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THE C^*-ALGEBRA GENERATED BY TWO PROJECTIONS

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Problems concerning pairs of projections play a fundamental rôle in the theory of operator algebras. Here we shall give a unified treatment of some of these problems, in terms of the representation theory of a universal C^*-algebra generated by two projections. While we shall obtain some mild improvements on known results, our main intention is to show how a representation-theoretic viewpoint clarifies some of the issues involved.

We begin by showing the existence of a C^*-algebra $C^*(p,q)$ generated by projections p, q, with the property that whenever P, Q are projections on a Hilbert space H, there is a representation π of $C^*(p,q)$ on H with $\pi(p) = P$ and $\pi(q) = Q$. A theorem of Pedersen [12] shows that this algebra has a concrete realisation as an algebra of 2×2-matrix-valued functions on $[0,1]$, so its representation theory is well-understood. We provide a short proof of Pedersen's theorem using the Mackey machine, and make some comments on the analogous algebra generated by three projections.

Next we consider questions of unitary equivalence of projections P, Q in a von Neumann algebra M, and in particular, of how to find a unitary $U \in M$ satisfying $UPU^* = Q$ and minimising $\|1 - U\|$. This is easily solved in $C([0,1], M_2(\mathbf{C}))$, although not necessarily in the subalgebra $C^*(p,q)$, and transferring the solution to M involves analysing the corresponding representation of $C^*(p,q)$. The resulting estimates on $\|1 - U\|$ in terms of $\|P - Q\|$ are sharp, and appear to be slightly better than those previously known, even in the case $\|P - Q\| \leq \delta < 1$.

Our other main application concerns the problem of unitary equivalence of pairs of projections – given two pairs $\{P, Q\}$, $\{P', Q'\}$, when is there a unitary U such that $UPU^* = P'$ and $UQU^* = Q'$? Dixmier showed that, when the projections $\{P, Q\}$ are in generic position (see Remark 3.3), the single self-adjoint operator $P + Q$ is a complete unitary invariant of the pair $\{P, Q\}$; since self-adjoint operators had already been classified up to unitary equivalence, this essentially solved the problem [6, §VI]. From our point of view, the generic position hypothesis is a representation-theoretic one, and his theorem can easily

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be proved by examining the corresponding element $p + q$ of $C^*(p, q)$. Further, applying this approach to an operator $\lambda P + Q$, for any $\lambda \in (0, \infty)$ *except* $\lambda = 1$, leads to a version of Dixmier's result which does not require any extra hypothesis on the pair $\{P, Q\}$.

We wish to stress that, while we may have obtained some minor improvements of known results, we do not claim that any of our work is highly original. Indeed, many of these results have been rediscovered several times already: see the bibliographic note on p. 18 of [5]. In addition to those who have written on the subject, it appears that the main ideas were known to others, including von Neumann, Mackey and Kadison. It is possible, however, that our C^*-algebraic approach has a slightly different flavour, and we hope our methods may be of use elsewhere.

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§1. The C^*-algebra generated by two projections.

PROPOSITION 1.1. *There is a unital C^*-algebra A generated by two projections p, q with the following universal property: whenever P, Q are a pair of projections in a unital C^*-algebra B, there is a unital homomorphism $\phi : A \to B$ such that $\phi(p) = P$ and $\phi(q) = Q$. Indeed, if u and v are the canonical generators of $C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$, take $A = C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$, $p = (\mathbf{1} - u)/2, q = (\mathbf{1} - v)/2$. The triple (A, p, q) is unique up to isomorphism.*

PROOF. The uniqueness is clear, so we just have to establish that $C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$ has the required universal property. By representing B concretely on a Hilbert space H, we may suppose that P and Q are projections on H. Then $U = \mathbf{1} - 2P$, $V = \mathbf{1} - 2Q$ are unitaries of order 2, and hence each defines a unitary representation of $\mathbf{Z}_2$. The pair U, V therefore defines a unitary representation of the free product $\mathbf{Z}_2 * \mathbf{Z}_2$, and there is a representation ϕ of $C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$ on H such that $\phi(u) = U, \phi(v) = V$. Since u, v generate $C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$, the range of ϕ lies in B, and the result follows.

REMARK 1.2 Since the algebra A in the proposition is essentially unique, we shall call it *the* C^*-algebra generated by two projections, and denote it $C^*(p, q)$. Further, we shall usually identify $C^*(p, q)$ with the algebra of matrix-valued functions described in the next theorem, which is due to Pedersen [12].

THEOREM 1.3. *There is an isomorphism of the C^*-algebra $C^*(p, q)$ generated by two projections onto*

$$A = \{f \in C([0, 1], M_2(\mathbf{C})) : f(0), f(1) \text{ are diagonal}\},$$

which carries the generating projections into the functions

$$p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q(x) = \begin{pmatrix} x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1-x \end{pmatrix}.$$

REMARK 1.4 This result seems more reasonable if we write $x = \cos^2\theta$: then $q(x)$ is the orthogonal projection of $\mathbf{C}^2$ onto the span of the vector $(\cos\theta, \sin\theta)$.

PROOF. One can easily verify that the formulas do define projections p, q lying in A, and there is therefore a homomorphism ϕ of the universal C^*-algebra $C^*(p, q)$ into A which sends the generators into the functions p, q. It is also easy to verify that for $x \in (0, 1)$ the operators $p(x), q(x)$ generate $M_2(\mathbf{C})$, that for $x = 0$ they generate the diagonal algebra $\mathbf{C}^2$, and that for $x = 1$ they together with $\mathbf{1}$ generate $\mathbf{C}^2$. Thus the range of ϕ is a rich subalgebra of A, and hence all of A by [7, 11.1.6]. It remains to show that ϕ is injective, or, equivalently, that every irreducible representation of $C^*(p, q)$ factors through ϕ; we shall do this with a straightforward application of the Mackey machine (e.g. [10]).

Recall that there is an identification of $C^*(p, q)$ with $C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$ under which $u = 1 - 2p$, $v = 2q$ are the generators of $\mathbf{Z}_2 * \mathbf{Z}_2 \subset U\,C^*(\mathbf{Z}_2 * \mathbf{Z}_2)$. Now the product uv has infinite order in $\mathbf{Z}_2 * \mathbf{Z}_2$, and every element of $\mathbf{Z}_2 * \mathbf{Z}_2$ is either a power (possibly negative) of uv or a product $(uv)^n v$; since conjugation by v sends uv to $vu = (uv)^{-1}$, this means $\mathbf{Z}_2 * \mathbf{Z}_2$ is isomorphic to the semidirect product $\mathbf{Z} \rtimes \mathbf{Z}_2$, in which $\mathbf{Z}_2$ acts as multiplication by ± 1. Thus the irreducible representations of $C^*(p, q)$ are in one-one correspondence with the irreducible unitary representations of $G = \mathbf{Z} \rtimes \mathbf{Z}_2$. To compute these, we consider the normal subgroup $\mathbf{Z}$, and look at the action of $\mathbf{Z}_2$ on $\mathbf{T} = \hat{\mathbf{Z}}$ by conjugation; the orbits are parametrised by $\{\exp(i\theta): 0 \le \theta \le \pi\}$. For $0 < \theta < \pi$, the stabiliser is trivial, and there is one irreducible representation of G whose restriction to $\mathbf{Z}$ lives on the orbit $\exp(\pm i\theta)$, namely $\mathrm{Ind}_{\mathbf{Z}}^{G} \exp(i\theta)$, which is two-dimensional because $|G : \mathbf{Z}| = 2$. For $\theta = 0$ or 1, the stabiliser is all of $\mathbf{Z}_2$, no induction is necessary, and we obtain a total of four one-dimensional irreducible representations of G; since A also has four, these must all factor through ϕ. The two-dimensional representations of G are characterised by their restrictions to $\mathbf{Z} = \langle uv \rangle$, so all we have to do is compute the irreducible components of the representations which send uv to

$$\phi(uv)(x) = (\mathbf{1} - 2p(x))(\mathbf{1} - 2q(x)) = \begin{pmatrix} 2x - 1 & 2\sqrt{x(1-x)} \\ -2\sqrt{x(1-x)} & 2x - 1 \end{pmatrix}.$$

But this matrix has eigenvalues $(2x - 1) \pm 2i\sqrt{x(1-x)}$, which if we write $x = \cos^2\theta$ (see Remark 1.4) become $\exp(\pm 2i\theta)$, and every orbit $\exp(\pm i\phi)$ for $0 < \phi < \pi$ has this form for some $x \in (0, 1)$. Thus all the irreducible representations of G factor through ϕ, and the result follows.

REMARK 1.5 This result is well-known, and the above proof is presumably what Blackadar has in mind in [1, problem 6.10.4]; the original proof of Pedersen had an algebraic flavour. It has been extensively used in operator theory (see, for example, [8, 13, 14]), and operator-theoretic proofs have also been given – for example, by Power [13], based on ideas of Halmos [9]. While we certainly do not claim this proof is more elementary, it is completely routine to anyone familiar with the Mackey machine.

COROLLARY 1.6. (1) *The spectrum of $C^*(p, q)$ is homeomorphic to the quotient of two copies of $[0, 1]$ in which the corresponding points of $(0, 1)$ have been identified.*
(2) *If two projections P, Q generate $B(H)$, then the dimension of H is at most 2.*

REMARK 1.7 While our approach to these results is not particularly original, it does help explain why problems involving 3 or more projections are inherently more complicated – for example, 3 projections can act irreducibly on an infinite-dimensional space, in contrast to Corollary 1.6(2) (see [4]). Indeed, the argument of Proposition 1.1 shows that $C^*(\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2)$ is a universal C^*-algebra generated by 3 projections. But, whereas $\mathbf{Z}_2 * \mathbf{Z}_2 \cong \mathbf{Z} \rtimes \mathbf{Z}_2$ is easy to analyse, any other free product, such as $\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2$, is non-amenable, and its reduced C^*-algebra is simple and non-nuclear [11, Theorem 1.1]. Now the full group algebra certainly has lots of finite-dimensional representations (take any 3 subspaces of any space), but this observation shows that $C^*(\mathbf{Z}_2 * \mathbf{Z}_2 * \mathbf{Z}_2)$ is non-nuclear, and implies immediately that it has infinite-dimensional irreducible representations. The structure of families of 3 or more projections has recently been studied by Sunder [15].

The universal property of $C^*(p,q)$ implies that any problem involving two projections on a Hilbert space H is a problem concerning a specific representation of $C^*(p,q)$ on H. Since we know the spectrum of $C^*(p,q)$, and it is a type I algebra, we can always analyse such a representation using direct integral theory. However, in most cases this seems to be unnecessary, and we can make do with the following simple lemma.

LEMMA 1.8. *Let $I = \{f \in C([0,1], M_2(\mathbf{C})) : f(0) = f(1) = 0\}$, and let π be a representation of $C^*(p,q)$ on H. Then π has a direct sum decomposition $\pi = \pi_c \oplus \pi_0 \oplus \pi_1$, in which π_c is nondegenerate on the ideal I, π_0 factors through the map $f \to f(0)$, and π_1 factors through $f \to f(1)$. Further, we can identify the summands as follows:*

(1) *If $\{f_n\}$ is an approximate identity in I, then $\pi(f_n)$ converges strongly to the projection onto $H_c = H(\pi_c)$;*
(2) *H_0 is the direct sum of the subspaces $H_0^p = \pi_0(p)H$ and $H_0^q = \pi_0(q)H$, and*

$$H_0^p = \ker \pi(q) \cap \operatorname{range} \pi(p),\ H_0^q = \ker \pi(p) \cap \operatorname{range} \pi(q);$$

(3) *H_1 is the direct sum of two subspaces $H_1^p = \pi_1(p)H = \pi_1(q)H$ and $H_1^{1-p} = \pi_1(1-p)H = \pi_1(1-q)H$, and*

$$H_1^p = \operatorname{range}\pi(p) \cap \operatorname{range}\pi(q), \quad H_1^{1-p} = \ker\pi(p) \cap \ker\pi(q).$$

PROOF. Take $H_c = (\pi(I)H)^-$, which is reducing for π because I is an ideal, and note that (1) then holds. Since $\pi|_I = 0$ on $H_c^\perp$ and I is the kernel of the homomorphism $f \to f(0) \oplus f(1)$ from $C^*(p,q)$ to $\mathbf{C}^2 \oplus \mathbf{C}^2$, the action of π on $H_c^\perp$ factors through this homomorphism. The resulting representation of $\mathbf{C}^2 \oplus \mathbf{C}^2$ decomposes as a direct sum $H_0^p \oplus H_0^q \oplus H_1^p \oplus H_1^{1-p}$, in which the action of $\pi(f)$ is given by

$$f(0)_{11}\mathbf{1} \text{ on } H_0^p, f(0)_{22}\mathbf{1} \text{ on } H_0^q, f(1)_{11}\mathbf{1} \text{ on } H_1^p, f(1)_{22}\mathbf{1} \text{ on } H_1^{1-p}.$$

If we take $H_0 = H_0^p \oplus H_0^q$, $H_1 = H_1^p \oplus H_1^{1-p}$, then the main assertion follows easily, and an inspection of $p(0)$, $q(0)$ and $p(1) = q(1)$ reveals that $H_0^p = \pi_0(p)H$, etc. It therefore remains to verify the identifications of the various subspaces in terms of $\pi(p)$ and $\pi(q)$.

We show first that $H_0^p = \ker\pi(q) \cap \operatorname{range}\pi(p)$. Because all the subspaces in question reduce π, it should be clear that $H_0^p = \ker\pi_0(q) \cap \operatorname{range}\pi_0(p)$ is contained in $\ker\pi(q) \cap \operatorname{range}\pi(p)$, so we suppose $\pi(q)\xi = 0 = \pi(1-p)\xi$, and try to prove $\xi \in H_0$. Now $H_0^\perp$ is the subspace on which $J = \{f \in C^*(p,q): f(0) = 0\}$ acts nondegenerately, so it is enough for us to show that $f(0) = 0$ implies $\pi(f)\xi = 0$. Consider $b = q + (1-p)$, and observe that by hypothesis we have $\pi(b)\xi = 0$. We have $\det b(x) = x \neq 0$ on $(0,1]$, and we can therefore find a sequence $\{a_n\}$ in $C^*(p,q)$ such that $\|a_n b\| \le 1$ and $a_n b(x) = 1$ for $x \in [\frac{1}{n}, 1]$. But we then have $\|f - f a_n b\| \to 0$ for any f satisfying $f(0) = 0$, and hence

$$\pi(f)\xi = \lim \pi(f a_n b)\xi = \lim \pi(f a_n)\pi(b)\xi = 0$$

whenever $f(0) = 0$. Thus $\xi \in H_0$, and it follows easily that ξ belongs to $\ker\pi(q)$ and range $\pi(p)$. The identification of H_0^q can be proved similarly, replacing b by $(1-q) + p$, and so can (3), using instead of b the elements $(1-p) + (1-q)$ and $p + q$ of $C^*(p,q)$, which are invertible except at $x = 1$.

§2. Unitary equivalence of projections in a von Neumann algebra.

THEOREM 2.1. *Suppose P, Q are projections in a von Neumann algebra M, and that there is an element W of M such that WW^* is the projection onto* $\ker P \cap \operatorname{range} Q$ *and W^*W is the projection onto* $\ker Q \cap \operatorname{range} P$. *Then there is a unitary $U \in M$ such that*

(a) $UPU^* = Q$;

(b) *U commutes with $|P - Q|$;*

(c) $|1 - U| = \sqrt{2}(1 - (1 - |P - Q|^2)^{\frac{1}{2}})^{\frac{1}{2}} \le \sqrt{2}|P - Q|$.

PROOF. Suppose $M \subset B(H)$, and let π be the representation of $C^*(p,q)$ on H such that $\pi(p) = P$, $\pi(q) = Q$. The idea is to solve the problem in the C^*-algebra $C^*(p,q)$, and then apply π to get a solution on H. For each pair $(p(x), q(x))$, the problem is easily solved: take

$$u(x) = \begin{pmatrix} \sqrt{x} & -\sqrt{1-x} \\ \sqrt{1-x} & \sqrt{x} \end{pmatrix}$$

and easy calculations give

$$|p(x) - q(x)|^2 = (p(x) - q(x))^2 = (1 - x)\mathbf{1},$$

$$|\mathbf{1} - u(x)|^2 = (\mathbf{1} - u(x))(\mathbf{1} - u(x)^*) = 2\mathbf{1} - u(x) - u(x)^* = 2(1 - \sqrt{x})\mathbf{1}.$$

(To see where $u(x)$ comes from, set $x = \cos^2\theta$: then $q(x)$ is the projection on the span of $(\cos\theta, -\sin\theta)$, and $u(x)$ is the usual rotation matrix.) However, we cannot necessarily apply π to u, because the function u is *not* in $C^*(p,q)$ – the matrix $u(0)$ is not diagonal.

We therefore consider the ideal

$$J = \{f \in C^*(p,q) : f(0) = 0\}.$$

Pointwise multiplication by the function u defines a multiplier $u \in M(J)$, and this element of $M(J)$ has all the required properties relative to the projections $p, q \in M(J)$. By Lemma 1.8, we can decompose $H = H_c \oplus H_0$, $\pi = \pi_c \oplus \pi_o$, where π_c is nondegenerate on J and π_0 factors through $f \to f(0)$. (Strictly speaking, π_c here is the representation $\pi_c \oplus \pi_1$ of that lemma.) The representation π_c extends uniquely to $M(J)$, and $\pi_c(u)$ is a unitary element of $B(H_c)$ which has all the required properties relative to the projections $\pi_c(p)$, $\pi_c(q)$.

To complete the proof of the theorem we have to handle the summand H_0, and this is where the hypothesis on W comes in. Indeed, we claim that the operator $U_1 = W - W^*$ has the required properties relative to $\pi_0(p)$, $\pi_0(q)$: the equality $U_1\pi_0(p)U_1^* = \pi_0(q)$ holds because $\pi_0(p)$, $\pi_0(q)$ are the projections onto $\ker Q \cap \operatorname{range} P$, $\ker P \cap \operatorname{range} Q$ respectively (see Lemma 1.8), the second condition holds because $|\pi_0(p) - \pi_0(q)| = \mathbf{1}$ commutes with everything, and the third because

$$|\mathbf{1} - (W - W^*)|^2 = 2\mathbf{1} - (W - W^*) - (W^* - W) = 2\mathbf{1}.$$

Thus we can take $U = \pi_c(u) + U_1$, and (a), (b), (c) hold. To see that U lies in M, we just observe that if $f_n \to \mathbf{1}$ strictly in $M(J)$, then $\pi(f_n)$ converges strongly to the projection onto H_c, and therefore $\pi_c(u) = \lim \pi_c(u)\pi(f_n)$ belongs to $\pi(C^*(p,q))'' \subset M$. The last inequality holds because $(1 - (1 - t^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq t$ for all $t \in [0, 1]$.

COROLLARY 2.2. *Suppose P, Q are projections in a von Neumann algebra M satisfying $\|P - Q\| < 1$. Then there is a unitary $U \in M$ such that $UPU^* = Q$ and*

$$\|1 - U\| = \sqrt{2}(1 - (1 - \|P - Q\|^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq \sqrt{2}\,\|P - Q\|.$$

PROOF. For $\xi \in \ker P \cap \operatorname{range} Q$ we have $\|(P - Q)\xi\| = \|\xi\|$, and thus the condition $\|P - Q\| < 1$ implies

$$\ker P \cap \operatorname{range} Q = \ker Q \cap \operatorname{range} P = \{0\}.$$

Therefore the theorem applies with $W = 0$. To see that (c) implies the norm condition, just observe that $f(t) = (1 - (1 - t^2)^{\frac{1}{2}})^{\frac{1}{2}}$ is increasing on $[0, 1]$, and therefore $\|f(|S|)\| = f(\|S\|)$ for all operators S with $\|S\| \leq 1$.

REMARK 2.3 The constant $\sqrt{2}$ is the best possible for arbitrary P, Q, and this result is very well-known. However, if we know a priori that $\|P - Q\| \leq \delta < 1$, then we can replace $\sqrt{2}$ by $\sqrt{2}(1 - (1 - \delta^2)^{\frac{1}{2}})^{\frac{1}{2}}/\delta$. To see this, we just need to check that for $0 \leq t \leq \delta \leq 1$, we have

$$(1 - (1 - t^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq (1 - (1 - \delta^2)^{\frac{1}{2}})^{\frac{1}{2}}\, t/\delta,$$

and this can be done by elementary algebra.

We shall now show that this constant is the best possible. Fix $\delta < 1$, and consider the pair $p(x), q(x)$ where $\delta = \sqrt{1 - x}$. Then $\|p(x) - q(x)\| = \delta$, and if $v \in U_2(\mathbf{C})$ satisfies $vp(x) = q(x)v$, then v must have the form

$$\begin{pmatrix} \lambda\sqrt{x} & -\mu\sqrt{1 - x} \\ \lambda\sqrt{1 - x} & \mu\sqrt{x} \end{pmatrix}$$

for some $\lambda, \mu \in \mathbf{C}$ with $|\lambda| = |\mu| = 1$; conversely, any such v satisfies $vp(x)v^* = q(x)$. The norm of $1 - v$ is the square root of the larger eigenvalue of $(1 - v^*)(1 - v) = 21 - v - v^*$, which is at least as large as

$$\begin{aligned} \tfrac{1}{2}\operatorname{tr}(21 - v - v^*) &= \tfrac{1}{2}\operatorname{tr}\begin{pmatrix} 2 - 2\sqrt{x}\operatorname{Re}\lambda & \sqrt{1 - x}(\mu - \bar{\lambda}) \\ \sqrt{1 - x}(\bar{\mu} - \lambda) & 2 - 2\sqrt{x}\operatorname{Re}\mu \end{pmatrix} \\ &= 2 - \sqrt{x}(\operatorname{Re}\lambda + \operatorname{Re}\mu) \\ &\geq 2 - 2\sqrt{x}. \end{aligned}$$

Thus

$$\|1 - v\| \geq \sqrt{2}\sqrt{1 - \sqrt{x}} = \sqrt{2}(1 - (1 - \delta^2)^{\frac{1}{2}})^{\frac{1}{2}},$$

with equality occurring when $\operatorname{Re}\lambda = \operatorname{Re}\mu = 1$, that is, when v is the operator $u(x)$ considered in the proof of the theorem. (This result was first proved by Davis and

Kahan [5, Proposition 4.3], but this proof seems easier.) Letting $\delta \to 1$ shows that $\sqrt{2}$ is best in general.

COROLLARY 2.4. *Suppose P, Q are two finite projections in a von Neumann algebra M. Then P, Q are equivalent in the sense of Murray and von Neumann (i.e. there exists $T \in M$ such that $TT^* = P$, $T^*T = Q$) if and only if there exists $W \in M$ such that WW^*, W^*W are the projections onto* $\ker P \cap \operatorname{range} Q$, $\ker Q \cap \operatorname{range} P$ *respectively. If so, there is a unitary $U \in M$ such that $UPU^* = Q$, $U|P-Q| = |P-Q|U$ and*

$$|1 - U| = \sqrt{2}(1 - (1 - |P - Q|^2)^{\frac{1}{2}})^{\frac{1}{2}} \leq \sqrt{2}|P - Q|.$$

PROOF. Let $\pi: C^*(p,q) \to B(H)$ be the representation with $\pi(p) = P$, $\pi(q) = Q$, and decompose $\pi = \pi_c \oplus \pi_0$ as in the proof of the theorem. As we saw there, the unitary u satisfies $\pi_c(upu^*) = \pi_c(q)$, so $\pi_c(p)$ is always equivalent to $\pi_c(q)$. Because P, Q are finite, this implies that P is equivalent to Q if and only if $\pi_0(p) = P - \pi_c(p)$ is equivalent to $\pi_0(q) = Q - \pi_c(q)$. But by Lemma 1.8

$$\operatorname{range} \pi_0(p) = \ker Q \cap \operatorname{range} P, \quad \operatorname{range} \pi_0(q) = \ker P \cap \operatorname{range} Q,$$

so $\pi_0(p)$ equivalent to $\pi_0(q)$ means precisely that there exists $W \in M$ as claimed. The result now follows immediately from the theorem.

COROLLARY 2.5. *Suppose that P, Q are equivalent finite projections in a von Neumann algebra M, or, more generally, that P, Q are projections in M and there exists $W \in M$ such that WW^*, W^*W are the projections onto* $\ker P \cap \operatorname{range} Q$, $\ker Q \cap \operatorname{range} P$ *respectively. Then there is an element V of M such that*

(a) $VV^* = Q$, $V^*V = P$;

(b) $V|P - Q| = |P - Q|V$;

(c) $|P - V| \leq \sqrt{2}|P - Q|$, $|Q - V| \leq \sqrt{2}|P - Q|$.

PROOF. By the previous corollary, the first hypothesis implies the existence of the partial isometry W. Now let $\pi = \pi_c \oplus \pi_0$ be as in the proof of the theorem, and consider

$$v(x) = \begin{pmatrix} \sqrt{x} & 0 \\ \sqrt{1-x} & 0 \end{pmatrix} \in M(J).$$

This has all the right properties relative to $p, q \in M(J)$, and much as before $V = \pi_c(v) + W$ is an element of M satisfying (a), (b), (c).

REMARK 2.6 Corollary 2.4 is a very minor improvement on [3, Lemma 1.4], in which the last inequality has 3 in place of $\sqrt{2}$, and Corollary 2.5 is slightly sharper than [2, Lemma 2.2], which is there deduced from [3, Lemma 1.4].

We finish this section by making some comments on the necessity of the hypothesis in Theorem 2.1 concerning the existence of the partial isometry W. If there is a unitary U satisfying $UPU^* = Q$, and if P, Q are finite, then there is such a W (Corollary 2.4), but for arbitrary P, Q this is not the case. For example, consider $H = L^2(\mathbf{Z})$ with the usual basis $\{e_n\}$, and let P, Q be the projections onto $\mathrm{sp}\{e_n : n \geq 0\}$, $\mathrm{sp}\{e_n : n \geq 1\}$: then the bilateral shift U satisfies $UPU^* = Q$, but $\ker Q \cap \operatorname{range} P = \mathrm{sp}\{e_0\}$ is not equivalent to $\ker P \cap \operatorname{range} Q = \{0\}$. However, the hypothesis is always necessary for the existence of a unitary U satisfying $UPU^* = Q$ *and* $U|P - Q| = |P - Q|U$:

Proposition 2.7. *Suppose P, Q are projections in a von Neumann algebra M, and there is a unitary $U \in M$ such that $UPU^* = Q$ and U commutes with $|P - Q|$. Then there exists $W \in M$ such that WW^*, W^*W are the projections onto* $\ker Q \cap \operatorname{range} P$, $\ker P \cap \operatorname{range} Q$ *respectively.*

Proof. We first observe that since U commutes with $|P - Q|$, it commutes with all its spectral projections, and hence in particular with the projection onto the eigenspace $\{\xi \in H : |P - Q|\xi = \xi\}$. By standard spectral theory we can see that, because $P - Q$ is self-adjoint, this is the span of the ± 1 eigenspaces of $P - Q$. Now

$$(P - Q)\xi = \xi \Rightarrow Q\xi = -(\mathbf{1} - P)\xi \Rightarrow (Q\xi|\xi) = -((\mathbf{1} - P)\xi|\xi),$$

and this implies $Q\xi = (\mathbf{1} - P)\xi = 0$, so $\xi \in \ker Q \cap \operatorname{range} P$; similarly, $(P - Q)\xi = -\xi$ implies $\xi \in \ker P \cap \operatorname{range} Q$. Let $\pi = \pi_c \oplus \pi_0$ be as in the proof of the theorem; then we have just shown that U commutes with the projection onto

$$(\ker Q \cap \operatorname{range} P) \oplus (\ker P \cap \operatorname{range} Q) = H_0 = \pi_0(p)H \oplus \pi_0(q)H.$$

As before, the projection onto H_c belongs to M, and therefore compressing U to the subspace H_0 gives a unitary $U_0 \in M$ such that $U_0\pi_0(p)U_0^* = \pi_0(q)$. Then $W = \pi_0(q)U_0\pi_0(p)$ has the required properties.

§3. Unitary equivalence of pairs of projections.

We now discuss our version of Dixmier's theorem. We shall show that for any $\lambda \in (0, \infty)$ except $\lambda = 1$, the operator $\lambda P + Q$ determines the pair of projections $\{P, Q\}$ up to unitary equivalence. The case $0 < \lambda < 1$ can be reduced to the case where $\lambda > 1$ by swapping P and Q, so we shall assume $\lambda > 1$.

Theorem 3.1. *Let $\{P, Q\}$ and $\{P', Q'\}$ be two pairs of projections on a Hilbert space, and suppose $\lambda > 1$. Then there is a unitary operator U such that $UPU^* = P'$ and $UQU^* = Q'$ if and only if the positive operator $\lambda P + Q$ is unitarily equivalent to $\lambda P' + Q'$.*

Proof. The forward implication is clear, so we shall suppose $\lambda P + Q$ is unitarily equivalent to $\lambda P' + Q'$. If π, ρ are the representations of $C^*(p, q)$ carrying the generators to $\{P, Q\}$, $\{P', Q'\}$ respectively, and $s = \lambda p + q$, then we are given that $\pi|_{C^*(s)}$ is equivalent to $\rho|_{C^*(s)}$, and have to show that π is equivalent to ρ. We may suppose without loss of generality that $\pi|_{C^*(s)} = \rho|_{C^*(s)}$. Since s is self-adjoint, the Gelfand transform induces an isomorphism of $C^*(s)$ onto $C(\sigma(s))$, so we begin by computing $\sigma(s)$. A function $f \in C^*(p, q)$ is invertible if and only if $f(x)$ is invertible for all x, and thus for

$$s(x) = \begin{pmatrix} \lambda + x & \sqrt{x(1-x)} \\ \sqrt{x(1-x)} & 1 - x \end{pmatrix}$$

we have

$$\sigma(s) = \bigcup_{x \in [0,1]} \sigma(s(x)) = \bigcup_{x \in [0,1]} \tfrac{1}{2}(1 + \lambda \pm \sqrt{(\lambda - 1)^2 + 4\lambda x}) = [0, 1] \cup [\lambda, \lambda + 1].$$

(Notice that the effect of choosing $\lambda > 1$ rather than $\lambda = 1$ is to split $\sigma(s)$ into two disjoint intervals.) Now for any representation π of a C^*-algebra A, $a = a^* \in A$ and $g \in C(\sigma(a))$, we have $\pi(g(a)) = g|_{\sigma(\pi(a))}(\pi(a))$; thus for any $g \in C(\sigma(s))$ we have

$$g(s)(x) = g|_{(1 + \lambda \pm \sqrt{(\lambda - 1)^2 + 4\lambda x})/2}(s(x)). \tag{3.1}$$

We now let $I = \{f \in C^*(p, q) : f(0) = f(1) = 0\}$, and decompose $\pi = \pi_c \oplus \pi_0 \oplus \pi_1$, $\rho = \rho_c \oplus \rho_0 \oplus \rho_1$ as in Lemma 2.8. We claim that these decompositions are compatible in the sense that $\mathscr{H}(\pi_j) = \mathscr{H}(\rho_j)$ for $j = c, 0, 1$. For suppose $f_n \in C_0((0, 1) \cup (\lambda, \lambda + 1))$ is an increasing sequence of positive functions such that $f_n \equiv 1$ on

$$\{(1 + \lambda \pm \sqrt{(\lambda - 1)^2 + 4\lambda x})/2 : x \in [\tfrac{1}{n}, 1 - \tfrac{1}{n}]\} \subset \sigma(s).$$

Then for $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$, (3.1) implies $f_k(s)(x) = 1$ for $k \geq n$, and hence $f_n(s) \to \mathbf{1}$ strictly in $M(I)$. Thus $\pi(f_n(s))$ converges strongly to the projection onto $\mathscr{H}(\pi_c)$, and similarly for $\rho(f_n(s))$; since $f_n(s) \in C^*(s)$ and $\pi = \rho$ on $C^*(s)$, the projections onto $\mathscr{H}(\pi_c)$ and $\mathscr{H}(\rho_c)$ must therefore coincide, i.e. $\mathscr{H}(\pi_c) = \mathscr{H}(\rho_c)$. Next we observe that for $j = 0, 1$, the matrices $s(j)$ and $\mathbf{1}$ generate the diagonal subalgebra of $M_2(\mathbf{C})$, and hence the quotient maps $f \to f(j)$ are surjective on $C^*(s)$. Since the representations π_j, ρ_j factor through these quotient maps, and $\pi = \rho$ on $C^*(s)$, we actually have $\pi_0 = \rho_0$ and $\pi_1 = \rho_1$. Thus the decompositions are compatible, as claimed, and it remains for us to show that π_c is unitarily equivalent to ρ_c. We shall do this by diagonalising s in $C([0, 1], M_2) \subset M(I)$ using the following lemma:

LEMMA 3.2. *Suppose $f \in C([0, 1], M_2)$ is self-adjoint and there is a continuous map $v : [0, 1] \to \mathbf{C}^2$ such that $v(x)$ is a unit eigenvector for $f(x)$. Let $p_1(x)$ denote the orthogonal projection onto $\mathbf{C}v(x)$. Then there is a continuous function $w : [0, 1] \to M_2$ such that $w(x)^*w(x) = p_1(x)$, $w(x)w(x)^* = \mathbf{1} - p_1(x)$, and we can* *write any* $g \in C([0,1], M_2)$ *in the form*

$$g = ap_1 + bw^* + cw + d(1 - p_1) \tag{3.2}$$

for some $a, b, c, d \in C([0,1])$.

PROOF. If $v(x) = (\alpha(x), \beta(x))$, then, since f is self-adjoint, $(-\overline{\beta(x)}, \overline{\alpha(x)})$ is also an eigenvector for $f(x)$. Thus if

$$u(x) = \begin{pmatrix} \alpha(x) & -\overline{\beta(x)} \\ \beta(x) & \overline{\alpha(x)} \end{pmatrix}$$

then $(u^*fu)(x)$ is diagonal, $p_1(x) = u(x)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}u(x)^*$ is the projection onto $\mathbf{C}v(x)$ and $w(x) = u(x)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}u(x)^*$ is a partial isometry intertwining $p_1(x)$ and $1 - p_1(x)$. Notice that we have written down formulas for $w(x)$ and $p_1(x)$ which are clearly continuous in x, and the last part follows by writing u^*gu as a matrix in $M_2(C([0,1]))$.

PROOF OF THEOREM 3.1 (CONTINUED). We want to apply this lemma to our function $s(x)$. Messy calculations show that, at least for $x \neq 0$ or 1,

$$v_1(x) = (-2\sqrt{x(1-x)},\ 2x + \lambda - 1 - \sqrt{(\lambda - 1)^2 + 4\lambda x})$$

is an eigenvector for $s(x)$ with eigenvalue $(1 + \lambda + \sqrt{(\lambda - 1)^2 + 4\lambda x})/2$, and

$$\|v_1(x)\|^2 = 2(\lambda - 1)^2 + 8\lambda x - 2(2x + \lambda - 1)\sqrt{(\lambda - 1)^2 + 4\lambda x}.$$

It follows from a few applications of L'Hôpital's rule that the corresponding unit eigenvector

$$v(x) = \|v_1(x)\|^{-1} v_1(x) \to (1, 0) \text{ as } x \to 0 \text{ or } 1,$$

which is an eigenvector for $s(0)$ and $s(1)$. Thus we can extend v to a continuous function on $[0,1]$ and apply the lemma to it: let p_1, w be the continuous functions we obtain. Observe that $p_1 = \chi_{[\lambda, \lambda+1]}(s)$ belongs to $C^*(s)$, and hence $\pi_c(p_1) = \rho_c(p_1) = P_1$, say. Let $V = \pi_c(w)$, $W = \rho_c(w)$ and define $U = W^*V + (1 - P_1)$; note that U is unitary and belongs to $C([0,1], M_2) \subset C_b((0,1), M_2) = M(I)$. Thus if $g \in C^*(p, q)$, the decomposition (3.2) is valid in $M(I)$, and we have

$$\begin{aligned} U\pi_c(g) &= U(aP_1 + bV^* + cV + d(1 - P_1)) \\ &= aW^*VP_1 + bW^*VV^* + c(1 - P_1)V + d(1 - P_1) \\ &= aP_1W^*V + bW^*(1 - P_1) + cWW^*V + d(1 - P_1) \\ &= (aP_1 + bW^* + cW + d(1 - P_1))U \\ &= \rho_c(g)U. \end{aligned}$$

Thus π_c is equivalent to ρ_c, as required.

REMARK 3.3. The key idea here is that an irreducible representation of $C^*(p, q)$ is determined up to equivalence by its restriction to $C^*(s)$. Dixmier's version of this result [6, §VI] used the operator $a = p + q$, but the subalgebra $C^*(a)$ does not distinguish between the irreducible components of the representation $f \to f(0)$, and he therefore needed extra assumptions. In fact he insisted that his projections P, Q be "in position p" – namely, that

$$\ker P \cap \ker(\mathbf{1} - Q) = 0 = \ker Q \cap \ker(\mathbf{1} - P) \tag{3.3}$$

$$\ker P \cap \ker Q = 0 = \ker(\mathbf{1} - Q) \cap \ker(\mathbf{1} - P) \tag{3.4}$$

In the notation of our proof, (3.3) says the summand π_0 does not appear, and (3.4) that π_1 does not appear (see Lemma 1.8); however, since $a(1)$ does generate the diagonal subalgebra of M_2, it seems that Dixmier's result is still valid without assumption (3.4). In his discussion of Dixmier's theorem, Halmos [9] also assumes (3.3) and (3.4) (he says "P and Q are in generic position"), and Sunder [15] has recently given a similar result, using a different operator, which only requires (3.4) and half of (3.3). (However, Sunder also discusses the analogous problem for larger families of projections).

ADDED IN PROOF. Bill Longstaff has pointed out to us that H. Behncke (Tohoku Math. J. 23 (1971), 349–352) had previously used the representation theory of $\mathbf{Z}_2 * \mathbf{Z}_2 \cong \mathbf{Z} \times \mathbf{Z}_2$ to study pairs of projections; however, he used direct integral theory rather than our C*-algebraic methods, and his applications were quite different from those in § 2 and § 3. In an earlier paper (Tohoku Math. J. 22 (1970), 181–183), he had also made remarks similar to our 1.7.

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