MATH. SCAND. 65 (1989), 259–277

SOME RESULTS IN SPECTRAL ANALYSIS AND SYNTHESIS AT INFINITY

ISMO SEDIG

Abstract.

Asymptotic spectra and Wiener spectra are considered. We show that the asymptotic spectrum of a bounded function coincides with the zero set of the closed ideal in $L^1(\mathbf{R})$ consisting of those integrable functions whose convolutions with the bounded vanish at infinity. This fact and known results in asymptotic spectral analysis allow us to characterize the bounded and uniformly continuous functions whose Wiener spectra are, e.g., finite. The characterization is quite similar to that of the asymptotic spectrum. Despite the similarities in spectral analysis, their spectral synthesis properties are different. In particular, it is shown that there exists a bounded function that cannot be "synthesized" from its asymptotic spectrum.

1. Introduction.

The asymptotic spectrum of a bounded function is usually defined in terms of its limit set. It is shown below that it may equally well be defined as the zero set of a certain closed ideal in the convolution algebra $L^1(\mathbf{R})$. This ideal is composed of those integrable functions whose convolutions with the bounded function vanish at infinity.

For this reason one may ask whether the characterization of the uniformly continuous functions in $L^\infty(\mathbf{R})$ whose asymptotic spectra are finite can be carried over to the Wiener spectrum. Below we examine this and related questions and show that the answers are affirmative. (For results in asymptotic spectral analysis, see Gripenberg et al. [4].) However, as far as spectral synthesis is concerned, the asymptotic spectrum and the Wiener spectrum are quite different. In particular, it turns out that the positive spectral synthesis result in Benedetto [1, Theorem 4.1] for the Wiener spectrum does not hold for the asymptotic spectrum.

The emphasis in Gripenberg et al. [4] is, naturally, on positive, i.e. one-sided, asymptotic spectra. In accordance with this, we will mainly consider one-sided Wiener spectra. The ordinary two-sided Wiener spectrum, as defined in Meyer

Received November 29, 1988

[8] and in Benedetto [1], can then be expressed as the union of the corresponding one-sided (positive and negative) Wiener spectra. This is, in fact, a special case of a property of ideals in $L^1(\mathsf{R})$.

As an application we obtain some results related to the asymptotic behavior of certain bounded functions.

2. Preliminaries. Definitions of Different Notions of Spectra.

Let I be an ideal in the convolution algebra $L^1(\mathsf{R})$, i.e., I is a subspace of $L^1(\mathsf{R})$ with the property that $g * f \in I$ for every $g \in L^1(\mathsf{R})$ and $f \in I$. The zero set $Z(I)$ of I is defined by

$$Z(I) = \bigcap_{f \in I} \{t \in \mathsf{R} \mid \hat{f}(t) = 0\}.$$

Here $\hat{f}$ stands for the Fourier transform of f, thus $\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{-itx}dx$.

The following property of ideals in $L^1(\mathsf{R})$ will be used below to establish a relationship between the different notions of Wiener spectra.

2.1 LEMMA. *Let I and J be ideals in $L^1(\mathsf{R})$. Then $I \cap J$ is an ideal in $L^1(\mathsf{R})$, and*

$$\text{(i)} \qquad Z(I \cap J) = Z(I) \cup Z(J).$$

PROOF. Clearly, $I \cap J$ is an ideal. Since $I \cap J \subset I$, $Z(I) \subset Z(I \cap J)$. Similarly, $Z(J) \subset Z(I \cap J)$ and, therefore, $Z(I) \cup Z(J) \subset Z(I \cap J)$. Suppose $t \notin Z(I) \cup Z(J)$. Then $\hat{f}(t)\hat{g}(t) \neq 0$ for some $f \in I$ and some $g \in J$. But then $f * g \in I \cap J$ and $(f * g)^\wedge (t) \neq 0$. This shows that $t \notin Z(I \cap J)$. Hence $Z(I \cap J) \subset Z(I) \cup Z(J)$, and so (i) holds.

Let $\varphi \in L^\infty(\mathsf{R})$ and put $I_\varphi = \{f \in L^1(\mathsf{R}) \mid f * \varphi \equiv 0\}$. I_φ is a closed ideal in $L^1(\mathsf{R})$ and its zero set is, by definition, the weak* spectrum $\sigma(\varphi)$ of φ, i.e.,

$$\sigma(\varphi) = \bigcap_{f \in I_\varphi} \{t \in \mathsf{R} \mid \hat{f}(t) = 0\}.$$

Let τ_y denote the translation operator, thus $\tau_y\varphi(x) = \varphi(x + y)$ for $x, y \in \mathsf{R}$. The (positive) limit set $\Gamma(\varphi)$ of φ is defined as follows:

$$\Gamma(\varphi) = \{\psi \in L^\infty(\mathsf{R}) \mid \text{ there exists a sequence } \alpha_k \to \infty \text{ such that } \tau_{\alpha_k}\varphi \to \psi \text{ weak* in } L^\infty(\mathsf{R})\}$$

Corollary 2.1 in Staffans [11] shows that the above definition of the limit set is equivalent to that given in [4] for functions in BUC(R), i.e., for the space of bounded and uniformly continuous functions on R. One defines the (positive) asymptotic spectrum $\sigma^{\infty}(\varphi)$ of a function φ in $L^{\infty}(\mathrm{R})$ by

$$\sigma^{\infty}(\varphi) = \overline{\bigcup_{\psi \in \Gamma(\varphi)} \sigma(\psi)}.$$

Equivalently, one may define $\sigma^{\infty}(\varphi)$ as the zero set of a closed ideal in $L^1(\mathrm{R})$. Put

$$A_{\varphi+} = \{f \in L^1(\mathrm{R}) | \lim_{x \to \infty} f * \varphi(x) = 0\}.$$

2.2 PROPOSITION. *Given $\varphi \in L^{\infty}(\mathrm{R})$, $A_{\varphi+}$ is a closed ideal in $L^1(\mathrm{R})$ whose zero set is the asymptotic spectrum of φ.*

As far as we know, Proposition 2.2 does not explicitly appear in the literature.

PROOF. Clearly, $A_{\varphi+}$ is a subspace of $L^1(\mathrm{R})$. Pick any $g \in L^1(\mathrm{R})$ and $f \in A_{\varphi+}$. Then by Levinson [7, Lemma 2.1], $\lim_{x \to \infty} g * f * \varphi(x) = 0$. This shows that $A_{\varphi+}$ is an ideal. If $f = \lim_{k \to \infty} f_k$ in $L^1(\mathrm{R})$, then $f * \varphi = \lim_{k \to \infty} f_k * \varphi$ uniformly on R. It follows that $A_{\varphi+}$ is closed.

Pick any $\psi \in \Gamma(\varphi)$ and $f \in A_{\varphi+}$. For some sequence $\alpha_k \to \infty$,

$$\begin{aligned} f * \psi(x) &= \lim_{k \to \infty} \int_{-\infty}^{\infty} f(x-y)\,\varphi(y+\alpha_k)dy \\ &= \lim_{k \to \infty} f * \varphi(x+\alpha_k) \equiv 0 \qquad (x \in \mathrm{R}). \end{aligned}$$

It follows that $\sigma(\psi) \subset Z(A_{\varphi+})$ and, therefore, $\sigma^{\infty}(\varphi) \subset Z(A_{\varphi+})$.

To prove the opposite inclusion, suppose that $t \in \mathrm{R} \setminus \sigma^{\infty}(\varphi)$. Let U be a neighborhood of t such that $U \cap \sigma(\psi) = \emptyset$ for every $\psi \in \Gamma(\varphi)$. Choose $f \in L^1(\mathrm{R})$ for which $\hat{f}(t) \neq 0$ and $\operatorname{supp}(\hat{f}) \subset U$. Let us show that $f \in A_{\varphi+}$. If this is not true, then one can find an $\varepsilon > 0$ and a sequence $\alpha_k \to \infty$ such that

$$|f * \varphi(\alpha_k)| = |f * (\tau_{\alpha_k}\varphi)(0)| > \varepsilon \quad (k \in \mathrm{N}).$$

But the weak* closure in $L^{\infty}(\mathrm{R})$ of the set $\{\tau_h \varphi \mid h \geqq 0\}$ is compact and metrizable in the induced weak* topology. See Staffans [11, Lemma 2.1]. Therefore, for some subsequence $\alpha_{k_j} \to \infty$ and a function ψ in $\Gamma(\varphi)$, $\lim_{j \to \infty} \tau_{\alpha_{k_j}} \varphi = \psi$ weak* in $L^{\infty}(\mathrm{R})$. In particular, $|f * \psi(0)| \geqq \varepsilon$. On the other hand, $\operatorname{supp}(\hat{f}) \cap \sigma(\psi) = \emptyset$ and, therefore, $f * \psi \equiv 0$. This contradiction shows that $f \in A_{\varphi+}$. But then $t \notin Z(A_{\varphi+})$, since $\hat{f}(t) \neq 0$. Hence $Z(A_{\varphi+}) \subset \sigma^{\infty}(\varphi)$ and consequently $\sigma^{\infty}(\varphi) = Z(A_{\varphi+})$.

Finally, given $\varphi \in L^\infty(\mathsf{R})$ consider the following subsets of $L^1(\mathsf{R})$:

$$J_{\varphi+} = \left\{ f \in L^1(\mathsf{R}) \mid \lim_{n\to\infty} \frac{1}{n} \int_0^n |f * \varphi(x)|\, dx = 0 \right\},$$

$$J_{\varphi-} = \left\{ f \in L^1(\mathsf{R}) \mid \lim_{n\to\infty} \frac{1}{n} \int_{-n}^0 |f * \varphi(x)|\, dx = 0 \right\},$$

$$J_{\varphi} = \left\{ f \in L^1(\mathsf{R}) \mid \lim_{n\to\infty} \frac{1}{2n} \int_{-n}^n |f * \varphi(x)|\, dx = 0 \right\}.$$

Clearly, $J_\varphi = J_{\varphi+} \cap J_{\varphi-}$. A straightforward calculation shows that $J_{\varphi+}, J_{\varphi-}$, and hence J_φ, are closed ideals in $L^1(\mathsf{R})$. One defines the positive Wiener spectrum $\sigma_{w_+}(\varphi)$ of φ as the zero set of J_{φ_+}. The negative Wiener spectrum $\sigma_{w+}(\varphi)$ of φ is defined to be the zero set of J_{φ_-}. Finally, the two-sided Wiener spectrum $\sigma_w(\varphi)$ of φ is, by definition, the zero set of J_φ.

Specializing Lemma 2.1 to $J_{\varphi+}$ and $J_{\varphi-}$ we get

$$\sigma_w(\varphi) = \sigma_{w+}(\varphi) \cup \sigma_{w-}(\varphi). \tag{2.1}$$

Moreover, from the obvious inclusions $I_\varphi \subset A_{\varphi+} \subset J_{\varphi+}$ and from Proposition 2.2, it follows that

$$\sigma_{w+}(\varphi) \subset \sigma^\infty(\varphi) \subset \sigma(\varphi). \tag{2.2}$$

Our final proposition in this section states that the one-sided Wiener spectrum is unaffected by the values that the underlying function takes on the other half axis. We let κ_s denote the characteristic function of S.

2.3 Proposition. *Let $\varphi \in L^\infty(\mathsf{R})$. Then the following assertions are true:*

(a) $\sigma_{w+}(\varphi) = \sigma_{w+}(\kappa_{\mathsf{R}^+}\varphi)$

(b) $\sigma_{w-}(\varphi) = \sigma_{w-}(\kappa_{\mathsf{R}^-}\varphi)$.

Proof. For every $f \in L^1(\mathsf{R})$ and $n \in \mathsf{N}$,

$$\int_0^n |f * (\kappa_{\mathsf{R}^+}\varphi)(x)|\, dx \leqq \int_0^n |f * (\kappa_{\mathsf{R}^-}\varphi)(x)|\, dx + \int_0^n |f * \varphi(x)|\, dx.$$

Moreover, $\lim_{x\to\infty} f * (\kappa_{\mathsf{R}^-}\varphi)(x) = 0$ and so $\lim_{n\to\infty} \frac{1}{n} \int_0^n |f * (\kappa_{\mathsf{R}^-}\varphi)(x)|\, dx = 0$. Suppose that $f \in J_{\varphi+}$. Then the preceding calculation shows that $f \in J_{(\kappa_{\mathsf{R}^+}\varphi)+}$. Hence $J_{\varphi+} \subset J_{(\kappa_{\mathsf{R}^+}\varphi)+}$. By a similar reasoning, we obtain the opposite inclusion. This proves (a). The proof of (b) is quite analogous.

In the next two sections we will mainly consider positive Wiener spectra. Most of the results that will be obtained hold with obvious modifications for the other Wiener spectra.

3. Spectral analysis for the Wiener spectrum.

Let $\varphi \in L^{\infty}(\mathsf{R})$. By the dual form of Wiener's Tauberian Theorem, $\sigma(\varphi)$ is empty if and only if $\varphi(x) = 0$ a.e. Obviously, this is not true for the positive Wiener spectrum. For example, any bounded function φ that is asympotically zero on the average, in the sense that $\lim_{n\to\infty} \frac{1}{n}\int_0^n |\varphi(x)|\,dx = 0$, has an empty positive Wiener spectrum. This follows from the fact that then $J_{\varphi+} = L^1(\mathsf{R})$. See Benedetto [1, Example 3.1]. But $\sigma_{w+}(\varphi)$ may be empty even if φ is not asymptotically zero on the average. As an example, consider $\varphi(x) = e^{ix^2}$ $(x \in \mathsf{R})$. For every $f \in L^1(\mathsf{R})$, $f * \varphi(x) = \varphi(x)(f\varphi)^{\wedge}(2x)$ and therefore, by the Riemann-Lebesgue Lemma, $f * \varphi$ vanishes at infinity. Thus even the asymptotic spectrum of φ is empty, although $\frac{1}{n}\int_0^n |\varphi(x)|\,dx = 1$ for every $n \in \mathsf{N}$. However, for uniformly continuous functions the sufficient condition stated above is also necessary:

3.1 THEOREM. *Let $\varphi \in \mathrm{BUC}(\mathsf{R})$. Then $\sigma_{w+}(\varphi)$ is empty if and only if*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{1}{n}\int_0^n |\varphi(x)|\,dx = 0.$$

PROOF. Let us assume that $\sigma_{w+}(\varphi) = \emptyset$. Since $J_{\varphi+}$ is a closed ideal in $L^1(\mathsf{R})$, it follows from Wiener's Tauberian Theorem that $J_{\varphi+} = L^1(\mathsf{R})$. Pick any $\varepsilon > 0$. By the uniform continuity of φ one can find a function $f \in J_{\varphi+}$ such that $\|\varphi - f * \varphi\|_{L^{\infty}(\mathsf{R})} < \varepsilon$. Then for every $n \in \mathsf{N}$

$$\frac{1}{n}\int_0^n |\varphi(x)|\,dx < \varepsilon + \frac{1}{n}\int_0^n |f * \varphi(x)|\,dx$$

Hence, $\limsup_{n\to\infty} \frac{1}{n}\int_0^n |\varphi(x)|\,dx \leqq \varepsilon$ and (i) follows.

3.2 COROLLARY. *Let $\varphi \in \mathrm{BUC}(\mathsf{R})$ and $f \in L^1(\mathsf{R})$. Suppose that $\hat{f}(t) \neq 0$ for every $t \in R$ and that $\lim_{n\to\infty} \frac{1}{n}\int_0^n |f * \varphi(x)|\,dx = 0$. Then φ satisfies the condition* (i) *in Theorem* 3.1.

Another corollary of Theorem 3.1 deals with bounded measures. Let $M(R)$ denote the space of bounded Borel measures on R. Recall that the Fourier-Stieltjes transform, $\hat{\mu}$, of a measure $\mu \in M(\mathsf{R})$ is defined by $\hat{\mu}(x) = \int_{-\infty}^{\infty} e^{-itx}\,d\mu(t)$ $(x \in \mathsf{R})$. It is shown in Hewitt and Stromberg [5, Example 3] that

$$\text{(3.1)} \qquad \lim_{n\to\infty} \frac{1}{n}\int_0^n |\hat{\mu}(x)|^2\,dx = \sum_{t\in\mathsf{R}} |\mu(\{t\})|^2.$$

If one combines (3.1) with Theorem 3.1, one gets the following result:

3.3 Corollary. *Let $\mu \in M(\mathrm{R})$. Then $\sigma_{w+}(\hat{\mu})$ is empty if and only if μ is continuous.*

Proof. By (3.1), μ is continuous if and only if $\lim_{n\to\infty} \frac{1}{n}\int_0^n |\hat{\mu}(x)|^2\,dx = 0$. On the other hand, $\lim_{n\to\infty} \frac{1}{n}\int_0^n |\hat{\mu}(x)|^2\,dx = 0$ if and, by the Schwarz inequality, only if $\lim_{n\to\infty} \frac{1}{n}\int_0^n |\hat{\mu}(x)|\,dx = 0$. An application of Theorem 3.1, justified by the uniform continuity of $\hat{\mu}$, then completes the proof.

It should be observed that Corollary 3.3 does not hold for the asymptotic spectrum. That is, there exists a continuous measure whose Fourier-Stieltjes transform has a nonempty asymptotic spectrum. To see this, take any continuous measure μ for which $\lim_{x\to\infty} \sup|\hat{\mu}(x)| > 0$ and then use Theorem 15.6.2(i) in [4].

The following theorem collects some elementary properties of the positive Wiener spectrum, all shared by both the weak* spectrum and the asymptotic spectrum.

3.4 Theorem. *Let $\varphi \in L^\infty(\mathrm{R})$, $\psi \in L^\infty(\mathrm{R})$, $f \in L^1(\mathrm{R})$, $a \in \mathrm{C}\setminus\{0\}$ and $y \in \mathrm{R}$. Then the following assertions are true:*

(a) *$\sigma_{w+}(\varphi)$ is closed.*
(b) $\sigma_{w+}(a\varphi) = \sigma_{w+}(\tau_y\varphi) = \sigma_{w+}(\varphi e^{iy\cdot}) - y = \sigma_{w+}(\varphi)$.
(c) $\sigma_{w+}(\varphi) \cap \{t \in \mathrm{R} \mid \hat{f}(t) \neq 0\} \subset \sigma_{w+}(f * \varphi)$.
(d) $\sigma_{w+}(f * \varphi) \subset \mathrm{supp}(\hat{f}) \cap \sigma_{w+}(\varphi)$.
(e) $\sigma_{w+}(\varphi + \psi) \subset \sigma_{w+}(\varphi) \cup \sigma_{w+}(\psi)$.
(f) *If $\sigma_{w+}(\varphi) \cap \sigma_{w+}(\psi) = \emptyset$, then $\sigma_{w+}(\varphi + \psi) = \sigma_{w+}(\varphi) \cup \sigma_{w+}(\psi)$.*
(g) *If $\sigma_{w+}(\varphi - \psi) = \emptyset$, then $\sigma_{w+}(\varphi) = \sigma_{w+}(\psi)$.*

Proof. The easy proofs of (a)–(f) are omitted, and (g) follows from (f).

3.5 Corollary. *Let $\varphi \in L^\infty(\mathrm{R})$, $\psi \in L^\infty(\mathrm{R})$ and suppose that $\lim_{n\to\infty} \frac{1}{n}\int_0^n |\varphi(x) - \psi(x)|\,dx = 0$. Then $\sigma_{w+}(\varphi) = \sigma_{w+}(\psi)$.*

Proof. As noted above, $J_{(\varphi-\psi)+} = L^1(\mathrm{R})$, or $\sigma_{w+}(\varphi - \psi) = \emptyset$. Now apply Theorem 3.4(g).

3.6 Corollary. *Let $\varphi \in L^\infty(\mathrm{R})$ and suppose that for some $a \in \mathrm{C}$*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{1}{n}\int_0^n |\varphi(x) - a|\,dx = 0.$$

If $a = 0$, then $\sigma_{w+}(\varphi) = \emptyset$, otherwise $\sigma_{w+}(\varphi) = \{0\}$.

In particular, if $\lim_{x\to\infty} \varphi(x)$ exists, then (i) holds with $a = \lim_{x\to\infty} \varphi(x)$.

PROOF. Define $\psi(x) \equiv a(x \in \mathsf{R})$. If $a = 0$ then $\sigma_{w+}(\psi) = \emptyset$, otherwise $\sigma_{w+}(\psi) = \{0\}$. Then use Corollary 3.5.

3.7 REMARK. It is well known that $\sigma(\varphi\psi) \subset \overline{\sigma(\varphi) + \sigma(\psi)}$ for every $\varphi, \psi \in L^\infty(\mathsf{R})$. However, this is not true for the positive Wiener spectrum. For example, take $\varphi(x) = \overline{\psi(x)} = e^{ix^2} (x \in \mathsf{R})$. Then $\varphi_{w+}(\varphi\psi) = \{0\}$, but $\overline{\sigma_{w+}(\varphi) + \sigma_{w+}(\psi)} = \emptyset$.

Let $\mathrm{BUC}^1(\mathsf{R}) = \{\varphi \in \mathrm{BUC}(\mathsf{R}) \mid \varphi$ is differentiable and $\varphi' \in \mathrm{BUC}(\mathsf{R})\}$. The following theorem relates the positive Wiener spectrum of a function in $\mathrm{BUC}^1(\mathsf{R})$ to that of its derivative. See also Theorems 15.4.18 and 15.6.3(v) in [4].

3.8 THEOREM. *Let $\varphi \in \mathrm{BUC}^1(\mathsf{R})$. Then $\sigma_{w+}(\varphi') \subset \sigma_{w+}(\varphi) \subset \sigma_{w+}(\varphi') \cup \{0\}$.*

PROOF. Let us first show that $J_{\varphi+} \subset J_{\varphi'+}$. Let $(\eta_k)_{k\in\mathsf{N}}$ be an approximate unit on R, that is, $\eta_k(x) = k\eta(kx)$ where η is a rapidly decreasing C^∞-function such that $\hat{\eta}(0) = 1$. Then

$$\varphi' = \lim_{k\to\infty} \eta_k * \varphi' = \lim_{k\to\infty} \eta'_k * \varphi,$$

uniformly on R. Therefore, $f * \varphi' = \lim_{k\to\infty} \eta'_k * f * \varphi$ uniformly on R, for every $f \in L^1(\mathsf{R})$. Now suppose that $f \in J_{\varphi+}$. Then for every $k \in \mathsf{N}$

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n |\eta'_k * f * \varphi(x)|\, dx = 0.$$

It follows that $\lim_{n\to\infty} \frac{1}{n} \int_0^n |f * \varphi'(x)|\, dx = 0$, or $f \in J_{\varphi'+}$. Hence $J_{\varphi+} \subset J_{\varphi'+}$ and consequently $\sigma_{w+}(\varphi') \subset \sigma_{w+}(\varphi)$.

To prove the second inclusion, suppose that $t \in \mathsf{R}\setminus(\sigma_{w+}(\varphi') \cup \{0\})$. Take a rapidly decreasing C^∞-function f such that $\hat{f}(t) \neq 0$ and $\operatorname{supp}(\hat{f}) \cap \sigma_{w+}(\varphi') = \emptyset$. It follows from Theorems 3.4(d) and 3.1 that $f \in J_{\varphi'+}$ or, equivalently, $f' \in J_{\varphi+}$. But $\widehat{f'}(t) = it\,\hat{f}(t) \neq 0$ and, therefore, $t \notin \sigma_{\varphi+}(\varphi)$. Hence $\sigma_{w+}(\varphi) \subset \sigma_{w+}(\varphi') \cup \{0\}$.

Recall that, by (2.2), $\sigma_{w+}(\varphi) \subset \sigma^\infty(\varphi) \subset \sigma(\varphi)$ for every $\varphi \in L^\infty(\mathsf{R})$. A partial converse is provided by the following proposition.

3.9 PROPOSITION. *Let $\varphi \in L^\infty(\mathsf{R})$ and suppose that t is an isolated point of $\sigma(\varphi)$. Then $t \in \sigma_{w+}(\varphi)$. In particular, $t \in \sigma^\infty(\varphi)$.*

Example 3.19(b) below shows that $\sigma^\infty(\varphi)$ may have isolated points that do not belong to $\sigma_{w+}(\varphi)$.

PROOF. It is well known that φ admits the unique decomposition $\varphi = ae^{it\cdot} + \psi$ where $a \in \mathbf{C} \setminus \{0\}$ and $t \notin \sigma(\psi)$. Apply Theorem 3.4(f) to conclude that $t \in \sigma_{w+}(\varphi)$.

Proposition 5.2(a) in Benedetto [1] shows that $\sigma(\varphi) = \sigma_w(\varphi)$ for every φ that is uniformly almost periodic. This is true for one-sided Wiener spectra as well:

3.10 PROPOSITION. *Let φ be uniformly almost periodic on* $\mathbf{R}$. *Then* $\sigma(\varphi) = \sigma_{w+}(\varphi)$.

PROOF. Suppose we have $J_{\varphi+} \subset I_\varphi$. Then $\sigma(\varphi) \subset \sigma_{w+}(\varphi)$ and hence the conclusion follows from (2.2).

Let us show that $J_{\varphi+} \subset I_\varphi$. Take any $f \in J_{\varphi+}$. Suppose that $\|f * \varphi\|_{L^\infty(\mathbf{R})} > 0$. Since closed ideals in $L^1(\mathbf{R})$ are translation invariant, we may assume that $|f * \varphi(0)| > 0$. On the other hand, $f * \varphi$ and therefore also $|f * \varphi|$ are uniformly almost periodic. Then the proof of Lemma VI.5.14 in Katznelson [6] shows that $\lim_{n\to\infty} \sup \frac{1}{n}\int_0^n |f * \varphi(x)|\,dx > 0$. But this is impossible, since $f \in J_{\varphi+}$. Thus $J_{\varphi+} \subset I_\varphi$.

3.11 COROLLARY. *Every closed subset of* $\mathbf{R}$ *is the positive Wiener spectrum of a bounded function.*

PROOF. Use Proposition 3.10 and the well-known fact that there exists a uniformly almost periodic function whose weak* spectrum coincides with any preassigned closed subset of $\mathbf{R}$.

3.12 COROLLARY. *Let φ be weakly almost periodic on* $\mathbf{R}$. *Then* $\sigma_{w+}(\varphi) = \sigma_{w-}(\varphi)$.

PROOF. Recall that φ admits the unique decomposition $\varphi = \varphi_1 + \varphi_2$ where φ_1 is uniformly almost periodic and $\lim_{n\to\infty} \frac{1}{2n}\int_{-n}^n |\varphi_2(x)|\,dx = 0$. See Eberlein [3, Theorem 1]. By Corollary 3.5 and Proposition 3.10 respectively,

$$\sigma_{w+}(\varphi) = \sigma_{w+}(\varphi_1) = \sigma(\varphi_1).$$

Another application of Proposition 3.10 and Corollary 3.5 (for the negative Wiener spectrum) yields

$$\sigma(\varphi_1) = \sigma_{w-}(\varphi_1) = \sigma_{w-}(\varphi).$$

Hence $\sigma_{w+}(\varphi) = \sigma_{w-}(\varphi)$.

Corollary 3.12 is not true for the asymptotic spectrum. In fact, there exists a nonnegative weakly almost periodic function φ such that $\lim_{x\to\infty} \sup \varphi(x) > 0$ and such that $\varphi(x) = 0$ for every $x < 0$. See the proof of Theorem 4.3.6 in Dunkl and Ramirez [2, pp. 44–45]. Hence the negative asymptotic spectrum of $\varphi, \sigma^{-\infty}(\varphi)$, defined e.g. as the zero set of the closed ideal $\{f \in L^1(\mathbf{R}) \mid \lim_{x\to-\infty} f * \varphi(x) = 0\}$, is empty. But, since φ is uniformly continuous and $\lim_{x\to\infty} \sup \varphi(x) > 0, \sigma^{\infty}(\varphi)$ is nonempty.

However, for Fourier-Stieltjes transforms (a subclass of weakly almost periodic functions) the analogue of Corollary 3.12 does hold: A classical theorem due to Rajchman says that, given $\mu \in M(\mathsf{R})$, $\lim_{x\to-\infty} \hat{\mu}(x) = 0$ if and only if $\lim_{x\to\infty} \hat{\mu}(x) = 0$. (The proof hinted at in Rudin [9, Exercise 6.7] extends without difficulty from $[0, 2\pi)$ to R.) On the other hand, for every $f \in L^1(\mathsf{R})$ $f * \hat{\mu}$ is also a Fourier-Stieltjes transform. Therefore, by Rajchman's theorem, $\lim_{x\to-\infty} f * \hat{\mu}(x) = 0$ if and only if $\lim_{x\to\infty} f * \hat{\mu}(x) = 0$. Thus we obtain the following:

3.13 THEOREM. *Let $\mu \in M(\mathsf{R})$. Then $\sigma^{-\infty}(\hat{\mu}) = \sigma^{\infty}(\hat{\mu})$.*

Note that, conversely, Rajchman's theorem can be deduced from Theorem 3.13. For example, if $\lim_{x\to\infty} \hat{\mu}(x) = 0$, then $\sigma^{\infty}(\hat{\mu}) = \emptyset$ and, therefore, $\sigma^{-\infty}(\hat{\mu}) = \emptyset$ also. From the uniform continuity of $\hat{\mu}$ it then follows that $\lim_{x\to-\infty} \hat{\mu}(x) = 0$.

We are now in a position to characterize the bounded and uniformly continuous functions whose positive Wiener spectra are "small" (e.g. finite). Our characterization in Theorem 3.16 below is very similar to that given in Gripenberg et al. [4, Theorem 15.6.2] for the asymptotic spectrum. In their characterization, asymptotically slowly varying and asymptotically drifting periodic functions appear. These are defined as follows. Let $\varphi \in \mathrm{BUC}(\mathsf{R})$. One says that φ is asymptotically slowly varying if for every $y \in \mathsf{R}$ $\lim_{x\to\infty} |(\tau_y\varphi - \varphi)(x)| = 0$. The function φ is called asymptotically drifting periodic with period p if $\lim_{x\to\infty} |(\tau_p\varphi - \varphi)(x)| = 0$. Let us take the average of $|\tau_y\varphi - \varphi|$ on R^+ and call φ asymptotically slowly varying on the average if

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n |(\tau_y\varphi - \varphi)(x)|\,dx \equiv 0 \quad (y \in \mathsf{R}). \tag{3.2}$$

Similarly, let us call φ asymptotically drifting periodic on the average with period p if

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n |(\tau_p\varphi - \varphi)(x)|dx = 0. \tag{3.3}$$

Obviously, every function that is asymptotically slowly varying is asymptotically slowly varying also on the average.

3.14 PROPOSITION. *Let* $\varphi \in \mathrm{BUC}^1(\mathsf{R})$. *Then* φ *is asymptotically slowly varying on the average if and only if*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{1}{n}\int_0^n |\varphi'(x)|dx = 0.$$

PROOF. Let us assume that φ' satisfies (i). For every $y \in \mathsf{R}$ and $n \in \mathsf{N}$

$$\frac{1}{n}\int_0^n |(\tau_y\varphi - \varphi)(x)|dx = \frac{1}{n}\int_0^n \left|\int_0^y \varphi'(u+x)du\right| dx$$
$$\leq \int_{\min(0,y)}^{\max(0,y)} \frac{1}{n}\int_0^n |\varphi'(u+x)|\, dxdu.$$

Therefore, it follows from the dominated convergence theorem and condition (i) that φ satisfies (3.2).

Conversely, suppose that φ satisfies (3.2). Define $f = \kappa_{(-\sqrt{2},0)} + \kappa_{(-1,0)}$. Then for every $x \in \mathsf{R}$

$$|f * \varphi'(x)| \leq |(\tau_{\sqrt{2}}\,\varphi - \varphi)(x)| + |(\tau_1\,\varphi - \varphi)(x)|,$$

so that $\lim_{n\to\infty} \frac{1}{n}\int_0^n |f * \varphi'(x)|dx = 0$. But it is easily seen that $\hat{f}(t) \neq 0$ for every $t \in \mathsf{R}$. Therefore, by Corollary 3.2, φ' satisfies condition (i).

3.15 COROLLARY. *Let* $\varphi \in \mathrm{BUC}(\mathsf{R})$ *and* $p \in \mathsf{R}$, *and define*

$$\psi(x) = \int_x^{x+p} \varphi(y)dy \quad (x \in \mathsf{R}).$$

Then ψ *is asymptotically slowly varying on the average if and only if* φ *is asymptotically drifting periodic on the average with period* p.

PROOF. $\psi \in \mathrm{BUC}^1(\mathsf{R})$ and $\psi' = \tau_p\varphi - \varphi$.

For the statements (a) – (c) of Theorem 3.16 below, recall that, by Theorem 3.1, $\sigma_{w^+}(\varphi) \neq \emptyset$ if and only if $\limsup_{n\to\infty} \frac{1}{n}\int_0^n |\varphi(x)|dx > 0$.

3.16 THEOREM. *Let* $\varphi \in \mathrm{BUC}(\mathsf{R})$. *Then the following is true:*

(a) $\sigma_{w^+}(\varphi) = \{0\}$ *if and only if* φ *is asymptotically slowly varying on the average and* $\sigma_{w^+}(\varphi) \neq \emptyset$.

(b) t_0 *is an isolated point of* $\sigma_{w^+}(\varphi)$ *if and only if* φ *is of the form* $\varphi = \theta\, e^{it_0\cdot} + \psi$, *where the function* θ *is asymptotically slowly varying on the average,* $\sigma_{w^+}(\theta) \neq \emptyset$ *and* $\sigma_{w^+}(\psi) = \sigma_{w^+}(\varphi)\backslash\{t_0\}$.

(c) $\sigma_{w^+}(\varphi) = \{t_1, \ldots, t_k\}$ *if and only if* φ *is of the form* $\varphi = \sum_{j=1}^{k} \theta_j e^{it_j\cdot}$, *where each*

function θ_j *is asymptotically slowly varying on the average and* $\sigma_{w+}(\theta_j) \neq \emptyset$.
(d) $\sigma_{w+}(\varphi) \subset \{kt_0 | k \in \mathsf{Z}\}$ *for some* $t_0 \neq 0$ *if and only if* φ *is asymptotically drifting periodic on the average with period* $2\pi/t_0$.

Combining Theorems 3.16(d) and 3.8 one obtains the following:

3.17 COROLLARY. *Let* $\varphi \in \mathrm{BUC}^1(\mathsf{R})$ *and let* $p \in \mathsf{R}$. *Then* φ *is asymptotically drifting periodic on the average with period* p *if and only if* φ' *is asymptotically drifting periodic on the average with period* p.

The proof of Theorem 3.16 is based on the following lemma:

3.18 LEMMA. *There exists a sequence* $(h_k)_{k \in \mathsf{N}}$ *in* $L^1(\mathsf{R})$ *such that for every* k, $\hat{h}_k \equiv 0$ *in a neighborhood of* 0, *and such that, given* $f \in L^1(\mathsf{R})$ *with* $\hat{f}(0) = 0$, $f = \lim_{k \to \infty} h_k * f$ *in* $L^1(\mathsf{R})$.

PROOF. Take any $h \in L^1(\mathsf{R})$ for which $\hat{h} \equiv 1$ in a neighborhood of 0. Define

$$h_k(x) = kh(kx) - \frac{1}{k} h\left(\frac{x}{k}\right) \quad (x \in \mathsf{R},\ k \in \mathsf{N}).$$

It is easily seen that each $\hat{h}_k$ vanishes in a neighborhood of 0. Also, for every $f \in L^1(\mathsf{R})$, f $= \lim_{k \to \infty} kh(k\cdot) * f$ in $L^1(\mathsf{R})$. Suppose that $\hat{f}(0) = 0$. Then, by Fubini's theorem,

$$\left\| \frac{1}{k} h\left(\frac{\cdot}{k}\right) * f \right\|_{L^1(\mathsf{R})} \leq \int_{-\infty}^{\infty} \frac{1}{k} \int_{-\infty}^{\infty} \left| h\left(\frac{x}{k} - \frac{y}{k}\right) - h\left(\frac{x}{k}\right) \right| |f(y)|\, dy dx$$
$$= \int_{-\infty}^{\infty} \|\tau_{-\frac{y}{k}} h - h\|_{L^1(\mathsf{R})} |f(y)|\, dy.$$

It follows that $\lim_{k \to \infty} \frac{1}{k} h\left(\frac{\cdot}{k}\right) * f = 0$ in $L^1(\mathsf{R})$. Therefore,

$$\|f - h_k * f\|_{L^1(\mathsf{R})} \leq \|f - kh(k\cdot) * f\|_{L^1(\mathsf{R})} + \left\| \frac{1}{k} h\left(\frac{\cdot}{k}\right) * f \right\|_{L^1(\mathsf{R})}$$
$$= o(1) \quad (k \to \infty).$$

PROOF OF THEOREM 3.16. (a) Let us assume that φ is asymptotically slowly varying on the average and that $\sigma_{w+}(\varphi) \neq \emptyset$. Choose an arbitrary $f \in L^1(\mathsf{R})$ with nonvanishing Fourier transform. It follows from the identity

$$f * (\tau_y \varphi - \varphi) = (\tau_y f - f) * \varphi \tag{3.4}$$

that $\tau_y f - f \in J_{\varphi+}$ for every $y \in \mathsf{R}$. Take any $t \in \sigma_{w+}(\varphi)$. Then

$$(\tau_y f - f)^\wedge(t) = \hat{f}(t)(e^{ity} - 1) \equiv 0 \quad (y \in \mathsf{R}).$$

Since $\hat{f}(t) \neq 0$, t must be 0. Thus $\sigma_{w+}(\varphi) = \{0\}$.

Conversely, suppose that $\sigma_{w+}(\varphi) = \{0\}$. Let f be as above. For every $y \in \mathsf{R}$ define $g_y = \tau_y f - f$. Then $\hat{g}_y(0) = 0$. Let $(h_k)_{k \in \mathsf{N}}$ be as in Lemma 3.18. Then, by Theorems 3.4(d) and 3.1,

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n |h_k * g_y * \varphi(x)|\, dx = 0 \quad (k \in \mathsf{N}). \tag{3.5}$$

But $g_y * \varphi = \lim_{k \to \infty} h_k * g_y * \varphi$ uniformly on R. Consequently, it follows from (3.5) and (3.4) that

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n |f * (\tau_y \varphi - \varphi)(x)|\, dx = 0.$$

Thus, by Corollary 3.2, φ is asymptotically slowly varying on the average.

(b) Let φ be of the given form. Then $\sigma_{w+}(\theta) = \{0\}$ and hence, by Theorem 3.4(b), $\sigma_{w+}(\theta e^{it_0 \cdot}) = \{t_0\}$. It follows from Theorem 3.4(f) that t_0 is an isolated point of $\sigma_{w+}(\varphi)$.

Assume that t_0 is an isolated point of $\sigma_{w+}(\varphi)$. One can find a function $f \in L^1(\mathsf{R})$ such that $\hat{f} \equiv 1$ in some neighborhood U of t_0 and $\operatorname{supp}(\hat{f}) \cap (\sigma_{w+}(\varphi) \setminus \{t_0\}) = \emptyset$. Define $\theta = (f * \varphi) e^{-it_0 \cdot}$ and $\psi = \varphi - \theta e^{it_0 \cdot}$. By Theorem 3.4(c), (d) and (b), $\sigma_{w+}(\theta) = \{0\}$. Therefore, θ is asymptotically slowly varying on the average. If $g \in L^1(\mathsf{R})$ and $\operatorname{supp}(\hat{g}) \subset U$, then $g = g * f$ in $L^1(\mathsf{R})$ and hence

$$g * \psi = g * (\varphi - f * \varphi) = (g - g * f) * \varphi \equiv 0.$$

Consequently, since we assume that $\hat{g}(t_0) \neq 0$, $t_0 \notin \sigma_{w+}(\psi)$. It follows that $\sigma_{w+}(\psi) = \sigma_{w+}(\varphi) \setminus \{t_0\}$.

(c) Apply (b) k times.

(d) Let φ be asymptotically drifting periodic on the average with period $2\pi/t_0$. Then the proof of the first part of (a), with an obvious modification, shows that $\sigma_{w+}(\varphi) \subset \{kt_0 \mid k \in \mathsf{Z}\}$.

Conversely, let $\sigma_{w+}(\varphi) \subset \{kt_0 \mid k \in \mathsf{Z}\}$. First, assume that $\sigma_{w+}(\varphi)$ is finite. By (c), φ is of the form $\varphi = \sum_{j=1}^{m} \theta_j e^{ik_j t_0 \cdot}$ where each k_j is an integer and each function θ_j is asymptotically slowly varying on the average. Then

$$\tau_{2\pi/t_0} \varphi - \varphi = \sum_{j=1}^{m} (\tau_{2\pi/t_0} \theta_j - \theta_j) e^{ik_j t_0 \cdot},$$

so that $\lim_{n\to\infty} \frac{1}{n} \int_0^n |(\tau_{2\pi/t_0}\varphi - \varphi)(x)|\,dx = 0$. Now suppose that $\sigma_{w+}(\varphi)$ is infinite. Pick any $\varepsilon > 0$. Choose $f \in L^1(\mathsf{R})$ such that $\|\varphi - f * \varphi\|_{L^\infty(\mathsf{R})} < \frac{\varepsilon}{2}$ and such that $\operatorname{supp}(\hat{f})$ is compact. Then $\sigma_{w+}(f * \varphi)$ is a finite subset of $\{kt_0 \mid k \in \mathsf{Z}\}$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n |(\tau_{2\pi/t_0}(f * \varphi) - f * \varphi)(x)|\,dx = 0. \tag{3.6}$$

Since

$$\begin{aligned} |(\tau_{2\pi/t_0}\varphi - \varphi)(x)| &\le |(\tau_{2\pi/t_0}\varphi - \tau_{2\pi/t_0}(f * \varphi))(x)| \\ &\quad + |(\tau_{2\pi/t_0}(f * \varphi) - f * \varphi)(x)| + |(f * \varphi - \varphi)(x)| \\ &< \varepsilon + |(\tau_{2\pi/t_0}(f * \varphi) - f * \varphi)(x)| \quad (x \in \mathsf{R}), \end{aligned}$$

it follows from (3.6) that $\limsup_{n\to\infty} \frac{1}{n} \int_0^n |(\tau_{2\pi/t_0}\varphi - \varphi)(x)|\,dx \le \varepsilon$. Thus φ is asymptotically drifting periodic on the average, with period $2\pi/t_0$.

This completes the proof.

3.19 EXAMPLES. (a) Let us give an example of a function that is asymptotically slowly varying (only) on the average. Let $\theta(x) = \kappa_{[-\frac{1}{2},\frac{1}{2}]}(x)\sin(2\pi x)$, and define $\varphi(x) = \int_0^x \sum_{k=1}^\infty \theta(u - 2^k)\,du$ $(x \in \mathsf{R})$. Then $\varphi \in \mathrm{BUC}^1(\mathsf{R})$ and

$$\begin{aligned} \int_0^n |\varphi'(x)|\,dx &\le \sum_{k=1}^{[^2\log n]} \int_{2^k - \frac{1}{2}}^{2^k + \frac{1}{2}} |\sin(2\pi x)|\,dx \\ &= \frac{2}{\pi}[^2\log n] = o(n) \quad (n \to \infty). \end{aligned}$$

It follows from Proposition 3.14 that φ is asymptotically slowly varying on the average. However, it is clear that φ is not asymptotically slowly varying. See also Theorem 15.3.3(i), (iv) in [4].

(b) Let us give an example of a function $\varphi \in \mathrm{BUC}(\mathsf{R})$ such that $\sigma_{w+}(\varphi) = \emptyset$ and $\sigma^\infty(\varphi) = \{0\}$. Thus, although 0 is an isolated point of $\sigma^\infty(\varphi)$, it does not belong to $\sigma_{w+}(\varphi)$ (cf. Proposition 3.9).

Let the function θ be as in (a) above. Define $\varphi(x) = \int_0^x \sum_{k=1}^\infty \frac{1}{k}\theta\left(\frac{u - 2^k}{k}\right)du$ $(x \in \mathsf{R})$. It is easily seen that $\varphi \in \mathrm{BUC}^1(\mathsf{R})$ and that $\lim_{x\to\infty} \varphi'(x) = 0$. Therefore, φ is asymptotically slowly varying. Also,

$$\limsup_{x\to\infty} |\varphi(x)| = |\varphi(2^k)| = \frac{1}{\pi} \quad (k \in \mathsf{N}).$$

Now apply Theorem 15.6.2(ii) in [4] to conclude that $\sigma^{\infty}(\varphi) = \{0\}$. On the other hand,

$$\int_0^n |\varphi(x)|\,dx < \sum_{k=1}^{(^2\log n]+1} \int_{2^k-\frac{k}{2}}^{2^k+\frac{k}{2}} |\varphi(x)|\,dx$$

$$< \frac{1}{\pi} \sum_{k=1}^{[^2\log n]+1} k = o(n) \quad (n \to \infty).$$

Thus, by Theorem 3.1, $\sigma_{w+}(\varphi) = \emptyset$.

Our final theorem in this section is the one-sided counterpart of Theorem 3.1 in Benedetto [1]. Given $\varphi \in L^{\infty}(\mathsf{R})$, put

$$\varphi_n = \kappa_{[o,n]}\,\varphi \quad (n \in \mathsf{N}).$$

3.20 Theorem. *Let $\varphi \in L^{\infty}(\mathsf{R})$. The positive Wiener spectrum of φ is the smallest closed subset E of R with the following property. If K is any compact subset of R such that $K \cap E = \emptyset$, then*

(i) $$\lim_{n\to\infty} \frac{1}{n} \int_K |\hat{\varphi}_n(t)|^2\,dt = 0.$$

The proof of Theorem 3.20 will be based on Lemma 3.21 below. The counterpart of Lemma 3.21 in the two-sided case is Proposition 3.2 in [1]. See also Lemma 1 in Meyer [8, p. 192].

3.21 Lemma. *Let $\varphi \in L^{\infty}(\mathsf{R})$ and $f \in L^1(\mathsf{R})$. Then*

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_n^{\infty} |f * \varphi_n(x)|\,dx = 0,$

(b) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_0^{n} |f * (\kappa_{\mathsf{R}^+}\,\varphi - \varphi_n)(x)|\,dx = 0,$

(c) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_{-\infty}^{0} |f * \varphi_n(x)|\,dx = 0,$

(d) $f \in J_{\varphi^+}$ if and only if $\displaystyle\lim_{n\to\infty} \frac{1}{n} \int_{-\infty}^{\infty} |f * \varphi_n(x)|^2\,dx = 0.$

Proof. (a) Take any $\varepsilon > 0$. Choose a function $g \in L^1(\mathsf{R})$ such that $\|f - g\|_{L^1(\mathsf{R})} < \|\varphi\|_{L^{\infty}(\mathsf{R})}^{-1}\varepsilon$ and such that $\operatorname{supp}(g) \subset [-a, a]$ for some $a > 0$. Then

$$\int_n^{\infty} |f * \varphi_n(x)|\,dx \leqq \int_n^{\infty} |(f - g) * \varphi_n(x)|\,dx + \int_n^{\infty} |g * \varphi_n(x)|\,dx$$

$$\leqq \|f - g\|_{L^1(\mathsf{R})} \|\varphi_n\|_{L^1(\mathsf{R})} + \int_n^{n+a} |g * \varphi_n(x)|\,dx$$

$$< n\varepsilon + a\,\|g\|_{L^1(\mathsf{R})}\,\|\varphi\|_{L^{\infty}(\mathsf{R})},$$

and, therefore, $\limsup_{n\to\infty} \frac{1}{n}\int_n^\infty |f * \varphi_n(x)|\,dx \leqq \varepsilon$, thus proving (a).

(b) Let $\tilde{f}(x) = f(-x)\,(x \in \mathrm{R})$. For every $n \in \mathrm{N}$,

$$\int_0^n |f * (\kappa_{\mathrm{R}^+}\varphi - \varphi_n)(x)|\,dx = \int_0^n \left|\int_n^\infty f(x-y)\varphi(y)\,dy\right|dx$$

$$\leqq \int_n^\infty |\varphi(y)| \int_0^n |\tilde{f}(y-x)|\,dxdy$$

$$\leqq \|\varphi\|_{L^\infty(\mathrm{R})} \int_n^\infty |\tilde{f}| * \kappa_{[0,n]}(y)\,dy.$$

Then take $\varphi(x) \equiv 1$ in (a), to obtain the desired conclusion.

(c) This is proved as assertion (a) above.

(d) By the proof of Proposition 2.3, $J_{\varphi+} = J_{(\kappa_{\mathrm{R}^+}\varphi)+}$. For every $n \in \mathrm{N}$

$$\int_0^n |f * (\kappa_{\mathrm{R}^+}\varphi)(x)|\,dx \leqq \int_0^n |f * (\kappa_{\mathrm{R}^+}\varphi - \varphi_n)(x)|\,dx + \int_0^n |f * \varphi_n(x)|\,dx$$

and

$$\frac{1}{n}\int_0^n |f * \varphi_n(x)|\,dx \leqq \left(\frac{1}{n}\int_{-\infty}^\infty |f * \varphi_n(x)|^2\,dx\right)^{1/2}.$$

Suppose that $\lim_{n\to\infty} \frac{1}{n}\int_{-\infty}^\infty |f * \varphi_n(x)|^2\,dx = 0$. Then the two inequalities above, together with assertion (b), show that $f \in J_{(\kappa_{\mathrm{R}^+}\varphi)+} = J_{\varphi+}$.

Conversely, let $f \in J_{(\kappa_{\mathrm{R}^+}\varphi)+}$. Then

$$(\|f\|_{L^1(\mathrm{R})}\|\varphi\|_{L^\infty(\mathrm{R})})^{-1}\int_{-\infty}^\infty |f * \varphi_n(x)|^2\,dx \leqq \int_{-\infty}^\infty |f * \varphi_n(x)|\,dx$$

$$\leqq \int_{-\infty}^0 |f * \varphi_n(x)|\,dx + \int_0^n |f * (\kappa_{\mathrm{R}^+}\varphi - \varphi_n)(x)|\,dx$$

$$+ \int_0^n |f * (\kappa_{\mathrm{R}^+}\varphi)(x)|\,dx + \int_n^\infty |f * \varphi_n(x)|\,dx = o(n) \quad (n \to \infty)$$

by (c), (b), the hypothesis and (a), respectively.

PROOF OF THEOREM 3.20. Take a compact $K \subset \mathrm{R}\setminus\sigma_{w+}(\varphi)$, $K \neq \emptyset$. Let us show that then (i) holds. Choose a function $f \in L^1(\mathrm{R})$ such that $\hat{f}(K) = \{1\}$ and such that $\operatorname{supp}(\hat{f}) \cap \sigma_{w+}(\varphi) = \emptyset$. Then $f \in J_{\varphi+}$, and therefore

$$\frac{1}{n}\int_K |\hat{\varphi}_n(t)|^2\,dt \leqq \frac{1}{n}\int_{-\infty}^\infty |\hat{f}(t)\,\hat{\varphi}_n(t)|^2\,dt$$

$$= \frac{2\pi}{n} \int_{-\infty}^{\infty} |f * \varphi_n(x)|^2 \, dx = o(1) \quad (n \to \infty)$$

by Plancherel's theorem and by Lemma 3.21(d), respectively.

Next, suppose that E is any closed subset of R with the property that (i) holds whenever K is compact and $K \cap E = \emptyset$. Let us show that $\sigma_{w+}(\varphi) \subset E$. Suppose this is not true, and pick any $t_0 \in \sigma_{w+}(\varphi) \setminus E$. Then choose a function $f \in L^1(\mathrm{R})$ for which $\hat{f}(t_0) \neq 0$, supp $(\hat{f})$ is compact and $\mathrm{supp}(\hat{f}) \cap E = \emptyset$. Since

$$\begin{aligned} \frac{1}{n} \int_{-\infty}^{\infty} |f * \varphi_n(x)|^2 dx &= \frac{1}{2\pi n} \int_{-\infty}^{\infty} |\hat{f}(t)\hat{\varphi}_n(t)|^2 \, dt \\ &\leq \frac{1}{2\pi} \|\hat{f}\|^2_{L^\infty(\mathrm{R})} \frac{1}{n} \int_{\mathrm{supp}(\hat{f})} |\hat{\varphi}_n(t)|^2 \, dt \\ &= o(1) \quad n \to \infty), \end{aligned}$$

Lemma 3.21(d) shows that $f \in J_{\varphi+}$. But this is impossible because $t_0 \in \sigma_{w+}(\varphi)$ and $\hat{f}(t_0) \neq 0$. Hence, $\sigma_{w+}(\varphi) \subset E$.

4. Synthesis at infinity.

Let $\varphi \in L^\infty(\mathrm{R})$. Recall that if $f \in L^1(\mathrm{R})$ and $\hat{f} \equiv 0$ in a neighborhood of $\sigma_{w+}(\varphi)$, then, by Theorems 3.4(d) and 3.1,

$$\lim_{n \to \infty} \frac{1}{n} \int_0^n |f * \varphi(x)| \, dx = 0. \tag{4.1}$$

In particular, if $\sigma_{w+}(\varphi) = \emptyset$, then (4.1) is true for every $f \in L^1(\mathrm{R})$. Similarly, if $f \in L^1(\mathrm{R})$ and $\hat{f} \equiv 0$ in a neighborhood of $\sigma^\infty(\varphi)$, then

$$\lim_{x \to \infty} f * \varphi(x) = 0. \tag{4.2}$$

One is thus led to consider whether (4.1) will hold, if one merely assumes that $\hat{f}$ vanishes on $\sigma_{w+}(\varphi)$, and whether (4.2) will hold, if it is only assumed that $\hat{f}$ vanishes on $\sigma^\infty(\varphi)$. Theorems 4.1 and 4.3 below provide answers to these synthesis problems at infinity.

4.1 THEOREM. *Let $\varphi \in L^\infty(\mathrm{R})$ and $f \in L^1(\mathrm{R})$. Suppose that $\hat{f}(\sigma_{w+}(\varphi)) = \{0\}$. Then* (4.1) *holds.*

The two-sided counterpart of Theorem 4.1 is contained in Benedetto [1, Theorem 4.1]. Although our Theorem 4.1 is an easy consequence of that of Benedetto, we will give a proof quite similar to that given in [1] for the two-sided case.

PROOF. Let us show that $\lim_{n\to\infty} \frac{1}{n}\int_{-\infty}^{\infty} |f * \varphi_n(x)|^2 dx = 0$, and then apply Lemma 3.21(d) to conclude that (4.1) is satisfied.

Take any $\varepsilon > 0$. There exist a compact $K \subset \mathsf{R}$ and an open $U \supset \sigma_{w+}(\varphi)$ such that

$$|\hat{f}(t)| < \sqrt{\frac{\varepsilon}{2}}\, \|\varphi\|_{L^\infty(\mathsf{R})}^{-1} \quad (t \in K^c \cup U)$$

(K^c is the complement of K in R). Then for every $n \in \mathsf{N}$

$$\text{(4.3)} \qquad \frac{1}{n}\int_{-\infty}^{\infty} |f * \varphi_n(x)|^2 dx = \frac{1}{2\pi n}\int_{-\infty}^{\infty} |\hat{f}(t)\,\hat{\varphi}_n(t)|^2\, dt$$

$$\leq \frac{\varepsilon}{2}\, \|\varphi\|_{L^\infty(\mathsf{R})}^{-2} \frac{1}{2\pi n}\int_{K^c\cup U} |\hat{\varphi}_n(t)|^2 dt + \frac{1}{2\pi n}\int_{K\cap U^c} |\hat{f}(t)\,\hat{\varphi}_n(t)|^2\, dt$$

and

$$\text{(4.4)} \qquad \frac{1}{2\pi n}\int_{K^c\cup U} |\hat{\varphi}_n(t)|^2 dt \leq \frac{1}{n}\int_{-\infty}^{\infty} |\varphi_n(x)|^2 dx$$

$$\leq \|\varphi\|_{L^\infty(\mathsf{R})}^2.$$

Furthermore, by Theorem 3.20, for every n large enough,

$$\text{(4.5)} \qquad \frac{1}{2\pi n}\int_{K\cap U^c} |\hat{f}(t)\hat{\varphi}_n(t)|^2\, dt < \frac{\varepsilon}{2}.$$

It then follows from (4.3), (4.4) and (4.5) that

$$\frac{1}{n}\int_{-\infty}^{\infty} |f * \varphi_n(x)|^2 dx < \varepsilon$$

for every n large enough. This completes the proof.

4.2 COROLLARY. *Let φ and f be as in Theorem* 4.1, *and assume that $f * \varphi$ is uniformly almost periodic on* R. *Then $f * \varphi \equiv 0$.*

PROOF. By Theorem 4.1, $f \in J_{\varphi+}$. Thus the conclusion follows from the proof of Proposition 3.10.

The second problem posed above is more difficult. In fact, there exists a bounded function that cannot be "synthesized' from its asymptotic spectrum:

4.3 THEOREM. *There exist a function $\varphi \in L^\infty(\mathsf{R})$ and a function $f \in L^1(\mathsf{R})$ for which $\hat{f}(\sigma^\infty(\varphi)) = \{0\}$, but $\limsup_{x\to\infty} |f * \varphi(x)| > 0$.*

However, note that, by (2.2) and Theorem 4.1, one has $\lim_{n\to\infty}\frac{1}{n}\int_0^n |f * \varphi(x)|dx = 0$.

The proof employs Malliavin's theorem, i.e., the fact that there exist a function $\psi \in L^\infty(\mathsf{R})$ and a function $f \in L^1(\mathsf{R})$ such that $\hat{f}(\sigma(\psi)) = \{0\}$, but $f * \psi(0) \neq 0$. Furthermore, we require that

$$\lim_{x\to\pm\infty} \psi(x) = 0. \tag{4.6}$$

For the fact that there exists such a nonsynthesizable ψ, see Theorem 1 in Sedig [10]. Now construct the function φ in Theorem 4.3 as follows. Let $(a_n)_{n\in\mathsf{N}}$ be a strictly increasing sequence of positive real numbers such that $\lim_{n\to\infty}(a_{n+1} - a_n) = \infty$. Let α_n denote the midpoint of $[a_n, a_{n+1}]$ $(n \in \mathsf{N})$. Define

$$\varphi(x) = \psi(x - \alpha_n),\ a_n < x \le a_{n+1}\ (n \in \mathsf{N})$$

and let $\varphi(x) = 0$ for $x \le a_1$.

4.4 LEMMA. *With the notation above,* $\sigma^\infty(\varphi) = \sigma(\psi)$.

PROOF. By the dominated convergence theorem, $\lim_{n\to\infty} \tau_{\alpha_n}\varphi = \psi$ weak* in $L^\infty(\mathsf{R})$. Thus $\psi \in \Gamma(\varphi)$, the limit set of φ, and, therefore, $\sigma(\psi) \subset \sigma^\infty(\varphi)$ by the definition of $\sigma^\infty(\varphi)$. To prove the inclusion in opposite direction, let us show that if $g \in L^1(\mathsf{R})$ and $g * \psi \equiv 0$, then $\lim_{x\to\infty} g * \varphi(x) = 0$. Then $I_\psi \subset A_{\varphi^+}$ and hence, by Proposition 2.2, $\sigma^\infty(\varphi) \subset \sigma(\psi)$.

Take any $\varepsilon > 0$. Choose $k_\varepsilon > 0$ for which $\int_{|x|>k_\varepsilon} |g(x)|\,dx < \frac{\varepsilon}{4}\|\psi\|^{-1}_{L^\infty(\mathsf{R})}$. Then for every $x \in \mathsf{R}$ and $n \in \mathsf{N}$

$$|g * \varphi(x)| < \frac{\varepsilon}{2} + \int_{-k_\varepsilon}^{k_\varepsilon} |g(y)|\,|\varphi(x-y) - \psi(x - y - \alpha_n)|\,dy. \tag{4.7}$$

By (4.6) one can find an $x_\varepsilon > 0$ such that $|\psi(x)| < \frac{\varepsilon}{4}\|g\|^{-1}_{L^1(\mathsf{R})}$ for every $|x| > x_\varepsilon$. Also, there exists an $n_\varepsilon \in \mathsf{N}$ which has the property that

$$a_n - a_{n-1} \ge 2(x_\varepsilon + k_\varepsilon) \quad (n \ge n_\varepsilon). \tag{4.8}$$

Suppose that $x \in \mathsf{R}$ and $x > a_{n_\varepsilon}$. Then

$$x \in (a_n, a_n + k_\varepsilon] \cup (a_n + k_\varepsilon, a_{n+1} - k_\varepsilon] \cup (a_{n+1} - k_\varepsilon, a_{n+1}]$$

for some $n \geq n_\varepsilon$. If $x \in (a_n, a_n + k_\varepsilon]$, then $\varphi(x-y) = \psi(x - y - \alpha_n)$ for every $y \in [-k_\varepsilon, x - a_n)$. Therefore, the integral in (4.7) is equal to

$$\int_{x-a_n}^{k_\varepsilon} |g(y)|\,|\varphi(x-y) - \psi(x-y-\alpha_n)|\,dy$$
$$= \int_{x-k_\varepsilon}^{a_n} |g(x-y)|\,|\psi(y-\alpha_{n-1}) - \psi(y-\alpha_n)|\,dy.$$

By (4.8), $y - \alpha_{n-1} > x_\varepsilon$ and $y - \alpha_n < -x_\varepsilon$ for every $y \in [x - k_\varepsilon, a_n]$. It follows that $|g * \varphi(x)| < \varepsilon$. By a similar reasoning, $|g * \varphi(x)| < \varepsilon$ if $x \in (a_{n+1} - k_\varepsilon, a_{n+1}]$. Finally, if $x \in (a_n + k_\varepsilon, a_{n+1} - k_\varepsilon]$, then $\varphi(x-y) = \psi(x-y-\alpha_n)$ for every $y \in [-k_\varepsilon, k_\varepsilon]$. Consequently, $|g * \varphi(x)| < \frac{\varepsilon}{2}$. Hence $\lim_{x\to\infty} g * \varphi(x) = 0$.

PROOF OF THEOREM 4.3. Let φ and f be the functions obtained above. Then, by Lemma 4.4, $\hat{f}(\sigma^\infty(\varphi)) = \{0\}$ and, by the dominated convergence theorem, $f * \psi(0) = \lim_{n\to\infty} f * \varphi(\alpha_n)$. But $f * \psi(0) \neq 0$ and hence $\limsup_{x\to\infty} |f * \varphi(x)| > 0$.

The proof suggests that there is a close relationship in spectral synthesis between the asymptotic spectrum and the weak* spectrum. In this regard, see Proposition 4.4 in Staffans [12].

ACKNOWLEDGEMENT. The author wishes to thank Docent O. J. Staffans for his kind interest and help.

REFERENCES

1. J. J. Benedetto, *The Wiener spectrum in spectral synthesis*, Stud. Appl. Math. LIV (1975), 91–115.
2. C. E. Dunkl and D. E. Ramirez, *Topics in Harmonic Analysis*, Appleton-Century Math. Ser., New York 1971.
3. W. F. Eberlein, *The point spectrum of weakly almost periodic functions*, Michigan Math. J. 3 (1955–1956), 137–139.
4. G. Gripenberg, S.-O. Londen and O. J. Staffans, *Volterra Integral and Functional Equations*, Cambridge Univ. Press, Cambridge, to appear.
5. E. Hewitt and K. R. Stromberg, *A remark on Fourier-Stieltjes transforms*, An. Acad. Brasil. Ciênc. 34 (1962), 175–180.
6. Y. Katznelson, *An Introduction to Harmonic Analysis*, Dover, New York 1976.
7. N. Levinson, *On the elementary character of Wiener's general Tauberian theorem*, J. Math. Anal. Appl. 42 (1973), 381–396.
8. Y. Meyer, *Le spectre de Wiener*, Studia Math. 27 (1966), 189–201.
9. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York 1974.
10. I. Sedig, *A note on Lipschitz functions and spectral synthesis*, Ark. Mat. 25 (1987), 141–145.
11. O. J. Staffans, *Tauberian theorems for a positive definite form, with applications to a Volterra equation*, Trans. Amer. Math. Soc. 218 (1976), 239–259.
12. O. J. Staffans, *On asymptotically almost periodic solutions of a convolution equation*, Trans. Amer. Math. Soc. 266 (1981), 603–616.

KOTKA INSTITUTE OF TECHNOLOGY
PÄÄSKYSENTIE 1
SF-48220 KOTKA
FINLAND