analytic and, in addition, the estimate

$$\|\hat{u}(z)\|_{H^{s+md}_z(\bar{\Omega},E)} \leqq C\Big(\|\hat{f}(z)\|_{H^s_z(\bar{\Omega},E)} + \|\hat{g}(z)\|_{\prod_k H^{s+md-k-1/2}_z(\Gamma,F_k)}\Big) \tag{3.7}$$

holds uniformly with respect to z. It will therefore be enough to show that $\hat{u}\in\mathscr{H}^{(s+md)}(\mathbf{C}_\rho,\bar{\Omega},E)$. To this end, take $z=\sigma+i\tau\in\mathbf{C}_\rho$ and derive from (3.7) the inequality

$$\begin{aligned}
&\|\hat{u}(z)\|^2_{H^{s+md}(\bar{\Omega},E)} + |z|^{2(s+md)/d}\|\hat{u}(z)\|^2_{H^0(\bar{\Omega},E)} \\
&\leqq C\Big(\|\hat{f}(z)\|^2_{H^s(\bar{\Omega},E)} + |z|^{2s/d}\|\hat{f}(z)\|^2_{H^0(\bar{\Omega},E)}\Big) \\
&+ C\sum_k\Big(\|\hat{g}_k(z)\|^2_{H^{s+md-k-1/2}(\Gamma,F_k)} + |z|^{2(s+md-k-1/2)/d}\|\hat{g}_k(z)\|^2_{H^0(\Gamma,F_k)}\Big),
\end{aligned}$$

which in turn implies that

$$\int \|\hat{u}(\sigma + i\tau)\|^2_{H^{s+md}(\bar{\Omega},E)} d\tau + \int |\sigma + i\tau|^{2(s+md)/d} \|\hat{u}(\sigma + i\tau)\|^2_{H^0(\bar{\Omega},E)} d\tau$$
$$\leqq C\left(\|\hat{f}\|^2_{\mathscr{H}^{(s)}(\mathbf{C}_\rho,\bar{\Omega},E)} + \|\hat{g}\|^2_{\prod_k \mathscr{H}^{(s+md-k-1/2}(\mathbf{C}_\rho,\Gamma,F_k)} \right).$$

This clearly yields what was to be proved.

4. Parabolic problems.

Initial-boundary value problems

$$(4.1) \qquad (P_\Omega(\partial_t) + G(\partial_t), T(\partial_t), (\gamma_t \partial_t^j)_{j<m}) u = (f, g, h)$$

are called here *parabolic* (in $\Omega \times \mathbf{R}_+$ or E^t), if the operator $(P_\Omega(\partial_t) + G(\partial_t), T(\partial_t))$ is parabolic. We are studying the solvability of parabolic problems in the space $H^{(s+md)}(\bar{\Omega} \times \bar{\mathbf{R}}_+, \rho, E^t)$. Therefore it is natural to assume that

$$f \in H^{(s)}(\bar{\Omega} \times \bar{\mathbf{R}}_+, \rho, E^t),$$
$$g \in \prod_{k=0}^{md-1} H^{(s+md-k-1/2)}(\Gamma \times \bar{\mathbf{R}}_+, \rho, F_k^t),$$
$$h \in \prod_{j=0}^{m-1} H^{s+md-jd-d/2}(\bar{\Omega}, E).$$

It will, however, turn out that such a mapping $u \mapsto (f, g, h)$ is not surjective.

In this concluding section we first derive two compatibility conditions on the data (f, g, h), necessary for the existence of a solution u; for simplicity we assume the parabolicity of the problem throughout the section. In the main theorems 4.8 and 4.9 we shall then see how these conditions work in solving parabolic problems (4.1).

4.1. Let us take $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathbf{R}}_+, \rho, E^t)$ and set $f = (P_\Omega(\partial_t) + G(\partial_t))u$, $h = (\gamma_t \partial_t^j u)_{0 \leqq j < m} = (h_j)_j$. It is now obvious that the total trace of u, denoted by $h^s = (\gamma_t \partial_t^j u)_{0 \leqq jd < s+md-d/2} = (h_j^s)_j$, can be given by means of f and h. In fact, by induction we find the following relation (cp. [24], [26]):

4.2. THEOREM. *With the above notation, the relation*

$$(4.2) \qquad h^s_{m+\nu} = \mathscr{M}_\nu f + \sum_{k=0}^{m-1} \mathscr{N}_{\nu-k} \sum_{j=0}^{m-1-k} (P_\Omega^{(m-j)} + G^{(m-j)}) h_{k+j}$$

is satisfied for any $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathbf{R}}_+, \rho, E^t)$ whenever $0 \leqq \nu d < s - d/2$, where the operators $\mathscr{M}_\nu$ and $\mathscr{N}_\nu$ are defined by $\mathscr{M}_0 = \gamma_t, \mathscr{M}_{-\nu} = 0$ and

$$\mathscr{M}_\nu = \gamma_t \partial_t^\nu - \sum_{j=0}^{m-1} (P_\Omega^{(m-j)} + G^{(m-j)}) \mathscr{M}_{\nu-m+j}$$

for $v = 1, 2, \ldots$, *and* $\mathcal{N}_0 = -I$ *in* $H^0(\bar{\Omega}, E)$, $\mathcal{N}_{-v} = 0$ *and*

$$\mathcal{N}_v = -\sum_{j=0}^{m-1} (P_\Omega^{(m-j)} + G^{(m-j)})\mathcal{N}_{v-m+j}$$

for $v = 1, 2 \ldots$.

4.3. We continue to use the previous notation and shall give two compatibility relations between the boundary values g and the total trace h^s of $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t)$. According to the preceding theorem, such a relation is de facto a relation between the data f, g, and h. The first one is a straightforward consequence of the coincidence of continuous operators in dense subspaces.

4.4. THEOREM. *If* $0 \leqq k \leqq md - 1$ *and* $v \in \mathsf{N}$ *such that* $s + md - (k + 1/2) - (vd + \delta/2) > 0$, *then for any* $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t)$ *the equation*

$$\gamma_t \partial_t^v g_k = \sum_{j=0}^{m-1} T_k^{(m-j)} h_{v+j}^s$$

holds in $H^{s+md-k-vd-1/2-d/2}(\Gamma, F_k)$.

4.5 When the case $s + md - k - vd = 1/2 + d/2$ occurs, more knowledge will be needed. In order to study this case further, fix a system of local trivializations (ψ_i), $(\zeta_{k,i})$ and a partition of unity (ϕ_i) associated with it as in Section 1. It should cause no confusion if we let $(u)^i$ denote jointly both $\psi_i \circ (\phi_i u) = (\psi_i^{-1})^* (\phi_i u)$ for sections u of $\tilde{E}$ or $\tilde{E}^t$ and $\psi_i' \circ (\phi_i' u)$ for sections u of $E|_\Gamma$ or $E^t|_{\Gamma \times \bar{\mathsf{R}}_+}$, and let $(u)^{k,i}$ denote correspondingly both $\zeta_{k,i} \circ (\phi_i u)$ for sections u of $\tilde{F}_k$ or $\tilde{F}_t^k$ and $\xi_{k,i}' \circ (\phi_i' u)$ for sections u of F_k or F_k^t.

For the second compatibility relation we now assume that in the trace operator (see Remark 2.5)

$$T_k^{(j)} = s_k^{(j)} \gamma^r + \sum_{0 \leqq l < r} S_{k,l}^{(j)} \gamma^l + T_k'^{(j)} \tag{4.3}$$

with $r = k - (m - j)d \geqq 0$ the coefficients $S_{k,l}^{(j)}$ are *differential operators* on Γ. Then it follows from Boutet de Monvel [5] (see also Rempel–Schulze [27]) that there exists a pseudo-differential operator $T_k^{(j)}(D_n)$ of order $r = k - (m - j)d \geqq 0$ with the transmission property at Γ such that the continuous operator $T_k^{(j)}(D_n)_\Omega$ from $H^{\sigma+r}(\bar{\Omega}, E)$ to $H^\sigma(\bar{\Omega}, \tilde{F}_k)$ for $\sigma \geqq 0$ satisfies the equation $T_k^{(j)} = \gamma^0 T_k^{(j)}(D_n)_\Omega$. Thus we also have $T_k = \gamma^0 T_k(D_n)_\Omega$, where

$$T_k(D_n) = \sum_{j=0}^{m-1} \partial_t^j T_k^{(m-j)}(D_n) \tag{4.4}$$

and $T_k(D_n)_\Omega$ is a continuous operator from $H^{(\sigma+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t)$ to $H^{(\sigma+md-k)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, \tilde{F}_k^t)$ for $\sigma \geqq 0$ (see the proof of Theorem 3.2).

Suppose now that $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathrm{R}}_+, \rho, E')$ with $s + md - k - vd = 1/2 + d/2$ (and $M_k > 0$). If we define, by using the notation of 1.3,

$$H^{(\sigma)}(\bar{X} \times \bar{\mathrm{R}}_+; H) = H^0(\bar{\mathrm{R}}_+; H^\sigma(\bar{X}; H)) \cap H^{\sigma/d}(\bar{\mathrm{R}}_+; H^0(\bar{X}; H))$$

for $\sigma \geqq 0$ and analogously the space $H^{(\sigma)}(\bar{X} \times \bar{\mathrm{R}}_+, \rho; H)$, then we have

$$v_{k,i} = \partial_t^v \sum_j T_k(D_n)_{\mathrm{R}_+^n}^{(j,i)}(u)^j \in H^{(s+md-k-vd)}(\bar{\mathrm{R}}_+^n \times \bar{\mathrm{R}}_+, \rho; \mathrm{C}^{M_k}) \tag{4.5}$$

for $i = 1, \ldots, i_1$, where

$$T_k(D_n)_{\mathrm{R}_+^n}^{(j,i)}(u)^j = r^+(\zeta_{k,i} \circ (T_k(D_n)(\psi_j^{-1} \circ e^+(u)^j))) \tag{4.6}$$

and

$$T_k(D_n)^{(j,i)} = \sum_{l=0}^{m-1} \partial_t^l T_k^{(m-l)}(D_n)^{(j,i)}. \tag{4.7}$$

Therefore, writing $\alpha_i = (\kappa_i^{-1})^* \phi_i$ (the pullback) and $\delta = 1/2 + d/2$, we see that

$$w_{k,i} = e^{-\rho t} \alpha_i v_{k,i} \in H^{(\delta)}(\bar{\mathrm{R}}_+^n \times \bar{\mathrm{R}}_+; \mathrm{C}^{M_k}).$$

By the continuity of the imbedding of $H^{(\delta)}(\bar{\mathrm{R}}_+^n \times \bar{\mathrm{R}}_+; \mathrm{C}^{M_k})$ into $H^{(\delta)}(\bar{\mathrm{R}}_+ \times \bar{\mathrm{R}}_+; H^0(\mathrm{R}^{n-1}; \mathrm{C}^{M_k}))$, we can thus deduce from Grisvard [11] the estimate (cp. [20, Chapter 4], [24])

$$\begin{aligned} &\int_0^\infty \|(\gamma_t w_{k,i})(\cdot, \sigma) - (\gamma^0 w_{k,i})(\cdot, \sigma^d)\|^2_{H^0(\mathrm{R}^{n-1}; \mathrm{C}^{M_k})} \frac{d\sigma}{\sigma} \\ &\leqq C \|w_{k,i}\|^2_{H^{(\delta)}(\bar{\mathrm{R}}_+^n \times \bar{\mathrm{R}}_+; \mathrm{C}^{M_k})}. \end{aligned} \tag{4.8}$$

Here we obtain, by (4.4)–(4.7),

$$\begin{aligned} \gamma_t w_{k,i} &= \alpha_i \gamma_t \partial_t^v \sum_j \sum_{l=0}^{m-1} \partial_t^l T_k^{(m-l)}(D_n)_{\mathrm{R}_+^n}^{(j,i)}(u)^j \\ &= \alpha_i \sum_l \sum_j T_k^{(m-l)}(D_n)_{\mathrm{R}_+^n}^{(j,i)}(\psi_j \circ (\phi_j h_{v+l}^s)) \\ &= \alpha_i \sum_l \sum_j r^+ (\zeta_{k,i} \circ (T_k^{(m-l)}(D_n)(e_\Omega(\phi_j h_{v+l}^s)))) \\ &= \alpha_i \sum_l \zeta_{k,i} \circ (T_k^{(m-l)}(D_n)_\Omega h_{v+l}^s) \\ &= \left(\sum_l T_k^{(m-l)}(D_n)_\Omega h_{v+l}^s\right)^{k,i}, \end{aligned}$$

and, writing $\alpha_i' = ((\kappa_i')^{-1})^* \phi_i'$,

$$
\begin{aligned}
\gamma^0 w_{k,i} &= e^{-\rho t} \alpha_i' \partial_t^\nu \sum_j \sum_{l=0}^{m-1} \partial_t^l \gamma^0 T_k^{(m-l)}(D_n)_{\mathsf{R}_+^n}^{(j,i)}(u)^j \\
&= e^{-\rho t} \alpha_i' \partial_t^\nu \sum_j \sum_l \partial_t^l \gamma^0 r^+ (\zeta_{k,i} \circ (T_k^{(m-l)}(D_n)(\psi_j^{-1} \circ e^+(u)^j))) \\
&= e^{-\rho t} \alpha_i' \partial_t^\nu \sum_j \zeta_{k,i}' \circ (\gamma^0 \sum_l \partial_t^l r_\Omega T_k^{(m-l)}(D_n) e_\Omega (\phi_j u)) \\
&= e^{-\rho t} \alpha_i' \partial_t^\nu (\zeta_{k,i}' \circ (\gamma^0 T_k(D_n)_\Omega u)) \\
&= e^{-\rho t} \partial_t^\nu (g_k)^{k,i}.
\end{aligned}
$$

It is now possible to state the second compatibility relation.

4.6. THEOREM. *Let $s > 0$, $k \in \mathsf{N}$, and $\nu \in \mathsf{N}$ be such that $M_k > 0$ and $s + md - k - \nu d = \delta = 1/2 + d/2$. Assume that the coefficients $S_{k,1}^{(j)}$ in the representation (4.3) of $T_k^{(j)}$ are differential operators on Γ, and let the operators $T_k^{(j)}(D_n)$ be given as above. Then there exists a constant $C > 0$ such that the estimate*

$$
\begin{aligned}
|||(g_k, h^s)|||_{s,\rho,E,F_k^t}^2 = \sum_{i=1}^{i_1} \int_0^\infty \Bigg\| \left(\sum_{l=0}^{m-1} T_k^{(m-l)}(D_n)_\Omega h_{\nu+l}^s \right)^{k,i} (\cdot, \sigma) \\
- \exp(-\rho\sigma^d)(\partial_t^\nu g_k)^{k,i}(\cdot, \sigma^d) \Bigg\|_{H^0(\mathsf{R}^{n-1};\mathsf{C}^{M_k})}^2 \frac{d\sigma}{\sigma} \\
\leqq C \, \|u\|_{H^{(s+md)}(\bar\Omega \times \bar{\mathsf{R}}_+,\rho,E^t)}^2
\end{aligned}
$$

holds for all $u \in H^{(s+md)}(\bar\Omega \times \bar{\mathsf{R}}_+, \rho, E^t)$.

PROOF. In view of the preceding reasoning, if suffices to consider the right side of (4.8). There we have $w_{k,i} = e^{-\rho t} \alpha_i v_{k,i}$, where by (4.4)–(4.7)

$$
v_{k,i} = \partial_t^\nu \sum_j \zeta_{k,i} \circ (T_k(D_n)_\Omega(\phi_j u)) = \partial_t^\nu \zeta_{k,i} \circ (T_k(D_n)_\Omega u),
$$

and hence $\alpha_i v_{k,i} = \partial_t^\nu (T_k(D_n)_\Omega u)^{k,i}$. Therefore we get (see 1.3)

$$
\begin{aligned}
&\|w_{k,i}\|_{H^{(\delta)}(\bar{\mathsf{R}}_+^n \times \bar{\mathsf{R}}_+;\mathsf{C}^{M_k})} \\
&\leqq C \, \|(T_k(D_n)_\Omega u)^{k,i}\|_{H^{(\delta+\nu d)}(\bar{\mathsf{R}}^n \times \bar{\mathsf{R}}_+,\rho;\mathsf{C}^{M_k})} \\
&\leqq C \, \| T_k(D_n)_\Omega u \|_{H^{(s+md-k)}(\bar\Omega \times \bar{\mathsf{R}}_+,\rho,F_k^t)} \\
&\leqq C \, \|u\|_{H^{(s+md)}(\bar\Omega \times \bar{\mathsf{R}}_+,\rho,E^t)}.
\end{aligned}
$$

4.7. Thus we have seen that, under the given hypotheses, the boundary trace g and the total initial trace h^s of a solution $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t)$ of the parabolic problem (4.1) satisfy necessarily the following (local resp. global) *compatibility relations* (I) and (II):

(I) *If* $s \geqq 0, 0 \leqq k \leqq md - 1$, *and* $\nu \in \mathsf{N}$ such that $s + md - k - \nu d > \delta = 1/2 + d/2$ *(and* $M_k > 0$*), then*

$$\gamma_t \partial_t^\nu g_k = \sum_{j=0}^{m-1} T_k^{(m-j)} h_{\nu+j}^s.$$

(II) *If* $s + md - k - \nu d = \delta$ *(and* $M_k > 0$*), then*

$$|||(g_k, h^s)|||_{s,\rho,E,F_k^t} < \infty.$$

Let us now suppose that (f, g, h) is given in the space $\Pi^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t, F^t)$, which is defined as the product space

$$H^{(s)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t) \times \prod_{k=0}^{md-1} H^{(s+md-k-1/2)}(\Gamma \times \bar{\mathsf{R}}_+, \rho, F_k^t) \times \prod_{j=0}^{m-1} H^{s+md-jd-d/2}(\bar{\Omega}, E).$$

The preceding considerations motivate us to set $h_j^s = h_j$ for $j = 0, \ldots, m-1$ and define $h_{m+\nu}^s$ for $0 \leqq \nu d < s - d/2$ by the formula (4.2) of Theorem 4.2. In this way we can set the *compatibility conditions* (I) and (II) on the data f, g, h, too. Note that condition (II) implies the same condition for any similar system of local trivializations and a partition of unity. From Theorems 4.6, 4.8, and 4.9 it follows for parabolic problems that condition (II) is also well-behaved with respect to the choice of the operators $T_k^{(j)}(D_n)$.

We are now ready to state and prove the main results. To this end, let $\Pi_{cc}^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t, \mathrm{F}^t)$ denote the space

$$\{(f, g, h) \in \Pi^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t, F^t) : \text{(I) and (II) hold for } (g, h^s)\}$$

and provide it with the norm given by

$$|||(f, g, h)|||_{s,\rho,E^t,F^t}^2 = \|(f, g, h)\|_{\Pi^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t, F^t)}^2 + \sum_{k+\nu d = s+md-\delta} |||(g_k, h^s)|||_{s,\rho,E,F_k^t}^2, \tag{4.9}$$

where $|||(g_k, h^s)|||_{s,\rho,E,F_k^t} = 0$ if $M_k = 0$.

4.8. THEOREM. *Let the problem*

$$(P_\Omega(\partial_t) + G(\partial_t), T(\partial_t), (\gamma_t \partial_t^j)_{j<m})u = (f, g, h)$$

be parabolic, and let $s \geqq 0$ *be given such that* $s \not\equiv d/2 \bmod d$. *Assume further that, if* *any* $0 \leqq k \leqq md - 1$ *satisfies the conditions* $M_k > 0$ *and* $s + md - k = \nu d + \delta$ *with* $\nu \in \mathbb{N}$ *and* $\delta = 1/2 + d/2$, *the coefficients* $S_{k,l}^{(j)}$ *in the representation* (4.3) *of* $T_k^{(j)}$ *are differential operators on* Γ *and the operators* $T_k^{(j)}(D_n)$ *are given as in* 4.5. *Then there exists* $\rho_0 > 0$ *such that for every* $\rho \geqq \rho_0$ *the a priori estimate*

$$\|u\|_{H^{(s+md)}(\bar{\Omega}\times \bar{\mathbb{R}}_+,\rho,E^t)} \leqq C\, |||(f,g,h)|||_{s,\rho,E^t,F^t}$$

holds for all $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathbb{R}}_+, \rho, E^t)$, *where*

$$f = (P_\Omega(\partial_t) + G(\partial_t))u, g = T(\partial_t)u, \text{ and } h = (\gamma_t\, \partial_t^j u)_{0 \leqq j < m}.$$

PROOF. We first recall that the continuous surjective trace operator

$$(\gamma_t\, \partial_t^j)_{0 \leqq jd < s+md-d/2} : H^{(s+md)}(\bar{\Omega} \times \bar{\mathbb{R}}_+, \rho, E^t) \to \prod_j H^{s+md-jd-d/2}(\bar{\Omega}, E)$$

has a continuous right inverse (see the references in 1.3). Therefore, if now $u \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathbb{R}}_+, \rho, E^t)$, there exists $v \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathbb{R}}_+, \rho, E^t)$ such that

$$(\gamma_t\, \partial_t^j v)_j = h^s = (\gamma_t\, \partial_t^j u)_j \tag{4.10}$$

and

$$\|v\|_{H^{(s+md)}(\bar{\Omega}\times\bar{\mathbb{R}}_+,\rho,E^t)} \leqq C\, \|h^s\|_{\prod_j H^{s-md-jd-d/2}(\bar{\Omega},E)}. \tag{4.11}$$

Since $s \not\equiv d/2 \bmod d$, we thus have $u - v \in H_{[0]}^{(s+md)}(\bar{\Omega} \times \bar{\mathbb{R}}_+, \rho, E^t)$. Now choose $\rho_0 > 0$ according to Theorem 2.9. Then we can apply Theorem 3.3 to obtain

$$\begin{aligned} &\|\mathscr{L}(u - v)\|_{\mathscr{H}^{(s+md)}(\mathbb{C}_\rho,\bar{\Omega},E)} \\ &\leqq C\Big(\|(P_\Omega(z) + G(z))\mathscr{L}(u - v)\|_{\mathscr{H}^{(s)}(\mathbb{C}_\rho,\bar{\Omega},E)} \\ &\qquad + \|T(z)\mathscr{L}(u - v)\|_{\prod_k \mathscr{H}^{(s+md-k-1/2)}(\mathbb{C}_\rho,\Gamma,F_k)}\Big). \end{aligned}$$

Hence, using 3.1, Theorem 3.2, and (4.11), we get

$$\begin{aligned} &\|u\|_{H^{(s+md)}(\bar{\Omega}\times\bar{\mathbb{R}}_+,\rho,E^t)} \\ &\leqq C\Big(\|h^s\|_{\prod_j H^{s+md-jd-d/2}(\bar{\Omega},E)} \\ &\qquad + \|(P_\Omega(\partial_t) + G(\partial_t))(u - v)\|_{H^{(s)}(\bar{\Omega}\times\bar{\mathbb{R}}_+,\rho,E^t)} \\ &\qquad + \|T(\partial_t)(u - v)\|_{\prod_k H_{[0]}^{(s+md-k-1/2)}(\Gamma\times\bar{\mathbb{R}}_+,\rho,F_k^t)}\Big). \end{aligned} \tag{4.12}$$

Here we have by Theorem 4.2

$$\|h^s_{m+\nu}\|_{H^{s+md-(m+\nu)d-d/2}(\bar\Omega,E)}$$

$$\leqq \|\mathcal{M}_\nu f\|_{H^{s-\nu d-d/2}(\bar\Omega,E)} + \sum_k \sum_{j=0}^{m-1-k} \|\mathcal{N}_{\nu-k}(P_\Omega^{(m-j)} + G^{(m-j)})h_{k+j}\|_{H^{s-\nu d-d/2}(\bar\Omega,E)}$$

$$\leqq C\left(\|f\|_{H^{(s)}(\bar\Omega\times\bar{\mathrm{R}}_+,\rho,E^t)} + \|h\|_{\prod_{j<m} H^{s+md-jd-d/2}(\bar\Omega,E)}\right).$$

Thus it follows from (4.11) that

$$\|(P_\Omega(\partial_t) + G(\partial_t))(u - v)\|_{H^{(s)}(\bar\Omega\times\bar{\mathrm{R}}_+,\rho,E^t)}$$

$$\leqq C\left(\|f\|_{H^{(s)}(\bar\Omega\times\bar{\mathrm{R}}_+,\rho,E^t)} + \|h\|_{\prod_{j<m} H^{s+md-jd-d/2}(\bar\Omega,E)}\right).$$

To estimate the last term of (4.12), assume that $M_k > 0$ and $s + md - k = \nu d + \delta$. Then

$$\|g_k - T_k(\partial_t)v\|^2_{H^{(s+md-k-1/2)}_{[0]}(\Gamma\times\mathrm{R}_+,\rho,F^t_k)}$$

$$= \|g_k - T_k(\partial_t)v\|^2_{H^{(s+md-k-1/2)}(\Gamma\times\bar{\mathrm{R}}_+,\rho,F^t_k)}$$

$$+ \int_0^\infty e^{-2\rho t}\|\partial_t^\nu(g_k - T_k(\partial_t)v)\|^2_{H^0(\Gamma,F_k)}\frac{dt}{t},$$

where

$$\int_0^\infty e^{-2\rho t}\|\partial_t^\nu(g_k - T_k(\partial_t)v)\|^2_{H^0(\Gamma,F_k)}\frac{dt}{t}$$

$$\leqq C\left(|\!|\!|(g_k, h^s)|\!|\!|^2_{s,\rho,E,F^t_k} + \|v\|^2_{H^{(s+md)}(\bar\Omega\times\bar{\mathrm{R}}_+,\rho,E^t)}\right)$$

by Theorem 4.6 and (4.10). Consequently,

$$\|T(\partial_t)(u - v)\|_{\prod_k H^{(s+md-k-1/2)}_{[0]}(\bar\Omega,E)}(\Gamma\times\mathrm{R}_+,\rho,F^t_k)$$

$$\leqq C\Bigg(\|g\|_{\prod_k H^{(s+md-k-1/2)}_{[0]}(\bar\Omega,E)}(\Gamma\times\bar{\mathrm{R}}_+,\rho,F^t_k)$$

$$+ \sum_k |\!|\!|(g_k, h^s)|\!|\!|_{s,\rho,E,F^t_k} + \|v\|_{H^{(s+md)}(\bar\Omega\times\bar{\mathrm{R}}_+,\rho,E^t)}\Bigg),$$

where, of course, the sum is taken over such k that $M_k > 0$ and

$s + md - k = \nu d + \delta$. Combining these partial estimates, we thus obtain the desired inequality.

REMARK. Note that, if $s + md - k = \nu d + \delta$ for some $0 \leqq k \leqq md - 1$, the condition $s \not\equiv d/2 \bmod d$ is satisfied. On the other hand, if $s - k \not\equiv \delta \bmod d$ for every k with $M_k > 0$, the a priori estimate takes the form

$$\begin{aligned}
&\|u\|_{H^{(s+md)}(\bar{\Omega} \times \bar{\mathrm{R}}_+, \rho, E^t)} \\
&\leqq C\Bigg(\|f\|_{H^{(s)}(\bar{\Omega} \times \bar{\mathrm{R}}_+, \rho, E^t)} \\
&+ \|g\|_{\prod_k H^{(s+md-k-1/2)}(\bar{\Omega}, E)(\Gamma \times \bar{\mathrm{R}}_+, \rho, F_k^t)} \\
&+ \|h\|_{\prod_{j<m} H^{s+md-jd-d/2}(\bar{\Omega}, E)} \Bigg).
\end{aligned}$$

4.9. THEOREM. *Let the hypotheses of Theorem 4.8 be satisfied. Then for* $(f, g, h) \in \Pi^{(s+md)}(\bar{\Omega} \times \bar{\mathrm{R}}_+, \rho, E^t, F^t)$, *the parabolic problem*

$$(P_\Omega(\partial_t) + G(\partial_t), T(\partial_t), (\gamma_t \partial_t^j)_{j<m})u = (f, g, h)$$

has a solution if and only if the data f, g, h *satisfy the compatibility conditions* (*I*) *and* (*II*) *given in* 4.7.

PROOF. The necessity of conditions (I) and (II) follows from Theorems 4.4 and 4.6.

In order to prove the sufficiency, assume that f, g, and h are compatible in the sense of (I) and (II). Set $h_j^s = h_j$ for $j = 0, \ldots, m - 1$ and define $h_{m+\nu}^s$ for $0 \leqq \nu d < s - d/2$ by (4.2). Then there exists $v \in H^{(s+md)}(\bar{\Omega} \times \bar{\mathrm{R}}_+, \rho, E^t)$ such that $(\gamma_t \partial_t^j v)_j = (h_j^s)_j = h^s$. To make use of this v, write $f' = (P_\Omega(\partial_t) + G(\partial_t))v$ and $g' = (g_k')_k = T(\partial_t)_v$.

It now follows from Theorem 4.2 that $\mathscr{M}_\nu f = \mathscr{M}_\nu f'$ for $0 \leqq \nu d < s - d/2$. This implies that

$$\gamma_t \partial_t^\nu (f - f') = 0 \qquad \text{for } 0 \leqq \nu d < s - d/2.$$

Consequently, $f - f' \in H_{[0]}^{(s)}(\bar{\Omega} \times \mathrm{R}_+, \rho, E^t)$, since $s \not\equiv d/2 \bmod d$.

If $0 \leqq \nu d < s + md - k - \delta$, then $\gamma_t \partial_t^\nu g_k' = \gamma_t \partial_t^\nu g_k$ by Theorem 4.4 and condition (I). In the case $s + md - k - \delta = \nu d$ (and $M_k > 0$) we get, by Theorem 4.6,

$$|||(g_k', h^s)|||_{s,\rho,E,F_k^t} < \infty,$$

and, by condition (II),

$$|||(g_k, h^s)|||_{s,\rho,E,F_k^t} < \infty.$$

It therefore follows (by means of local trivializations of F_k) that $t^{-1/2}e^{-\rho t}\partial_t^{\nu}(g_k - g'_k) \in L^2(\mathsf{R}_+; H^0(\Gamma, F_k))$, and hence

$$g_k - g'_k \in H_{(0)}^{(s+md-k-1/2)/d}(\mathsf{R}_+, \rho; H^0(\Gamma, F_k)).$$

Since immediately $g_k - g'_k \in H^0(\mathsf{R}_+, \rho; H^{s+md-k-1/2}(\Gamma, F_k))$, we obtain

$$g_k - g'_k \in H_{[0]}^{(s+md-k-1/2)}(\Gamma \times \bar{\mathsf{R}}_+, \rho, F_k^t).$$

We next apply the Laplace transform calculus given in Section 3. What we now know is that (see 3.1) $\mathscr{L}(f - f') \in \mathscr{H}^{(s)}(\mathsf{C}_\rho, \bar{\Omega}, E)$ and $\mathscr{L}(g - g') \in \prod_k \mathscr{H}^{(s+md-k-1/2)}\mathsf{C}_\rho, \Gamma, F_k)$. Hence according to Theorem 3.4, there exists $\hat{w} \in \mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \bar{\Omega}, E)$ such that $(P_\Omega(z) + G(z))\hat{w} = \mathscr{L}(f - f')$ and $T(z)\hat{w} = \mathscr{L}(g - g')$. An application of the inverse transformation $\mathscr{L}^{-1}$ of $\mathscr{L}$ then yields $w = \mathscr{L}^{-1}\hat{w} \in H_{[0]}^{(s+md)}(\bar{\Omega} \times \mathsf{R}_+, \rho, E^t)$ which satisfies the equations

$$(P_\Omega(\partial_t) + G(\partial_t))w = f - f', \qquad T(\partial_t)w = g - g'.$$

Thus we obtain a solution u by setting $u = v + w$, and the proof is complete.

4.10. CONCLUSION. *Under the assumptions of Theorem* 4.8, *there exists* $\rho_0 > 0$ *such that for any* $\rho \geqq \rho_0$ *the operator* $(P_\Omega(\partial_t) + G(\partial_t), T(\partial_t), (\gamma_t\partial_t^j)_{j<m})$ *of the parabolic problem* (4.1) *is an isomorphism from* $H^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, \mathrm{E}^t)$ *to the space* $\Pi_{cc}^{(s+md)}(\bar{\Omega} \times \bar{\mathsf{R}}_+, \rho, E^t, F^t)$ *defined in* 4.7.

REFERENCES

1. M. S. Agranovič, *Boundary value problems for systems with a parameter*, Math.USSR-Sb. 13 (1971), 25–64.
2. M. S. Agranovič and M. I. Višik, *Elliptic problems with a parameter and parabolic problems of general type*, Russian Math. Surveys 19 (1964), no. 3, 53–157.
3. V. P. Bogatova and V. P. Glushko, *Solvability of inititial-boundary value problems for higher order parabolic equations with degeneracy in the space variable*, Soviet Math. Dokl. 34 (1987), 503–506.
4. L. Boutet de Monvel, *Comportement d'un opérateur pseudo-différentiel sur une variété à bord I, II*, J. Analyse Math. 17 (1966), 241–304.
5. L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. 126 (1971), 11–51.
6. A. Bove, B. Franchi and E. Obrecht, *Parabolic problems with mixed time dependent lateral conditions*, Comm. Partial Differential Equations 7 (1982), 1253–1288.
7. Čan Zuĭ Ho and G. I. Èskin, *Boundary value problems for parabolic systems of pseudodifferential equations*, Soviet Math. Dokl. 12 (1971), 739–743.
8. J. Chazarain and A. Piriou, *Introduction à la Théorie des Équations aux Dérivées Partielles Linéaires*, Gauthier-Villars, Paris, 1981.
9. T. Donaldson, *A Laplace Transform Calculus for Partial Differential Operators*, Mem. Amer. Math. Soc. 143, 1974.
10. F. R. Gantmacher, *Matrizenrechnung I*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1965.
11. P. Grisvard, *Caractérisation de quelques espaces d'interpolation*, Arch. Rational Mech. Anal. 25 (1967), 40–63,

12. G. Grubb, *Functional Calculus of Pseudo-Differential Boundary Problems* (Progr. Math. 65), Birkhäuser, Boston, 1986.
13. G. Grubb, *Complex powers of pseudo-differential boundary value problems with the transmission property*, in *Pseudo-Differential Operators* (Proc. Oberwolfach, 1986), eds. H. O. Cordes, B. Gramsch and H. Widom (Lecture Notes in Math. 1256), pp. 169–191, Springer-Verlag, Berlin-Heidelberg, 1987.
14. G. Grubb and V. A. Solonnikov, *Reduction of basic initial-boundary value problems for the Stokes equations to initial-boundary value problems for systems of pseudodifferential equations*, Report, Copenhagen Univ. Math. Dept. 2, 1987. (Translation of article published in Zap. Nauchn. Sem. L.O.M.I. 163, pp. 37–48, Leningrad, 1987.)
15. L. Hörmander, *Linear Partial Differential Operators* (Grundlehren Math. Wiss. 116), Springer-Verlag, Berlin-Heidelberg, 1963.
16. L. Hörmander, *Fourier integral operators. I*, Acta Math. 127 (1971), 79–183.
17. L. Hörmander, *The Analysis of Linear Partial Differential Operators I, III* (Grundlehren Math. Wiss. 256, 274), Springer-Verlag, Berlin-Heidelberg, 1983, 1985.
18. I. D. Iliev, *Initial boundary value problems for a class of higher order evolution equations*, C. R. Acad. Bulgare Sci. 35 (1982), 585–587.
19. O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type* (Transl. Math. Monographs 23), American Mathematical Society, Providence, R. I., 1968.
20. J. L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications I, II* (Grundlehren Math. Wiss. 181, 182), Springer-Verlag, Berlin-Heidelberg, 1972.
21. R. S. Palais, *Seminar on the Atiyah–Singer Index Theorem* (Ann. of Math. Stud. 57), Princeton University Press, Princeton, N. J., 1965.
22. A. Piriou, *Problèmes aux limites généraux pour des opérateurs différentiels paraboliques dans un domaine borné*, Ann. Inst. Fourier (Grenoble) 21 (1971), 59–78.
23. V. T. Purmonen, *On parametrically quasi-elliptic boundary problems*, Ann. Acad. Sci. Fenn. Ser. A I Math. 5 (1980), 237–262.
24. V. T. Purmonen, *Quasi-parabolic initial-boundary value problems*, Ann. Acad. Sci. Fenn. Ser. A I Math. 6 (1981), 29–62.
25. V. T. Purmonen, *On quasi-parabolic problems with normal boundary conditions*, Ann. Acad. Sci. Fenn. Ser. A I Math. 7 (1982), 279–290.
26. V. T. Purmonen, *On the compatibility of the initial and boundary values in quasi-parabolic problems*, Bericht (Report), Univ. Jyväskylä Math. Inst. 32, 1986.
27. S. Rempel and B.-W. Schulze, *Index Theory of Elliptic Boundary Problems*, Akademie-Verlag, Berlin, 1982.
28. B.-W. Schulze, *A symbolic calculus for a class of parabolic boundary value problems*, Mathematical structures–computational mathematics–mathematical modelling, 2, 279–286, Bulgarian Academy of Sciences, Sofia, 1984.
29. L. Schwartz, *Théorie des distributions à valeurs vectorielles (I)*, Ann. Inst. Fourier (Grenoble) 7 (1957), 1–141.
30. L. N. Slobodeckiĭ, *Generalized Sobolev spaces and their application to boundary problems for partial differential equations*, Amer. Math. Soc. Transl. (2) 57 (1966), 207–275.
31. F. Treves, *Introduction to Pseudodifferential and Fourier Integral Operators 1, 2*, Plenum Press, New York, 1980.
32. M. I. Višik and G. I. Èskin, *Parabolic convolution equations in a bounded region*, Amer. Math. Soc. Transl. (2) 95 (1970), 131–162.
33. L. R. Volevič and B. P. Panejah, *Certain spaces of generalized functions and embedding theorems*, Russian Math. Surveys 20 (1965), no. 1, 1–73.

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