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CHARACTERIZATION OF PERFECT INVOLUTION GROUPS

TORBEN MAACK BISGAARD

0. Introduction.

By Herglotz' theorem, every positive definite function on the group Z of integers is the trigonometric moment sequence of a unique measure on the unit circle. A slight change in the definition of positive definiteness leads to the following result of Jones et al. [7]: If φ: $\mathrm{Z} \to \mathrm{R}$ is such that the kernel $(m, n) \to \varphi(m + n)$ is positive semidefinite, there is a measure μ on $\mathrm{R} \setminus \{0\}$ such that $\varphi(n) = \int x^n \, d\mu(x)$, $n \in \mathrm{Z}$. In this case the measure need not be uniquely determined (an example of an indeterminate two-sided moment sequence is [2, 6.4.6]).

To place the moment problems of Herglotz and of Jones et al. into a common frame, consider any abelian semigroup $(S, +)$ with zero, equipped with an involution $s \to s^*$. Call a function φ: $S \to \mathrm{C}$ *positive definite* if the kernel $(s, t) \to \varphi(s + t^*)$ is positive semidefinite, and let S^* be the space of multiplicative complex functions σ on S satisfying $\sigma(0) = 1$ and $\sigma(s^*) = \overline{\sigma(s)}$. Two questions arise:

Is it true that for every positive definite function φ on S there is a Radon measure μ on S^* such that $\varphi(s) = \int \sigma(s) d\mu(\sigma)$, $s \in S$?

If so, is μ always uniquely determined?

We shall call S *semiperfect* if the answer to the first question is affirmative; *perfect* if, in addition, the answer to the second question is affirmative.

Every bounded positive definite function on S is represented by a unique Radon measure on S^* (in fact concentrated on the bounded members of S^*) [8]. Thus S is perfect if every positive definite function on S is bounded. This is so for abelian inverse semigroups (with the natural involution), hence in particular for abelian groups with the conventional involution $s^* = -s$ (the Bochner-Weil

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theorem for discrete groups) as well as for idempotent abelian semigroups (semilattices) with the involution $s^* = s$.

An involution semigroup admitting unbounded positive definite functions may be perfect, non-perfect semiperfect, or non-semiperfect, as shown by the following examples where the semigroup operation is addition and the involution is the identity ($s^* = s$):

The semigroup $\mathbb{Q}_+$ of nonnegative rationals is perfect [2, 6.5.6]. So is $\mathbb{Q}$ [2, 6.5.10] and so, more generally, is every countable rational vector space (by the countable direct sum theorem for perfect semigroups [2, p. 224]).

The semigroup $\mathbb{N}_0$ of nonnegative integers is semiperfect (Hamburger's theorem) but is not perfect since there are indeterminate moment sequences.

For $k \geqq 2$ the semigroups $\mathbb{N}_0^k$ and $\mathbb{Z}^k$, associated with the multidimensional moment problem and its two-sided analogue, are non-semiperfect ([1], [2, 6.4.8]).

The purpose of the present paper is to characterize those abelian *groups* with involution which are perfect, and those which are semiperfect. For each property, it turns out, there are just two "forbidden *-homomorphic images".

Section 1 contains definitions and basic observations. In Section 2 we establish a key lemma, on extension of completely positive definite functions on countable torsion-free abelian groups carrying the identical involution. The main theorem is proved in Section 3, and Section 4 discusses, among other things, the prospects of extension of the result to semigroups.

1. Preliminaries.

Every semigroup appearing in this paper is abelian, so (except when being formal) we write just "semigroup" for "abelian semigroup". The same applies to "group"; thus "free group" means "free abelian group". In the absence of any indication to the contrary, the semigroup operation is addition (+).

A **-semigroup* $(S, *)$ consists of a semigroup S with zero and an *involution*, that is, an involutory automorphism of S, written $s \to s^*$. If $(T, *)$ is another *-semigroup, a homomorphism p: $S \to T$ is called a **-homomorphism* provided that $p(0) = 0$ and $p(s^*) = p(s)^*$. A **-subsemigroup* is a subsemigroup containing 0 and stable under the involution. The terms **-group* and **-subgroup* should be self-explanatory.

A *character* on a *-semigroup S is a *-homomorphism of S into the *-semigroup $(\mathbb{C}, \cdot, ^-)$, the bar denoting complex conjugation. The set of characters on S is denoted by S^* and is equipped with the topology of pointwise convergence. We let $E_+(S^*)$ denote the set of Radon measures (i.e. tight Borel measures) μ on S^* satisfying $\int |\sigma(s)|\, d\mu(\sigma) < \infty$ for all $s \in S$, and define the generalized Laplace transform $\mathscr{L}$: $E_+(S^*) \to \mathbb{C}^S$, written $\mu \to \mathscr{L}\mu$, by $\mathscr{L}\mu(s) = \int \sigma(s) d\mu(\sigma)$ for $\mu \in E_+(S^*)$, $s \in S$; if S has to be specified, we write $\mathscr{L}_S$ rather than just $\mathscr{L}$. A function

φ: $S \to \mathsf{C}$ is a *moment function* if $\varphi = \mathscr{L}\mu$ for some $\mu \in E_+(S^*)$; any such μ is said to *represent* φ. A moment function is *determinate* if it has only one representing measure; otherwise, *indeterminate*.

EXAMPLE 1. On every group G the mapping $s \to -s$ is an involution, called the *inverse involution* and denoted by $-\mathrm{id}$. For this involution, what we call characters are the usual group characters, $E_+(G^*)$ consists of all measures on the compact space $G^* (= \hat{G})$, and $\mathscr{L}$ is just the Fourier transform.

EXAMPLE 2. On every semigroup S with zero, the mapping $s \to s$ is an involution, called the *identical involution* and denoted by id. For this involution the characters are those homomorphisms of S into $(\mathsf{R}, \cdot)$ not identically zero. We emphasize that even if the underlying semigroup is a group, characters need not be bounded; so they are not the same as usual group characters. For example, the characters on $(\mathsf{Z}, \mathrm{id})$ are the functions $\zeta_x (x \in \mathsf{R} \setminus \{0\})$ given by $\zeta_x(n) = x^n$; of these, only ζ_1 and ζ_{-1} are bounded.

A complex function φ on a $*$-semigroup S is *positive definite* if for every choice of finitely many s_j in S the square matrix $(\varphi(s_j + s_k^*))$ is positive semidefinite. The set of positive definite functions on S is denoted by $\mathscr{P}(S)$. Every moment function is positive definite since if $\varphi = \mathscr{L}\mu$, $s_j \in S$, and $c_j \in \mathsf{C}$ then $\sum c_j \overline{c_k} \varphi(s_j + s_k^*) = \int |\sum c_j \sigma(s_j)|^2 d\mu(\sigma) \geqq 0$. If, conversely, every positive definite function on S is a moment function, we say S is *semiperfect*. If every moment function on S is determinate, S is *determinate*; otherwise, *indeterminate*. Finally, S is *perfect* if every positive definite function on S is a determinate moment function (i.e., if S is semiperfect and determinate). The following homomorphism theorems, countable direct sum theorem, and product theorem will be used later on:

Every $*$-homomorphic image of a perfect $*$-semigroup is perfect [2, 6.5.5];

Every $*$-homomorphic image of a semiperfect $*$-semigroup is semiperfect (by a variation of the proof of [2, 6.5.5]);

The direct sum of a sequence of perfect $*$-semigroups is perfect [2, p. 224];

The product of a perfect $*$-semigroup and a finitely generated semiperfect $*$-semigroup is semiperfect [3].

(The involution on a direct sum of $*$-semigroups is understood to be the one rendering the canonical embeddings $*$-preserving).

If p: $S \to T$ is a $*$-homomorphism the *dual mapping* p^*: $T^* \to S^*$ is defined by $p^*(\tau) = \tau \circ p, \tau \in T^*$. Now p^* is continuous, so if ν is a finite Radon measure on T^* then the image measure ν^{p^*} is a Radon measure on S^*. If $\nu \in E_+(T^*)$ then $\nu^{p^*} \in E_+(S^*)$ and $\mathscr{L}_S(\nu^{p^*}) = (\mathscr{L}_T \nu) \circ p$.

If S is a $*$-semigroup and $a \in S$, we define the shift operator E_a: $\mathsf{C}^S \to \mathsf{C}^S$ by $E_a \varphi(s) = \varphi(a + s)$ for $\varphi \in \mathsf{C}^S$, $s \in S$. In particular, E_0 is the identity operator I on C^S. A function φ: $S \to \mathsf{C}$ is *completely positive definite* if every *shift* of φ (that is, every function of the form $E_a\varphi$ with $a \in S$) is positive definite. On a 2-divisible semigroup (in particular, on a rational vector space) carrying the identical involution, every positive definite function is completely positive definite.

LEMMA 1. *Every shift of a positive definite function is a linear combination of positive definite functions.*

PROOF. Let $\varphi \in \mathscr{P}(S)$ and $a \in S$ be given. The functions $\varphi_n = (I + i^n E_a + i^{-n} E_{a*} + E_{a+a*})\varphi$ $(n = 0, 1, 2, 3)$ are positive definite and $E_a\varphi = \frac{1}{4}\sum_{n=0}^{3} i^{-n}\varphi_n$.

2. An extension lemma.

In this section, U denotes a countable rational vector space. Every rational vector space occurring in this section we consider with the identical involution and make it a topological group by imposing on it the trace of the finest locally convex topology on the enveloping real vector space.

LEMMA 2. *For a subgroup X of U the following three conditions are equivalent:*
(i) *X is closed in U;*
(ii) *$X \cap T$ is closed in T for every finite-dimensional subspace T of U;*
(iii) *There exist a subspace V of U and a free subgroup F of U/V such that $X = \pi^{-1}(F)$ where π: $U \to U/V$ is the quotient mapping.*
Note. When (iii) *holds, X is isomorphic to $V \times F$* [6, p. 74].

PROOF. The rest being trivial, let us show (ii) $\Rightarrow$ (iii). Let (U_n) be an increasing sequence of finite-dimensional subspaces of U with union U. By the well-known structure of closed subgroups of R^d, for each n there exist a subspace V_n of U_n and a free subgroup F_n of U_n/V_n such that $X \cap U_n = \pi_n^{-1}(F_n)$ where π_n: $U_n \to U_n/V_n$ is the quotient mapping. Define $V = \cup V_n$. For $j < k$ we have $V_j = V_k \cap U_j$ since V_j is the greatest subspace of U_j contained in X; it follows that $V_n = V \cap U_n$ for each n. Hence there are injective homomorphisms ι_n: $U_n/V_n \to U/V$ such that $\iota_n \circ \pi_n = \pi | U_n$ where π: $U \to U/V$ is the quotient mapping. Writing $F = \cup \iota_n(F_n)$, check that $X = \pi^{-1}(F)$. Since the groups $F \cap \pi(U_n) = \iota_n(F_n)$ are free, so is F [9, p. 378].

LEMMA 3. *If X is a subgroup of U and if φ: $X \to \mathsf{R}$ is completely positive definite then φ is midpoint convex in the sense that $\varphi(x + a) \leqq \frac{1}{2}(\varphi(x) + \varphi(x + 2a))$ for all $x, a \in X$.*

More generally, for all $x, a \in X$, $n \in \mathsf{N}$, and $m \in \{0, 1, \ldots, n\}$ we have

$$(1) \qquad \varphi(x + ma) \leqq \frac{n - m}{n}\varphi(x) + \frac{m}{n}\varphi(x + na).$$

Hence, if V is a subspace of U and if φ: $V \to \mathsf{R}$ is positive definite then

$\varphi((1-\lambda)x+\lambda y) \leqq (1-\lambda)\varphi(x)+\lambda\varphi(y)$ for all $x, y \in V$ and $\lambda \in [0,1] \cap \mathbf{Q}$; it follows that φ is continuous.

Finally, if X is any closed subgroup of U and if φ: $X \to \mathbf{R}$ is positive definite then φ is continuous.

NOTE. In the last statement, the assumption that X be closed is indispensable. For example, if $U = \mathbf{Q}$ and $X = \{p/3^k \mid p \in \mathbf{Z}, k \in \mathbf{N}\}$ then X admits discontinuous characters, such as ξ given by $\xi(p/3^k) = (-1)^p$.

PROOF. Concerning the first statement: Since $E_x\varphi$ is positive definite, the matrix $\begin{pmatrix} \varphi(x) & \varphi(x+a) \\ \varphi(x+a) & \varphi(x+2a) \end{pmatrix}$ is positive semidefinite, so $\varphi(x) \geqq 0, \varphi(x+2a) \geqq 0$, and $\varphi(x+a) \leqq (\varphi(x)\varphi(x+2a))^{\frac{1}{2}} \leqq \frac{1}{2}(\varphi(x)+\varphi(x+2a))$.

To show (1), note that the midpoint convexity of φ implies $\varphi(x+ka) \leqq \frac{1}{2}(\varphi(x+(k-1)a)+\varphi(x+(k+1)a))$ for $k = 1,\ldots,n-1$. Multiplying the k'th inequality by $((n-2m)k+nm-n|k-m|)/2n$ and summing gives the desired result.

For the third statement, recall that every positive definite function on a rational vector space is completely positive definite. Write $\lambda = m/n$ with $n \in \mathbf{N}$, $m \in \mathbf{Z}$, and apply (1) to $X = V$, $a = (y-x)/n$ to get the inequality stated. The continuity of φ follows by an argument similar to the familiar proof that every convex function on a real vector space is continuous for the finest locally convex topology.

To prove the last statement, take V, F, π as in Lemma 2. The cosets of V being open in X, it suffices to show that for each coset A of V in X the function $\varphi \mid A$ is continuous. This amounts to showing the continuity of $E_a\varphi \mid V$ for some $a \in A$. By Lemma 1, $E_a\varphi \mid V$ is a linear combination of positive definite functions on V, hence continuous by what was shown above.

LEMMA 4. *If X is a subgroup of U and if φ: $X \to \mathbf{R}$ is completely positive definite then φ extends to a continuous function on the closure of X.*

PROOF. With no loss of generality, suppose that X spans U. In a first step, assume $U = \mathbf{Q}^k$ for some $k \in \mathbf{N}$. We shall show that for every bounded subset B of X there exists $a \geqq 0$ such that

$$(2) \qquad |\varphi(x)-\varphi(y)| \leqq a\,\|x-y\|_\infty \quad \text{for all } x, y \in B$$

with the notation $\|z\|_\infty = \max\{|z_1|,\ldots,|z_k|\}$ for $z = (z_1,\ldots,z_k) \in \mathbf{Q}^k$. The existence of the desired extension follows.

Since X contains a basis of $\mathbf{Q}^k$, there is no loss of generality in assuming that $(1,0,\ldots,0), \ldots, (0,\ldots,0,1)$ are in X. Again with no loss of generality, assume $B \subset [-\frac{1}{2},\frac{1}{2}]^k$. We first show

$$(3) \qquad |\varphi(x)-\varphi(0)| \leqq c\,\|x\|_\infty \text{ for all } x \in X \cap [-1,1]^k$$

where $c = k \sup\{\varphi(s) - \varphi(0) \mid s \in \{-1, 0, 1\}^k\}$. It suffices to prove

$$\varphi(x) - \varphi(0) \leqq c\,\|x\|_\infty \text{ for all } x \in X \cap [-1, 1]^k$$

since this inequality applied to $-x$, along with the midpoint convexity of φ (Lemma 3), yields $\varphi(0) - \varphi(x) \leqq \varphi(-x) - \varphi(0) \leqq c\,\|x\|_\infty$.

For convenience, reverse the signs of suitable coordinates such that the coordinates of $x = (x_1, \ldots, x_k)$ are nonnegative. Put $S = \{0, 1\}^k$. The subgroup F of U generated by $S \cup \{x\}$ is torsion-free and finitely generated, hence free, so the set $E = F \cap [0, 1]^k$ is finite.

We shall construct a family $(\pi_t)_{t \in E}$ of probability measures on E such that

$$t = \int u \, d\pi_t(u) \tag{4}$$

$$\varphi(t) \leqq \int \varphi \, d\pi_t \tag{5}$$

$$\pi_t = \varepsilon_t \text{ if and only if } t \in S \tag{6}$$

where ε_t denotes the Dirac measure at t. Given $t = (t_1, \ldots, t_k) \in E$, choose $s = (s_1, \ldots, s_k) \in S$ such that $|t_i - s_i| \leqq \frac{1}{2}$ for $i = 1, \ldots, k$. Then the point $r = 2t - s$ is in E and the measure $\pi_t = \frac{1}{2}(\varepsilon_r + \varepsilon_s)$ satisfies (4). Moreover, (5) follows from the midpoint convexity of φ, and (6) holds since if $t \notin S$ then $t \neq s$ whereas if $t \in S$ then the choice of s forces $r = s = t$.

Define a sequence $(\mu_n)_{n \geq 0}$ of probability measures on E by $\mu_0 = \varepsilon_x$ and $\mu_{n+1} = \sum_{t \in E} \mu_n(\{t\})\pi_t$; choose an accumulation point μ of (μ_n). By (4), (5), induction, and going to the limit,

$$x = \int t \, d\mu(t) \tag{7}$$

$$\varphi(x) \leqq \int \varphi \, d\mu \tag{8}$$

$$\mu = \sum_{t \in E} \mu(\{t\})\pi_t. \tag{9}$$

From (6) and (9) it follows that $\mu(S) = \mu(S) + \sum_{t \in E \setminus S} \mu(\{t\})\pi_t(S)$, that is $\pi_t(S) = 0$ for all $t \in \operatorname{supp}(\mu) \setminus S$. Suppose $\operatorname{supp}(\mu) \setminus S$ is nonempty and choose a vertex v of the polytope $\operatorname{conv}(\operatorname{supp}(\mu) \setminus S)$. Since v is in $\operatorname{supp}(\mu) \setminus S$, we have $\pi_v(S) = 0$. From (9) it follows that $\operatorname{supp}(\pi_v) \subset \operatorname{supp}(\mu)$; so $\operatorname{supp}(\pi_v) \subset \operatorname{supp}(\mu) \setminus S$. This fact together with $v = \int u \, d\pi_v(u)$ and the choice of v implies $\pi_v = \varepsilon_v$, contradicting (6).

The contradiction shows that μ is concentrated on S. Now (7) and (8) imply $\varphi(x) - \varphi(0) \leqq \int (\varphi(t) - \varphi(0))d\mu(t) \leqq k^{-1}c\mu(S \setminus \{0\})$. The desired inequality follows since $\mu((S\setminus\{0\}) \leqq \int(\sum s_i)d\mu(s) \leqq \sum \int s_i d\mu(s) = \sum x_i \leqq k\,\|x\|_\infty$. This proves (3).

To derive (2), let $x, y \in X \cap [-\frac{1}{2}, \frac{1}{2}]^k$ be given and assume $x \neq y$. Denote by n the greatest integer such that $\|ny - (n-1)x\|_\infty \leqq 1$. Then

$$(10) \qquad n\,\|y - x\|_\infty = \|n(y - x)\|_\infty \geqq \|(n+1)y - nx\|_\infty - \|y\|_\infty > \tfrac{1}{2}.$$

Because x and $ny - (n-1)x$ are in $X \cap [-1, 1]^k$, the inequalities (3), (10) imply

$$\varphi(ny - (n-1)x) - \varphi(x) \leqq |\varphi(ny - (n-1)x) - \varphi(0)| + |\varphi(x) - \varphi(0)|$$
$$\leqq c\,\|ny - (n-1)x\|_\infty + c\,\|x\|_\infty \leqq 2c \leqq na\,\|y - x\|_\infty$$

where $a = 4c$. Combine this with (1) (applied to $a = y - x$, $m = 1$) to obtain $\varphi(y) - \varphi(x) \leqq a\,\|y - x\|_\infty$. A similar argument gives $\varphi(x) - \varphi(y) \leqq a\,\|x - y\|_\infty$, proving (2).

In the general case, use the axiom of choice to find a pair (Y, Φ) consisting of a subgroup Y of $\bar{X}$ containing X and a completely positive definite function Φ on Y extending φ, and maximal for the natural ordering of such pairs. Let (U_k) be an increasing sequence of finite-dimensional subspaces of U with union U. By the first part of the proof, for each k the function $\Phi\,|\,(Y \cap U_k)$ extends to a continuous, hence completely positive definite, function Φ_k on $(Y \cap U_k)^-$. Evidently $\Phi_j = \Phi_k\,|\,(Y \cap U_j)^-$ for $j < k$; so Φ extends to a completely positive definite function on $\bigcup (Y \cap U_k)^-$. The maximality of (Y, Φ) now implies $Y = \bigcup (Y \cap U_k)^-$. By Lemma 2 it follows that Y is closed, so $Y = \bar{X}$. By Lemma 3, Φ is continuous.

REMARK 1. The appeal to the axiom of choice in the proof of Lemma 4 is convenient, but not necessary. By an appropriate choice of seminorms to replace $\|\cdot\|_\infty$, the above proof of the finite-dimensional case can be extended so as to cover the general case.

3. The result

One more lemma is needed for the proof of the theorem:

LEMMA 5. *No uncountable rational vector space carrying the identical involution is semiperfect.*

PROOF. Let the space in question be $\mathbf{Q}^{(I)}$ for some set I. Consider the function φ: $\mathbf{Q}^{(I)} \to R$ given by

$$\varphi(s) = \exp\left(\frac{1}{2}\sum_{i \in I} s_i^2\right) \quad \text{for } s = (s_i) \in \mathbf{Q}^{(I)}.$$

Denoting by γ the lognormal distribution, we have

$$\varphi(s) = \int \exp\langle\cdot, s\rangle d\left(\bigotimes_{j\in J} \gamma\right), \quad s \in \mathbf{Q}^J$$

for every finite subset J of I, identifying $\{s \in \mathbf{Q}^{(I)} \mid s_i = 0 \text{ for } i \notin J\}$ with $\mathbf{Q}^J$; here $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbf{R}^J$. Hence φ is positive definite. If $\mathbf{Q}^{(I)}$ is semiperfect it follows that

$$\varphi(s) = \int \exp\langle\cdot, s\rangle d\mu, \quad s \in \mathbf{Q}^{(I)}$$

for some Radon probability measure μ on $\mathbf{R}^I$. For any finite subset J of I, the image measure μ^{π_J} under the projection π_J: $\mathbf{R}^I \to \mathbf{R}^J$, like the measure $\bigotimes_{j\in J} \gamma$, represents $\varphi \mid \mathbf{Q}^J$; by the perfectness of $\mathbf{Q}^J$ it follows that $\mu^{\pi_J} = \bigotimes_{j\in J} \gamma$. Thus μ is a Radon product measure $\bigotimes_{i\in I} \gamma$. Since an uncountable family of noncompactly supported Radon measures cannot have a Radon product measure, it follows that I is countable.

THEOREM. *Let G be an abelian group with involution and let W be a rational vector space, the dimension of which is the smallest uncountable cardinal; consider W with the identical involution.*

(i) *G is perfect if and only if neither W nor* $(\mathbf{Z}, \mathrm{id})$ *is a *-homomorphic image of G.*

(ii) *G is semiperfect if and only if neither W nor* $(\mathbf{Z}^2, \mathrm{id})$ *is a *-homomorphic image of G.*

PROOF. Since $(\mathbf{Z}, \mathrm{id})$ is not perfect (see §0) and since neither $(\mathbf{Z}^2, \mathrm{id})$ nor W is semiperfect (§1 and Lemma 5), necessity of the conditions follows from the homomorphism theorems (§1).

In what follows we assume that neither $(\mathbf{Z}^2, \mathrm{id})$ nor W be a *-homomorphic image of G, and show that G is semiperfect. That done, we shall complete the proof by showing that if $(\mathbf{Z}, \mathrm{id})$ is not a *-homomorphic image of G then G is determinate.

Considering the quotient groups $S = G/\{s + s^* \mid s \in G\}$ and $T = G/\{s - s^* \mid s \in G\}$ with the inverse involution and the identical involution, respectively, see that the quotient mappings are *-homomorphisms.

Let U be the enveloping rational vector space of the greatest torsion-free quotient group of T, and let p_1: $G \to S$ and p_2: $G \to U$ denote the canonical mappings. Equip the product group $S \times U$ with the involution $(s, u)^* = (-s, u)$ and define a *-homomorphism p: $G \to S \times U$ by $p(x) = (p_1(x), p_2(x))$. Let π_1: $S \times U \to S$ and π_2: $S \times U \to U$ be the projections and note that $p_1 = \pi_1 \circ p$, $p_2 = \pi_2 \circ p$.

The rational vector space U is countable-dimensional. To see this, choose a family $(e_j)_{j\in J}$ in G such that $(p_2(e_j))$ is a linear basis of U. If J is uncountable, choose a mapping m of J onto W and let $M\colon U \to W$ the $\mathbb{Q}$-linear mapping determined by $M(p_2(e_j)) = m(j)$. The *-homomorphism $M \circ p_2$ maps G onto W, contrary to assumption.

The proof that G is semiperfect consists in several steps:

(a) *Every positive definite function on G factors through p.*

That is to say, every positive definite function on G is constant on each coset of the kernel $\ker(p)$. By Lemma 1 it suffices to show that every positive definite function on G is constant on $\ker(p)$. Let $\varphi\colon G \to \mathbb{C}$ be positive definite and define $\psi = \varphi \mid \ker(p)$.

Since $\ker(p) \subset \ker(p_1) = \{s + s^* \mid s \in G\}$ and since $E_{s+s^*}\varphi$ is positive definite for each $s \in G$, the function ψ is completely positive definite.

To show $\psi(a) = \psi(0)$ for any given $a \in \ker(p)$, note that $\ker(p) \subset \ker(p_2)$, so the definition of p_2 gives $ka = s - s^*$ for some $k \in \mathbb{N}$ and $s \in G$. Choosing $t \in G$ such that $a = t + t^*$, we find $a = a^*$ and $2ka = ka + ka^* = s - s^* + (s - s^*)^* = 0$. The function Ψ defined on $(\mathbb{Z}, \mathrm{id})$ by $\Psi(n) = \psi(na)$ is completely positive definite, hence midpoint convex (Lemma 3). Since Ψ is periodic it follows that Ψ is constant. In particular, $\psi(0) = \Psi(0) = \Psi(1) = \psi(a)$. This proves (a). Obviously the function on $p(G)$ involved in the factoring will again be positive definite.

In the following we consider $S \times U$ with the product of the discrete topology on S and the topology on U from §2. Let H be the closure of $p(G)$ in $S \times U$.

b) *Every positive definite function on $p(G)$ extends to a unique positive definite function on H.*

We first show that every positive definite function on $p(G)$ extends to a continuous (hence positive definite) function on H. The cosets of $\{0\} \times U$ being open in $S \times U$, it suffices to show that for any positive definite function φ on $p(G)$ and any coset A of $\{0\} \times U$ the function $\varphi \mid (p(G) \cap A)$ extends to a continuous function on $(p(G) \cap A)^-$. Thanks to Lemma 1, we need only consider the case $A = \{0\} \times U$. Now $p(G) \cap A = p(\ker(p_1)) = p(\{s + s^* \mid s \in G\}) = \{t + t^* \mid t \in p(G)\}$, so $E_a\varphi$ is positive definite for each $a \in p(G) \cap A$ and therefore $\varphi \mid (p(G) \cap A)$ is completely positive definite. Identifying $p(G) \cap A$ with a subgroup of U in the natural way, we get the result by Lemma 4.

Since $H \cap (\{0\} \times U)$ is isomorphic to a closed subgroup of U, every positive definite function on $H \cap (\{0\} \times U)$ is continuous (Lemma 3). Using again Lemma 1 and the fact that the cosets of $\{0\} \times U$ are open in $S \times U$, we find that every positive definite function on H is continuous, proving the uniqueness statement of (b).

Define $K = S \times \pi_2(H)$ and note that K contains H.

(c) *Every positive definite function on H extends to a positive definite function on K.*

Observe that for $s \in S$ and $y \in H$ we have $(s, \pi_2(y)) + (s, \pi_2(y))^* = (0, 2\pi_2(y)) = y + y^*$. Hence

$$\{x + x^* \mid x \in K\} = \{y + y^* \mid y \in H\}. \tag{11}$$

Now let φ: $H \to \mathbf{C}$ be positive definite and define Φ: $K \to \mathbf{C}$ by $\Phi \mid H = \varphi$, $\Phi \mid (K \setminus H) = 0$. We claim that Φ is positive definite. This is equivalent to saying that whenever α is a complex measure on K with finite support then

$$\int \Phi d(\alpha * \tilde{\alpha}) \geqq 0$$

where $*$ denotes convolution and the adjoint operation $\tilde{}$ is defined by conjugate linearity and the condition that $\tilde{\varepsilon}_x = \varepsilon_{x^*}$ for Dirac measures ε_x.

If $x_1, \ldots, x_n$ are representatives of the finitely many cosets of H meeting the support of α (x_i and x_j representing distinct cosets if $i \neq j$) then

$$\alpha = \sum_{i=1}^{n} \varepsilon_{x_i} * \beta_i$$

where each β_i is a complex measure on H with finite support. Hence

$$\alpha * \tilde{\alpha} = \sum_{i,j=1}^{n} \varepsilon_{x_i + x_j^*} * \beta_i * \tilde{\beta}_j.$$

Terms with $i \neq j$ are supported by cosets of H other than H itself since if $x_i + x_j^* \in H$ then (11) gives $x_i - x_j = (x_i + x_j^*) - (x_j + x_j^*) \in H$. The remaining terms are supported by H since (11) implies $x_i + x_i^* \in H$. It follows that

$$(\alpha * \tilde{\alpha}) \mid H = \sum_{i=1}^{n} \varepsilon_{x_i + x_i^*} * \beta_i * \tilde{\beta}_i.$$

Using (11), choose $y_1, \ldots, y_n \in H$ such that $x_i + x_i^* = y_i + y_i^*$. Seeing that the measures $\gamma_i = \varepsilon_{y_i} * \beta_i$ are supported by H, use the positive definiteness of φ to conclude $\int \Phi d(\alpha * \tilde{\alpha}) = \sum_{i=1}^{n} \int \varphi d(\gamma_i * \tilde{\gamma}_i) \geqq 0.$

(d) *K is semiperfect.*

We first show $\pi_2(H) = \overline{p_2(G)}$. Since π_2 is continuous, $\pi_2(H) = \pi_2(\overline{p(G)}) \subset \overline{\pi_2(p(G))} = \overline{p_2(G)}$. For the converse inclusion, note that by Lemma 2 there exist a subspace V of U and a free subgroup F of U/V such that $\overline{p_2(G)} = \pi^{-1}(F)$ where π: $U \to U/V$ is the quotient mapping. Since $(0, 2p_2(x)) = p(x + x^*) \in p(G) \subset H$

for $x \in G$, we have $\{0\} \times 2p_2(G) \subset H$. It follows that

$$\{0\} \times 2\overline{p_2(G)} \subset H, \tag{12}$$

hence $2\overline{p_2(G)} \subset \pi_2(H)$. Also, $p_2(G) = \pi_2(p(G)) \subset \pi_2(\overline{p(G)}) = \pi_2(H)$. Since F is discrete in U/V, the cosets of V are open in $\overline{p_2(G)}$, so each of them meets $p_2(G)$. Therefore

$$F = \pi(p_2(G)) \tag{13}$$

and $\overline{p_2(G)} = \pi^{-1}(F) = p_2(G) + V = p_2(G) + 2V \subset p_2(G) + 2\overline{p_2(G)} \subset \pi_2(H)$, proving $\pi_2(H) = \overline{p_2(G)}$.

Since $\pi^{-1}(F)$ is isomorphic to $V \times F$, the *-group K is *-isomorphic to $S \times V \times F$, the first factor carrying the inverse involution and the other two the identical involution. In view of the assumption that $(\mathbf{Z}^2, \mathrm{id})$ not be a *-homomorphic image of G, (13) implies $F = \mathbf{Z}^\delta$ with $\delta \in \{0, 1\}$. In particular, F is finitely generated and semiperfect. By one of the product theorems in §1, $S \times V$ is perfect; by the other it follows that K is semiperfect.

We are now in a position to prove that G is semiperfect. If φ: $G \to \mathbf{C}$ is positive definite then by (a), (b), (c) there is a positive definite function Φ on K such that $\varphi = \Phi \circ p$, and by (d) we have $\Phi = \mathscr{L}\nu$ for some $\nu \in E_+(K^*)$. Hence $\nu^{p^*} \in E_+(G^*)$ and $\varphi = \mathscr{L}(\nu^{p^*})$.

In the remainder of the proof we assume that $(\mathbf{Z}, \mathrm{id})$ not be a *-homomorphic image of G, and show that G is determinate (hence perfect).

Referring to the proof of (d), see that the present assumption implies $F = \{0\}$, hence $\overline{p_2(G)} = V$. Since $p_2(G)$ spans U, it follows that $V = U$. By (12), $\{0\} \times U = \{0\} \times 2U \subset H$, and since $S = p_1(G) = \pi_1(p(G)) \subset \pi_1(\overline{p(G)}) = \pi_1(H)$, it follows that $H = S \times U$.

The mapping $(\sigma, \upsilon) \to \sigma \otimes \upsilon$ given by $\sigma \otimes \upsilon(s, u) = \sigma(s)\upsilon(u)$ is easily seen to be a homeomorphism of $S^* \times U^*$ onto H^*.

We claim that p^*: $H^* \to G^*$ (the dual of p: $G \to H$) is a homeomorphism of H^* onto G^*. To see that p^* is one-to-one and onto, let $\gamma \in G^*$ be given. The character $\gamma/|\gamma|$ equals 1 on $\{s + s^* \mid s \in G\}$ and therefore has the form $\sigma \circ p_1$ for a unique $\sigma \in S^*$. The character $|\gamma|$ equals 1 on $\{s - s^* \mid s \in G\}$ and takes values in the uniquely divisible group $(]0, \infty[, \cdot)$ and therefore has the form $\upsilon \circ p_2$ for a unique $\upsilon \in U^*$. Now $\sigma \otimes \upsilon$ is the unique character on H satisfying $\gamma = p^*(\sigma \otimes \upsilon)$.

To see that $p^{*^{-1}}$ is continuous, consider a net $(\sigma_i \otimes \upsilon_i)$ in H^* such that $(p^*(\sigma_i \otimes \upsilon_i))$ converges to some $\gamma \in G^*$. Define $\sigma \otimes \upsilon = p^{*^{-1}}(\gamma)$ and $\gamma_i = p^*(\gamma_i \otimes \upsilon_i)$. Given $s \in S$, choose $x \in G$ such that $s = p_1(x)$. Then $\sigma_i(s) = \gamma_i(x)/|\gamma_i(x)| \to \gamma(x)/|\gamma(x)| = \sigma(s)$, so $\sigma_i \to \sigma$. Given $u \in U$, choose $x \in G$ and $k \in \mathbf{N}$ such that $ku = p_2(x)$. Then $\upsilon_i(u) = |\gamma_i(x)|^{1/k} \to |\gamma(x)|^{1/k} = \upsilon(u)$, so $\upsilon_i \to \upsilon$.

Next, we claim that if $\mu \in E_+(G^*)$ then $\mu^{p^{*^{-1}}} \in E_+(H^*)$. Given $s \in S$ and $u \in U$, choose $x \in G$ and $k \in \mathbf{N}$ such that $ku = p_2(x)$. Then

$$\int_{H^*} |\eta(s,u)| d\mu^{p^{*-1}}(\eta) = \int_{G^*} |\gamma(x)|^{1/k} d\mu(\gamma) \leqq \mu(G^*)^{(k-1)/k} \left(\int_{G^*} |\gamma(x)| \, d\mu(\gamma) \right)^{1/k},$$

proving the claim since the right side is finite by assumption.

We can now show that G is determinate. Since $\mathscr{L}_G$: $E_+(G^*) \to \mathscr{P}(G)$ is the composite of the three mappings

$$E_+(G^*) \xrightarrow[\mu \to \mu^{p^{*-1}}]{} E_+(H^*) \xrightarrow[\mathscr{L}_H]{} \mathscr{P}(H) \xrightarrow[\Phi \to \Phi \circ p]{} \mathscr{P}(G),$$

it suffices to verify that each of these is one-to-one. Since p^{*-1} is one-to-one, so is $\mu \to \mu^{p^{*-1}}$; injectivity of $\mathscr{L}_H$ is part of the fact that $H = S \times U$ is perfect. That $\Phi \to \Phi \circ p$ is one-to-one was shown in (b).

The rational vector space appearing in the following consequence of the Theorem is understood to carry the topology described in §2.

COROLLARY. *A torsion-free abelian group carrying the identical involution is perfect if and only if it is countable and is dense in the enveloping rational vector space.*

PROOF. Let G be the group in question and U the enveloping rational vector space. The space W of the Theorem is a homomorphic image of G if and only if G is uncountable (see the relevant portion of the proof of the Theorem). Now assume that G is countable. By Lemma 2, $\bar{G} = \pi^{-1}(F)$ for some subspace V of U with quotient mapping π: $U \to U/V$ and some free subgroup F of U/V. The cosets of V being open in $\bar{G}$, we have $F = \pi(G)$. Since F is the greatest free quotient of G then $\mathbf{Z}$ is not a homomorphic image of G if and only if $F = \{0\}$, which is equivalent to G being dense in U.

4. Comments.

REMARK 2. If a countable $*$-semigroup S is *$*$-divisible* in the sense that for each $s \in S$ there exist $t \in S$ and $(m,n) \in \mathbf{N}_0^2$ with $m + n \geqq 2$ such that $s = mt + nt^*$ then S is perfect [4]. By the homomorphism theorem and the countable direct sum theorem it follows that a countable $*$-semigroup is perfect if it is the sum of its $*$-divisible $*$-subsemigroups. The condition is satisfied by countable abelian inverse semigroups, by $\mathbf{Q}_+$, and by the perfect $*$-semigroups constructible from these by means of the homomorphism theorem and the countable direct sum theorem.

The purpose of the present remark is to point out the existence of a countable perfect $*$-group which does not satisfy the above condition.

For a semigroup carrying the identical involution, $*$-divisibility is equivalent to the condition that each element be divisible by infinitely many integers.

Pontrjagin [9] exhibited an indecomposable torsion-free abelian group G of rank 2 of which no nonzero element is divisible by infinitely many integers. This group, when considered with the identical involution, is not the sum of its *-divisible *-subsemigroups (or else it would have nonzero elements divisible by infinitely many integers) but is nonetheless perfect since if $\mathbf{Z}$ were a homomorphic image of G then G would be decomposable (cf. [6, p. 74]).

REMARK 3. A countable *-group is perfect (resp. semiperfect) if and only if $(\mathbf{Z}, \mathrm{id})$ (resp. $(\mathbf{Z}^2, \mathrm{id})$) is not a *-homomorphic image of it. In the Theorem, an uncountable identical-involution rational vector space occurs as a "forbidden *-homomorphic image", apparently because of the failure of the class of Radon measures to be stable under the formation of uncountable products. To overcome this by widening the class of representing measures permitted, agree to call a *-semigroup S *quasiperfect* [4] if for each positive definite function φ on S there is a unique measure μ, on the σ-field in S^* induced by the evaluations $\sigma \to \sigma(s)$: $S^* \to \mathbf{C}$ $(s \in S)$, such that $\varphi(s) = \int \sigma(s) d\mu(\sigma)$, $s \in S$. Every *-divisible *-semigroup is quasiperfect [4]. In particular, every rational vector space, with any involution whatsoever, is quasiperfect. Thus, in order for a *-group G to be quasiperfect, the condition that an uncountable identical-involution rational vector space not be a *-homomorphic image of G is *not* necessary. The condition that $(\mathbf{Z}, \mathrm{id})$ not be a *-homomorphic image of G *is* necessary. Whether it is also sufficient, we do not know.

REMARK 4. Let S be a cancellative *-semigroup and $G = S - S$ the quotient group, with the involution extending that of S. Whether G is perfect, non-perfect semiperfect, or non-semiperfect can be determined from our Theorem and it is natural to examine to what extent the conclusion can be transferred to S.

If S is perfect, so is G, being a *-homomorphic image of $S \times S$ (cf. the proof of [2, 6.5.10]). It can be shown that if S is semiperfect, so is G.

The perfectness or semiperfectness of G, however, does not imply that S have the same property:

The *-group $G = (\mathbf{Z}^2, *)$ with the involution $(m, n)^* = (n, m)$ is semiperfect [3]. Yet its generating *-subsemigroup $S = (\mathbf{N}_0^2, *)$, associated with the complex moment problem, is not semiperfect [2, 6.3.5].

Boas [5] noticed that if a finite sequence $\varphi(0), \ldots, \varphi(n)$ is *strictly completely positive definite* in the sense that the matrices $(\varphi(j + k))_{j,k \leqq [\frac{n}{2}]}$ and $(\varphi(j + k + 1))_{j,k \leqq [\frac{(n-1)}{2}]}$ are positive definite then for all sufficiently large choices of $\varphi(n + 1)$ the extended sequence $\varphi(0), \ldots, \varphi(n + 1)$ is again strictly completely positive definite. He inferred that every sequence φ: $\mathbf{N}_0 \to \mathbf{R}$ satisfying a certain growth condition is an indeterminate Stieltjes moment sequence. The method of Boas can be applied to every discrete subsemigroup of $[0, \infty[$ and shows that any such semigroup is indeterminate. So if S is a subsemigroup of $\mathbf{Q}_+$ of the form

$S = \{s_0, s_1, \ldots\}$ with $0 = s_0 < s_1 < \ldots$ and $s_{n+1} - s_n \to 0$ as $n \to \infty$ (such as $S = \{2^{-k}p \mid 2^k k \leqq p\} = \{0, 1, \frac{3}{2}, 2, \frac{9}{4}, \frac{5}{2}, \frac{11}{4}, 3, \ldots\}$) then S is indeterminate (hence non-perfect) despite the fact that $G = S - S$, being a dense subgroup of $\mathbf{Q}$, is perfect.

REMARK 5. By the Corollary in §3, a nonzero subgroup of $(\mathbf{Q}, \mathrm{id})$ is perfect if and only if it is dense in $\mathbf{Q}$. For a nonzero subsemigroup of $\mathbf{Q}_+$, however, denseness in $\mathbf{Q}_+$ is neither necessary nor sufficient for perfectness: The non-dense subsemigroup $\mathbf{Q}_+ \setminus]0, 1]$ can be shown to be perfect, and one can construct a dense subsemigroup of $\mathbf{Q}_+$ which is not perfect.

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MATEMATISK INSTITUT
UNIVERSITETSPARKEN 5
DK-2100 KØBENHAVN Ø
DENMARK