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INFINITE TENSOR PRODUCTS OF UPPER TRIANGULAR MATRIX ALGEBRAS

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Let $n \geqq 2$ be an integer and let $T(n)$ be the algebra of $n \times n$ complex matrices which have zero entries below the main diagonal. Under the operator norm $T(n)$ is a Banach algebra, and for a sequence (n_k) of such integers there is a natural way to associate a unital Banach algebra

$$T((n_k)) = T(n_1) \otimes T(n_2) \otimes \ldots$$

which is an infinite tensor product in the sense of inductive limits.

In what follows we determine the group Aut $T((n_k))$ of Banach algebra automorphisms of $T((n_k))$. The quotient group Out $T(n_k))$, obtained from the normal subgroup of pointwise inner automorphisms, turns out to be the discrete group of permutations π such that $n_k = n_{\pi(k)}$, $k = 1, 2, \ldots$. Thus, up to composition by pointwise inner automorphisms the set of outer automorphisms may be uncountable, finite, or even trivial. In fact we describe all isomorphisms and epimorphisms between these Banach algebras.

We also determine the structure of the complete lattice Id $T((n_k))$ of all closed two-sided ideals of $T((n_k))$, with the natural lattice operations. The abstract framework needed concerns primary approximately finite lattices, and we develop a little general theory in this direction, inspired by Arveson's unique factorization theory for primary completely distributive *metric* lattices. It turns out that the unordered set $\{n_1, n_2, \ldots\}$ is a complete lattice isomorphism invariant for the AF lattice Id $T((n_k))$ and hence a complete Banach algebra isomorphism invariant for the algebras.

The algebras $T((n_k))$ can be regarded as the approximately finite versions of reflexive operator algebras associated with certain commutative subspace lattices defined on an infinite tensor product Hilbert space. Such algebras were introduced and studied by Arveson [1, Chapter 3]. He obtained complete

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similarity invariants for these algebras as a conequence of a unique factorization theory mentioned above. We obtain a similar result in the class of approximately finite lattices and our proof derives directly from Arveson's arguments. However the arguments simplify considerably in our setting.

We can define $T((n_k))$ as a subalgebra of the well known Glimm algebra, or UHF C*-algebra,

$$M((n_k)) = M(n_1) \otimes M(n_2) \otimes \ldots.$$

Here $M(n)$ indicates the full $n \times n$ complex matrix algebra and the infinite tensor product is the C*-algebra direct limit of the direct injective unital system $M(n_1) \to M(n_1 n_2) \to \ldots$, under natural embeddings. The isomorphism theory and automorphism groups of these algebras are well understood (see [4], [5], [7], [9], for example) and, being approximately finite C*-algebras, K_0 theory is also available as a complete invariant. Thus $M((n_k))$ and $M((m_k))$ are isomorphic if and only if the sequences of partial products $n_1 n_2 \ldots n_k$ and $m_1 m_2 \ldots m_k$, satisfy the Glimm divisibility criterion: each term from one sequence must divide some term of the other. In other language, (n_k) and (m_k) must determine the same supernatural number. It follows then that $T((n_k))$ and $T((m_k))$ may fail to be isomorphic even though their associated UHF algebras are isomorphic, just as with finite tensor products. We note that the K_0 group of $T((n_k))$ coincides with the K_0 group of the diagonal subalgebra, from which it follows that K-theory provides poor invariants for the algebras $T((n_k))$. However unlike the UHF algebras, which are simple, there is a rich ideal structure, and this structure can serve to study morphisms and the automorphism group. For example the automorphisms that fix the ideal lattice are precisely the pointwise inner automorphisms.

The results above and related matters are organized in the following way. In section one we define approximately finite lattices and note relevant examples and key properties such as zero-one laws for factorizations. In section two we determine the ideal lattice of $T((n_k))$ as an AF lattice. Here we use standard approximation techniques associated with natural expectation mappings on the containing UHF algebra. We have used similar methods in [8] to study ideals in another class of non-self-adjoint subalgebras of AF algebras, namely in nest subalgebras associated with a maximal projection nest in the diagonal. Sections three and four use ideas of Arveson and develop the structure of prime elements in finite and approximately finite primary lattices, respectively. In section five we determine the nature of isomorphisms, epimorphisms and the automorphism group. In the final section we compute K_0.

For general lattice theory the reader may consult the standard reference Birkhoff [3], where ideal completions of lattices are discussed a little. Arveson's results are also described in his lecture notes [2].

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1. Approximately finite lattices.

Let L_0 be a lattice with respect to meet and join operations $\vee$ and $\wedge$ respectively. An ideal of L_0 is a subset J which is closed under joins and is such that if $a \leqq b$, with $a \in \mathrm{L}_0$ and $b \in J$, then $a \in J$.

The collection of all ideals, including the empty set, forms a complete lattice $\hat{L}_0$ known as the *ideal completion* of L_0. The lattice L_0 is injectively embedded in $\hat{L}_0$ as the sublattice of principal ideals of L_0.

We say that a complete lattice L is *approximately finite* if there is a countable sublattice $L_0 \subset L$ such that L is isomorphic to $\hat{L}_0$ as a lattice. More precisely we require that the natural injection $L_0 \to \hat{L}_0$ extends to an isomorphism $L \to \hat{L}_0$.

Let $L_1 \subset L_2 \subset \ldots$ be a chain of finite sublattices of L_0 with union equal to L_0. Then there is a one-to-one correspondence between elements J of $\hat{L}_0$ and certain chains of ideals $J_1 \subset J_2 \subset \ldots$, where each J_k is an ideal in L_k. The correspondence is given by

$$J \to J \cap L_1, J \cap L_2, \ldots,$$

and so we require that the chain have the fullness property,

$$J_k = (\cup J_m) \cap L_k, \text{ for all } k.$$

Approximately finite lattices often arise naturally as the direct limit of a direct system of finite lattices. In fact the class of such limit lattices, which we shall define in terms of an ideal completion, coincides with the class of AF lattices, as we now indicate.

An injective direct system of finite lattices is a sequence of finite lattices $M_1, M_2, \ldots$ together with injective embeddings

$$M_1 \to M_2 \to \ldots.$$

The collection M_{00} of increasing sequences (m_k), with $m_k \in M_k$, and which are eventually constant, forms a lattice in a natural way. Identifying eventually equal sequences we obtain a countable lattice M_0 in which each lattice M_i is naturally and injectively embedded, say $M_i \to \alpha(M_i)$. Moreover M_0 is the union of the chain $\alpha(M_1) \subset \alpha(M_2) \subset \ldots$. We define the *direct* limit L of the original system to be the ideal completion of M_0, and we write $L = \lim_k M_k$.

We usually consider lattices which possess both a first and last element, denoted by 0 and 1 respectively, and refer to such as unital lattices. A morphism between unital lattices is said to be unital if it maps 0 to 0 and 1 to 1.

An element c of a lattice is *join-irreducible*, or *prime*, if $c = a \vee b$ implies that $a = c$ or $b = c$, and a unital lattice is primary if the unit 1 is prime. An element c is *meet-irreducible* if $c = a \wedge b$ implies $a = c$ or $b = c$. If the first element 0 of a unital lattice is meet-irreducible then we say that the lattice itself is meet-irreducible. There is an elementary duality between the theory of primary lattices and meet-irreducible lattices that arises through the converse lattice, $(L, <)$ say, of the lattice $(L, \leqq)$; $a < b$ in $(L, <)$ if and only if $b \leqq a$ in $(L, \leqq)$, $a \wedge b$ in $(L, <)$ is the supremum $a \vee b$ in $(L, \leqq)$, and $a \vee b$ in $(L, <)$ is the infimum $a \wedge b$ in $(L, \leqq)$. It is easy to check that $(L, <)$ is primary if and only if $(L, \leqq)$ is meet-irreducible.

A finite lattice is primary if the supremum of all elements strictly less than 1 is also strictly less than 1, and is meet-irreducible if the infimum of all elements strictly greater than zero is also strictly greater than zero.

We now give some examples to illustrate the concepts above.

EXAMPLES 1. For $n = 2, 3, \ldots$ write $L(n)$ for the totally ordered unital lattice $\{0, 1, \ldots, n-1\}$. In particular $L(2)$ is the trivial unital lattice. These lattices are primary and meet-irreducible.

2. For $n, m = 2, 3, \ldots$ let $L(n) \times L(m)$ be the product lattice of $L(n)$ and $L(m)$ with the product partial ordering. For $n, m > 2$ these lattices are neither primary nor meet-irreducible.

3. A subset A of the product set $\{1, \ldots, n-1\} \times \{1, \ldots, m-1\}$ for $n, m \geqq 2$, is said to be *increasing* if (j_1, j_2) belongs to A whenever $j_1 \leqq k_1$ and $j_2 \leqq k_2$ for some element (k_1, k_2) in A. The totality of increasing sets, together with the empty set (which is also regaded as an increasing set), forms a lattice of sets (under the set operations) which we denote by $\mathrm{Inc}(n, m)$. Thus $\mathrm{Inc}(n, 2)$ and $\mathrm{Inc}(2, n)$ are just copies of $L(n)$. Similarly we can define $\mathrm{Inc}(n_1, \ldots, n_r)$ for integers $n_1, \ldots, n_r$ that are greater than unity, and there are natural unital injections

$$\mathrm{Inc}(n_1, \ldots, n_r) \to \mathrm{Inc}(n_1, \ldots, n_s)$$

for $r < s$. Here the increasing set A gets mapped to the increasing set $A \times N_{r+1} \times \ldots \times N_s$, where $N_j = \{1, \ldots, n_j - 1\}$. Note that the lattice $\mathrm{Inc}(n_1, \ldots, n_r)$ is generated by r sublattices $L_1, \ldots, L_r$ where L_k is a copy of the nest lattice $L(n_k)$. These lattices are primary and meet-irreducible.

4. For a sequence (n_k), of integers $n_k \geqq 2$, we can define the direct limit AF lattice associated with the system

$$\mathrm{Inc}(n_1, n_2) \to \mathrm{Inc}(n_1, n_2, n_3) \to \ldots.$$

We see later that such lattices are primary and meet-irreducible. The lattice can be thought of as the infinite tensor product of the nest lattices $L(n_1), L(n_2), \ldots.$

5. Let A be a partially ordered set with a last element a, and let L be a unital lattice. Then the collection, $\mathrm{Inc}(A, L)$ say, of increasing functions from A to L,

forms a unital lattice. Thus f belongs to $\mathrm{Inc}(A, L)$ if $f\colon A \to L$ and $f(b) \leqq f(c)$ if $b \leqq c$. If L is a finite meet-irreducible lattice then $\mathrm{Inc}(A, L)$ is also meet-irreducible. For if 0^+ is the unique successor of 0 in L then the function f, such that $f(1) = 0^+$ and $f(b) = 0$ for all $b \neq 1$, is the unique successor of the zero function.

For example, if L is a lattice then $\mathrm{Inc}(L, L(2))$ is the lattice of increasing subsets of L.

The lattice structure that we will be concerned with in later sections is the lattice $\mathrm{Id}\,A$ of closed ideals of a unital Banach algebra A. Here the join operation is closed linear span and meet is intersection. Clearly $\mathrm{Id}\,A$ is a complete unital lattice. We shall look at a class of inductive limit Banach algebras where the ideal lattice $\mathrm{Id}\,A$ can be identified as a direct limit of explicit finite lattices. This *identification* is fairly standard analysis, but the analysis of the *structure* of $\mathrm{Id}\,A$ requires quite a bit of lattice theory. The payoff is that the structure of meet-irreducible elements can be made quite explicit (see Theorem 4.2) and this has considerable implications for the nature of isomorphisms and automorphisms of the algebra A.

We complete the present section by considering complete distributivity and factorizations in the context of AF-lattices.
This information will be needed for the lattice theory in section 4.

PROPOSITION 1.1. *Let L be an* AF *lattice and let $c_1, c_2, \ldots$ and b be elements of L. Then* $\bigvee_{j=1}^{\infty} (b \wedge c_j) = b \wedge \left(\bigvee_{j=1}^{\infty} c_j\right)$.

PROOF. This is immediate because L is a complete lattice.

DEFINITION 1.2. Sublattices L_1 and L_2 of a lattice are said to be independent if the following property holds: if $a \wedge b \leqq a' \vee b'$, with a, a' in L_1 and b, b' in L_2, then $a \leqq a'$ or $b \leqq b'$.

DEFINITION 1.3 (Arveson [1]). Let L be a complete unital lattice. *A factorization* of L is a sequence of sublattices $L_1, L_2, \ldots$ such that

(i) $L = L_1 \vee L_2 \vee \ldots$

(ii) For every j the lattices L_j and $\bigvee_{k \neq j} L_k$ are independent

(iii) $\bigcap_{n=1}^{\infty} (L_n \vee L_{n+1} \vee \ldots) = \{0, 1\}$.

Similarly we shall say that $L_1, \ldots, L_n$ is a factorization of L if (i) and (ii) hold. Property (iii) is called the zero-one law for the sequence $L_1, L_2, \ldots$. The next proposition shows how zero-one laws arise naturally in certain direct limit AF lattices.

PROPOSITION 1.4. *Let $L_1, L_2, \ldots$ be unital sublattices of a lattice L such that for each n* the lattices *$L_1, \ldots, L_n$ form a factorization of the lattice that they generate. If $L = \lim_n (L_1 \vee \ldots \vee L_n)$ then $L_1, L_2, \ldots$ is a factorization of L.*

PROOF. Let $M_k = L_1 \vee \ldots \vee L_k$ so that L is (isomorphic to) the AF lattice $\lim_k M_k$. This means that L is identified with the lattice of ideals of the countable sublattice $L_0 = \bigcup_{k=1}^{\infty} M_k$. Moreover each such ideal β of L_0 is associated uniquely with the increasing sequence $\beta \cap M_1, \beta \cap M_2, \ldots$. In view, of this correspondence we can establish properties of elements β in L by arguing locally with the finite lattice of ideals in M_k.

First we obtain property (ii) of Definition 1.3. Let $N_r = \bigvee_{j \neq r} L_j$, and note that N_r is simply the sublattice of ideals β of L_0 such that $\beta \cap M_n = \beta \cap N_r^n$, where $N_r^n = L_1 \vee \ldots \vee L_{r-1} \vee L_{r+1} \vee \ldots \vee L_n$, for $n \geqq r$. Let β, β' be elements of N_r, regarded as ideals of L_0, and let α, α' be principal ideals of L_0 determined by p, p' in L_r respectively. Furthermore assume that $\alpha \wedge \beta \leqq \alpha' \vee \beta'$. Let $q \in \beta \cap M_n$. Since $(\alpha \wedge \beta) \cap M_n \subset (\alpha' \vee \beta') \cap M_n$ there exists q' in $\beta' \cap M_n$ and p'' in $\alpha' \cap M_n$ such that $p \wedge q = p'' \vee q' \leqq p' \vee q'$. By the independence of L_r and N_r^n, if $p \nleqq p'$ then $q \leqq q'$. Thus if $p \nleqq p'$, or equivalently, $\alpha \nleqq \alpha'$, then $\beta' \cap M_n \subset \beta' \cap \mathrm{M}_n$. Since n is arbitrary property (ii) now follows.

Similarly it can be shown that if $\beta \in L_n \vee L_{n+1} \vee \ldots$, then, for $n \leqq m$, $\beta \cap M_m$ is an ideal in $L_n \vee \ldots \vee L_m$ and for $n > m$, $\beta \cap M_m = \{0\}$ or M_m. Hence for $\gamma \in \bigcap_{n=1}^{\infty} (L_n \vee L_{n+1} \vee \ldots)$ we have $\gamma \cap M_m$ for all m, and so property (iii) holds.

DEFINITION 1.5. We say that the factorization $L_1, L_2, \ldots$ of the AF lattice L is a coherent factorization if L is isomorphic to the approximately finite lattice $\lim_n (L_1 \vee \ldots \vee L_n)$, as in the statement of Proposition 1.4.

PROPOSITION 1.6. *Let L_1, $L_2, \ldots$ be a coherent factorization of the unital AF lattice L, and let $p_k \in L_k$ for $k = 1, 2, \ldots$. Then either $\wedge_k p_k$ is the zero element or $p_k = 1$ for all but a finite number of k.*

PROOF. Let $\beta = \wedge_k p_k$ which is identified with the ideal $\{x \in L_0 : x \leqq p_k$ for all $k\}$, where L_0 is as in the proof of Proposition 1.4. Let $x \in \beta \cap M_r$, where $M_r = L_1 \vee \ldots \vee L_r$, as before. Then $x \wedge 1 \leqq 0 \vee p_k$ for all k, and so, by the independence of the lattices M_r and L_k for $k > r$, it follows that $x \leqq 0$ or $1 \leqq p_k$. Thus if $p_k \neq 1$ for an infinity of k, then $x = 0$. Hence $\beta = 0$.

Our last proposition in this section is also an elementary conequence of local arguments. A similar assertion holds with primary replaced by meet-irreducible.

PROPOSITION 1.7. *Let $L = \lim_k L_k$ be the* AF *lattice determined by finite primary unital lattices L_k. Then L is primary.*

2. Id $T((n_k))$ as an AF lattice.

The following notation will be useful. Let (n_k) be a sequence of integers, with $n_k \geqq 2$ for all k, to avoid trivialities. Let $A = T((n_k)) = \bigotimes_{k=1}^{\infty} T(n_k)$, $B = M((n_k)) = \bigotimes_{k=1}^{\infty} M(n_k)$, $C = C((n_k)) = \bigotimes_{k=1}^{\infty} C(n_k)$, where $C(n_k)$ is the diagonal algebra $T(n_k) \cap T(n_k)^*$. Also, for $r = 1, 2, \ldots$, let us write A_r, B_r and C_r for the finite tensor product algebras associated with the r-tuple $n_1, \ldots, n_r$, regarded as the canonical subalgebras of A, B and C respectively.

We now define some important expectation maps on the algebra B. For $r < s$ let $U_{r,s}$ be the unitary group of the diagonal algebra $C(n_{r+1}) \otimes \ldots \otimes C(n_s) \subseteq C$, and let du denote Haar measure on $U_{r,s}$. The linear contractive map $\Phi_{r,s}$ defined on B_s by

$$\Phi_{r,s}(x) = \int_{U_{r,s}} u^* x\, u\, du, \qquad x \text{ in } B_s,$$

is a projection and has range equal to the subalgebra $M(n_1) \otimes \ldots \otimes M(n_r) \otimes C(n_{r+1}) \otimes \ldots \otimes C(n_s)$. Since $\Phi_{r,t}$ extends $\Phi_{r,s}$ when $s < t$, we can define Φ_r on B as the pointwise limit

$$\Phi_r(x) = \lim_{n \to \infty} \Phi_{r,r+n}(x).$$

The map Φ_r is a contractive projection onto the subalgebra $B_r \otimes C(n_{r+1}) \otimes \ldots$. In particular $\Phi_r(x) \to x$ as $r \to \infty$ for every x in B.

PROPOSITION 2.1. *Let J be a closed subspace of B that is a C-module. Then J is the closed union of the subspaces $J \cap B_n$, $n = 1, 2, \ldots$. In particular this holds true for ideals J of the subalgebra A.*

PROOF. Note that if x belongs to J then so does $\Phi_r(x)$ for every r. On the other hand the bimodule $\Phi_r(J)$ is finitely generated over C, and the proposition is easily obtained in this case.

The synthesis property expressed in the last proposition is required to identify the ideal lattice of A. In fact the same feature holds for appropriate modules in general approximately finite C^*-algebras (see [8]).

Let us introduce a twisted partial ordering on the set of pairs

$$\delta(n) = \{(i,j)\colon\ 1 \leqq i \leqq j \leqq n\},$$

which reflects the ideal structure of $T(n)$. We write $(i,j) \leqq (k,l)$ when $i \geqq k$ and $j \leqq l$. If S is an increasing subset of $\delta(n)$ with respect to this ordering then the set J of matrices in $M(n)$ supported by S is an ideal. Conversely every ideal arises in this way. More generally we have the following elementary proposition.

We write 2 for the trivial unital lattice $L(2)$, and we use the notation of example 5 in section 1.

PROPOSITION 2.2. (i) *The ideal lattice* $\mathrm{Id}\, T(n)$ *is isomorphic to* $\mathrm{Inc}(\delta(n), 2)$.

(ii) *If A is any complex algebra then the ideal lattice* $\mathrm{Id}\,(T(n) \otimes A)$ *is isomorphic to the lattice* $\mathrm{Inc}(\delta(n), \mathrm{Id}\, A)$.

In particular $T(n_1) \otimes T(n_2)$ has an ideal lattice which is isomorphic to $\mathrm{Inc}(\delta(n_1), \mathrm{Inc}(\delta(n_2), 2))$, and we write this more simply as $\mathrm{Inc}(\delta(n_1), \delta(n_2), 2)$. Similarly the r-fold tensor product $T(n_1) \otimes \ldots \otimes T(n_r)$ has an ideal lattice denoted by $\mathrm{Inc}(\delta(n_1), \ldots, \delta(n_r), 2)$.

There are natural embeddings

$$\mathrm{Inc}(\delta(n_1), \ldots, \delta(n_r), 2) \to \mathrm{Inc}(\delta(n_1), \ldots, \delta(n_s), 2),$$

when $r \leqq s$, which are most easily identified by checking first that $\mathrm{Inc}(\delta(n_1), \ldots, \delta(n_r), 2)$ is isomorphic to $\mathrm{Inc}(\delta(n_1) \times \ldots \times \delta(n_r), 2)$, the lattice of increasing subsets of the partially ordered product space $\delta(n_1) \times \ldots x\, \delta(n_r)$. The embeddings above correspond precisely to the embedding $\mathrm{Id}\, A_r \to \mathrm{Id}\, A_{r+1}$ of the ideal lattice of $\mathrm{Id}\, A_r$. (Here an ideal J in $\mathrm{Id}\, A_r$ is identified with the ideal $\bar{J}$ in $\mathrm{Id}\, A_{r+1}$ that it generates).

THEOREM 2.3. *The ideal lattice of* $T((n_k))$ *is isomorphic to the approximately finite lattice* $\lim_k \mathrm{Inc}(\delta(n_1) \times \ldots \times \delta(n_k), 2)$.

PROOF. We have observed that the limit lattice in the statement of the theorem is isomorphic to $\lim_k \mathrm{Id}\, A_k$.

By Proposition 2.1 we can identify $\mathrm{Id}\, A$ with the set of sequences $J \cap A_1$, $J \cap A_2, \ldots$, for J in $\mathrm{Id}\, A$. An increasing sequence $J_1, J_2, \ldots$ of ideals J_k of A_k, is such a sequence precisely when $J_r = A_r \cap (\cup_k J_k), r = 1, 2, \ldots$. Let us call such a sequence an inductive sequence of ideals. Then, more precisely, Proposition 2.1 allows us to identify $\mathrm{Id}\, A$ with the lattice of increasing inductive sequences of ideals. From the definition of direct limits of lattices, we see that $\mathrm{Id}\, A$ is isomorphic to $\lim_k \mathrm{Id}\, A_k$.

We have already observed that the limit lattice of a unital direct system of primary lattices is primary. Similar reasoning or direct argument with Proposition 2.1 shows that the ideal lattice $\mathrm{Id}\, T((n_k))$ is meet-irreducible.

REMARK. Similar reasoning applies in the context of nest subalgebras of AF algebras considered in [8]. For example it is possible to define a natural upper triangular subalgebra, $\mathrm{TM}((n_k))$ say, of $M((n_k))$, which is the inductive limit algebra $\lim_k T(n_1 \ldots n_k)$, with respect to certain natural embeddings 'by refinement'. For this algebra we can obtain the identification

$$\mathrm{Id}\, \mathrm{TM}((n_k)) = \lim_k \mathrm{Inc}(\delta(n_1 \ldots n_k), 2).$$

3. Finite primary lattices

We now collect together some elementary facts concerning finite factorizations and finite primary lattices. The arguments here are partly derived from Arveson's paper [1].

PROPOSITION 3.1. *Let M be a finite unital lattice with unital sublattices $L_1,\dots,L_n$ which form a factorization of M. If each factor L_k is primary then M is primary.*

PROOF. Let $e_1,\dots,e_n$ be the largest non unis in $L_1,\dots,L_n$ respectively. We show that $e_1 \vee \dots \vee e_n$ is the largest non unit in L, and hence that L is primary. Note first that $e_1 \vee \dots \vee e_n$ is strictly less than 1. For otherwise $1 \wedge 1 \leqq e_1 \vee \dots \vee e_n)$ and so by independence $1 \leqq e_2 \vee \dots \vee e_n$. Continuing with this argument obtain the contradiction $1 \leqq e_n$. On the other hand let $a \in L$, $a \neq 1$. Then $a = a_1 \vee \dots \vee a_r$ with each a_j of the form $x_{j1} \wedge \dots \wedge x_{jn}$, with $x_{ji} \in L_i$, for all i. For each j there exists x_{jk} with $x_{jk} \leqq e_k$. (For otherwise $a_j = 1$, and $a = 1$). Thus $a_j \leqq e_1 \vee \dots \vee e_n$ and so $a \leqq e_1 \vee \dots \vee e_n$, as desired.

In view of Proposition 1.7 we now deduce that if $L_1, L_2 \dots$ is a coherent factorization of the approximately finite lattice L, then L is primary if each factor L_k is primary.

COROLLARY 3.2. *Let M be a finite unital lattice with unital sublattices $L_1,\dots,L_n$ which form a factorization of M. If p is a prime element of L_i for some i, and if M is primary, then p is a prime element of M.*

PROOF. Let p be a non-zero prime element of L_i and define $N = p \wedge M$, $N_k = p \wedge L_k$, for $k = 1,\dots,n$. We claim that $N_1,\dots,N_n$ is a factorization of N.

Clearly, $N_1,\dots,N_n$ generate N. Fix r and elements a, a' in L_r, b, b' in $\bigvee_{j \neq r} L_j$, and assume that

$$(p \wedge a) \wedge (p \wedge b) \leqq (p \wedge a') \vee (p \wedge b').$$

If $r = i$ then $(p \wedge a) \wedge b = (p \wedge a) \wedge (p \wedge b) \leqq ((p \wedge a') \vee b') \wedge p \leqq (p \wedge a') \vee b'$. Hence $p \wedge a \leqq p \wedge a'$ or $b \leqq b'$. On the other hand if $r \neq i$ then $a \wedge (p \wedge b) = (p \wedge a) \wedge (p \wedge b) \leqq (p \wedge a') \vee (p \wedge b')$, and so $a \leqq a'$ or $p \wedge b \leqq p \wedge b'$. In both cases we have the desired alternative, $p \wedge a \leqq p \wedge a'$ or $p \wedge b \leqq p \wedge b'$.

We next show that each of the lattices N_k is primary, and the corollary will follow from Proposition 3.1.

Assume that $p = (p \wedge a) \vee (p \wedge b)$ with a and b in N_k. If $k = i$ then $p \wedge a = p$ or $p \wedge b = p$ because p is prime in L_i. On the other hand if $k \neq i$ then $p \wedge 1 = p = p \wedge (a \vee b) \leqq a \vee b = 0 \vee (a \vee b)$ and so, by independence, $p \leqq 0$ or $1 \leqq a \vee b$. Hence $1 = a \vee b$ and $a = 1$ or $b = 1$ because M is primary. Hence $p \wedge a = p$ or $p \wedge b = b$ as required.

COROLLARY 3.3. *Let M be a unital primary lattice with unital primary sublattices $L_1, \ldots, L_n$ which form a factorization of M. Let p be an element of the form $p = \bigwedge_{r=1}^{l} p_r$ where each p_r is a prime in M_r. Then p is prime in M.*

PROOF. By Proposition 3.1 it suffices to show that each of the sublattices $p \wedge L_i$ is primary. Suppose then that a, b are elements of L_i such that $p = (p \wedge a) \vee (p \wedge b)$, and $p \neq 0$. Let $q_i = \bigwedge_{r \neq i} p_r$ so that $p_i \wedge q_i = p = ((p_i \wedge a) \vee (p_i \wedge b)) \wedge q_i$. Since the lattices L_i and $\bigvee_{j \neq i} L_j$ are independent it follows that $p_i = (p_i \wedge a) \vee (q_i \wedge b)$ and hence $p_i = p_i \wedge a$ or $p_i = p_i \wedge b$, since p_i is prime. Hence $p = p \wedge a$ or $p = p \wedge b$, and $p \wedge L_i$ is primary.

The converse to the last corollary is also valid; every prime element p of the lattice M is of the form $p_1 \wedge \ldots \wedge p_n$ where each p_k is prime in L_k. We see this in the next section where we obtain an analogous representation for prime elements in certain approximately finite lattices admitting a factorization $L_1, L_2, \ldots$ by finite primary sublattices.

4. Prime elements and unique factorisation.

Our context in this section concerns approximately finite lattices L which arise as in the statement of Proposition 1.4, that is, L is isomorphic to the approximately finite lattice $\lim_n (L_1 \vee \ldots \vee L_n)$ associated with the sequence $L_1, L_2, \ldots$ which is a factorisation of L by finite lattices. We call such a factorisation a *coherent factorisation* of the AF lattices L.

A factorisation is said to be *indecomposable* when none of the sublattices L_k admits a nontrivial factorisation. The following theorem is the counterpart of a theorem of Arveson for distributive metric lattices [1, Theorem 3.3.2].

THEOREM 4.1. *Let $L_1, L_2, \ldots$ and $N_1, N_2, \ldots$ be two indecomposable coherent factorisation of the approximately finite unital primary lattice L. Then there is a permutation π such that $N_k = L_{\pi(k)}$ for all k.*

However, we will not be able to use this theorem in the context of the ideal lattice of $T((n_k))$ since this lattice is not primary. It is however meet-irreducible, and we have the following theorem.

THEOREM 4.2. *Let $L_1, L_2, \ldots$ and $N_1, N_2, \ldots$ be two indecomposable coherent factorisations of the approximately finite meet-irreducible lattice L. Then there is a permutation π such that $N_k = L_{\pi(k)}$ for all k.*

These unique factorisation theorems follow from Theorem 4.3 which is the key result of this section. In fact we only use this theorem in what follows.

THEOREM 4.3. *Let $L_1, L_2, \ldots$ be a coherent factorisation of the approximately finite unital lattice L.*

(i) *L has nonzero prime elements if and only if L_k is primary for almost every k. Moreover the nonzero primes are precisely the elements p of the form*

$$p = p_1 \wedge p_2 \wedge \ldots \wedge p_m$$

where p_k is a nonzero prime in L_k, for $1 \leqq k \leqq m$, $m = 1, 2, \ldots$.

(ii) *The nontrivial meet-irreducible elements of L are precisely those of the form*

$$p = \vee p_k$$

where p_k is a nontrivial meet-irreducible element in L_k, for $k = 1, 2, \ldots$.

PROOF. We first show that for each prime $p \neq 0$ we have

$$p = \wedge \{a_k : a_k \geqq p, a_k \in L_k\}.$$

(This is the AF version of Theorem 3.1.2 in [2]). Let p_0 denote the infimum and let $p_n = \wedge \{a : a \geqq p, a \in L_{n+1} \vee L_{n+2} \vee \ldots\}$. Then $p_0 \wedge p_n \geqq p$, and in fact it will be enough to show that for each n, $p \geqq p_0 \wedge p_n$. To see that this is enough, note that

$$p = p_0 \wedge p_n = \vee(p_0 \wedge p_n) = p_0 \wedge (\vee p_n) = p_0 \wedge 1.$$

The last two equalities here follow from Proposition 1.1 and property (iii) of Definition 1.3 respectively.

Suppose then that $x \geqq p$. We show that $x \geqq p_0 \wedge p_n$. Let $\beta_1, \ldots, \beta_l$ be an enumeration of the elements of the form $x_1 \wedge \ldots \wedge x_{n-1}$ with x_i in L_i. Consider the collection N of elements of the form

$$\bigvee_{k=1}^{l} (\beta_k \wedge c_k)$$

with c_k in $L_n \vee L_{n+1} \vee \ldots$. Then N is a complete lattice. Hence for some $c_1, \ldots c_l$ we have

$$p \leqq x = \bigvee_{k=1}^{l} (\beta_k \wedge c_k).$$

Since p is a prime element it follows that $p \leqq \beta_k \wedge c_k$ for some k and so $p \leqq \beta_k$ and $p \leqq c_k$. We have $p_0 \leqq \beta_k$ and $p_n \leqq c_k$ and so $p_0 \wedge p_n \leqq \beta_k \wedge c_k \leqq x$ as required.

Let

$$p_k = \wedge \{a_k : a_k \geqq p, a_k \in L_k\},$$

with $p_k = 1$ if this set is empty. Suppose that $p_k = a \vee b$ with a, b in L_k. Let

$q_k = \bigwedge_{i \neq k} p_i$. Then $p = p_k \wedge q_k = (a \vee b) \wedge q_k = (a \wedge q_k) \vee (b \wedge q_n)$ and so $p = a \wedge q_k$ or $p = b \wedge q_k$. Suppose that $p = a \wedge q_k$. Then $p = p_k \wedge q_k = a \wedge q_k \leqq a \vee 0$. By independence $p_k \leqq a$ (since $q_k \neq 0$). Also $a \leqq a \vee b = p_k$, and so $p_k = a$. The other case, namely $p = b \wedge q_n$ leads to $p_n = b$.

Thus p_k is prime for all k, and by Proposition 1.6, $p_k = 1$ for all but a finite number of k. On the other hand the results in section 3 show that p is prime in L if p has the form given in (i).

(iii) We can argue exactly as above for a meet-irreducible element p $\neq 0$ by replacing prime by meet-irreducible, $\vee$ *by* $\wedge$, $\wedge$ *by* $\vee$, and 0 and 1 by 1 and 0, respectively. In this way we see that $p = \vee p_k$ with p_k a meet-irreducible element of L_k for all k. It remains to show that $p = \vee p_k$ is meet-irreducible for every choice of meet-irreducible elements $p_k \in L_k$.

First note that $p_1 \vee \ldots \vee p_n$ is meet-irreducible for each n. This follows by applying Corollary 3.3 to the lattice $L_1 \vee \ldots \vee L_n$ with the converse partial ordering. The infinite case follows in an elementary way from local arguments.

For suppose that $p = a \wedge b$ and p, a, b are represented by the ideals π, α, respectively, of the countable lattice L_0, where L_0 is the union of the finite $L_{(n)} = L_1 \vee \ldots \vee L_n$. Then

$$\pi \cap L_{(n)} = (\alpha \cap \beta) \cap L_{(n)} = (\alpha \cap L_{(n)}) \cap (\beta \cap L_{(n)}).$$

However, by independence $\pi \cap L_{(n)}$ is the principal ideal for the meet-irreducible element $p_1 \vee \ldots \vee p_n$, and so either $\alpha \cap L_{(n)} = \pi \cap L_{(n)}$ or $\beta \cap L_{(n)} = \pi \cap L_{(n)}$. This is true for all n and so $\alpha = \pi$ or $\beta = \pi$, as required.

Notice that we have shown that L is primary (resp. meet-irreducible) if and only if each factor L_k, of the coherent factorisation, is primary (resp. meet-irreducible).

PROOFS OF THEOREMS 4.1 AND 4.2. Let L_m, N_n be as in Theorem 4.1 or 4.2. Set $L_{m,n} = L_m \cap N_n$. Then we claim that it follows from Theorem 4.3 that $L_{m,1}, L_{m,2}, \ldots$ and $L_{1,n}, L_{2,n}, \ldots$ are coherent factorisations of L_m and N_n respectively. In fact the arguments for this are virtually identical to those in the proof of Theorem 3.3.1 in [1] and so we omit the details. Since each L_m is indecomposable it follows that the factors $L_{m,k}$ are all trivial except for a single factor, $L_{m,\pi(m)}$ say. Similarly the factors $L_{1,\pi(m)}, L_{2,\pi(m)}, \ldots$ are all trivial except for a single factor which must be $L_{m,\pi(m)}$. Thus $L_m = L_{m,\pi(m)} = \vee_j L_{j,\pi(m)} = N_{\pi(m)}$. So, with π replaced by π^{-1}, the proofs are complete.

REMARK. We have obtained the unique factorisation theorem, Theorem 4.1, without recourse to Arveson's factorisation theorem for distributive metric lattices. It seems logical to make the elementary context independent of the topological one. However it may well be possible to deduce our theorem from Arveson's by constructing normal valuations on AF lattices.

5. Isomorphisms and the automorphism group of $T((n_k))$.

The following theorem characterizes the Banach algebra isomorphisms and epimorphisms between the algebras $T((n_k))$ and $T((m_k))$ where, as usual, (n_k) and (m_k) are sequences of positive integers greater than unity.

THEOREM 5.1. (i) *$T((n_k))$ and $T((m_k))$ are isomorphic if and only of there is a permutation π such that $m_k = n_{\pi(k)}$, $k = 1, 2, \dots$.*

(ii) *There is an onto unital homomorphism from $T((n_k))$ to $T((m_k))$ if and only if there is an injection π: $\mathsf{N} \to \mathsf{N}$ such that $m_k \leqq n_{\pi(k)}$ for all k.*

Let $L((n_k))$ be the approximately finite unital lattice $\lim_k \operatorname{Inc}(\delta(n_1), \dots, \delta(n_k), 2)$ so that by Proposition 2.3 $L((n_k))$ and Id $T((n_k))$ are isomorphic. By Proposition 1.7 $L((n_k))$ is meet-irreducible. There are canonical identifications of the lattice $L_j = \operatorname{Inc}(\delta(n_j), 2)$ as a unital sublattice of $L((n_k))$ and, by Proposition 1.4 L_1, L_2, ... is a factorization of $L((n_k))$. However the factorization is not indecomposable. Each sublattice L_j admits a factorization $L_j^\alpha \vee L_j^\beta$, where L_j^α and L_j^β are copies of the nest lattice $L(n_j)$:

$$L_j^\alpha = \{\phi_t \in \operatorname{Inc}(\delta(n_j), 2)\colon\ \phi_t((i,j)) = 1 \Leftrightarrow 1 \leqq i \leqq t\}$$

$$L_j^\beta = \{\psi_t \in \operatorname{Inc}(\delta(n_j), 2)\colon\ \psi_t((i,j)) = 0 \Leftrightarrow t \leqq j \leqq 1\}$$

Thus $L((n_k))$ is a meet-irreducible unital approximately finite lattice with indecomposable coherent factorisation L_1^α, L_1^β, L_2^α, L_2^β, ...

The set of non zero meet-irreducible ideals of $T(n)$ is isomorphic to $\delta(n)$. The point (i,j) in $\delta(n)$ corresponds to the ideal spanned by the matrix units $e_{k,1}$ where $k < i$ or $1 > j$. Theorem 4.3 shows that the set of nontrivial meet-irreducible ideals of $T((n_k))$ is canonically isomorphic to the product set

$$\delta((n_k)) = \delta(n_1) \times \delta(n_2) \times \dots$$

with the product partial ordering.

LEMMA *If $n_k \geqq 1$ and $m_k \geqq 1$ for all k then there is an order isomorphism $\gamma: \delta((n_k)) \to \delta((m_k))$ if and only if there is a permutation π such that $m_k = n_{\pi(k)}$, for all k.*

PROOF. Let $t_k = (1, 1)$ and note that the interval $[(1, n_k), (1, 1)]$ is a chain in $\delta(n_k)$. Moreover, if $t = (0, 0, \dots, t_k, 0, \dots)$, where the j'^{th} zero is the zero element (n_j, n_j) of $\delta(n_j)$, then the interval $[0, t]$ in $\delta((n_k))$ is a chain of length n_k. Moreover it is easy to see that every segment $[0, t']$ which is totally ordered, is of this form. Thus α induces a bijection of such segments, and it follows that $m_k = n_{\pi(k)}$ for some permutation π.

PROOF OF THEOREM 5.1. Since a Banach algebra isomorphism between $T((n_k))$

and $T((m_k))$ induces a bijection between the sets of meet-irreducible ideals, part (i) follows from the discussion above.

Suppose now that $m_k \leqq n_{\pi(k)}$ for all k for some fixed injection π. Then there is a contractive unital algebra homomorphism $\sigma_k: T(n_{\pi(k)}) \to T(m_k)$, and the contractive maps $\sigma_l \otimes \ldots \otimes \sigma_k$, from $T((n_{\pi(k)}))$ to $T((m_k))$, converge pointwise to an epimorphism.

On the other hand let σ be an epimorphism from $T((n_k))$ to $T((m_k))$. Then $\ker \sigma$ is a meet-irreducible ideal and so by Theorem 4.3 (ii), $\ker \sigma = \tilde{J}_1 \vee \tilde{J}_2 \vee \ldots$, where $\tilde{J}_k$ is the ideal in $T((n_k))$ generated by a meet-irreducible ideal J_k of $T(n_k)$. Thus there are integers $1 \leqq r_k \leqq n_k$ and a natural isomorphism

$$T((n_k))/\tilde{J}_1 \vee \ldots \vee \tilde{J}_l \to T(r_1) \otimes \ldots \otimes T(r_l) \otimes T(m_{l+1}) \otimes T(m_{l+2}) \otimes \ldots$$

Since $\ker \sigma$ is the closed union of the ideals $\tilde{J}_1 \vee \ldots \vee \tilde{J}_l$, it follows that $T((n_k))/\ker \sigma$ is isometrically isomorphic to $T((r_k))$. Now (ii) follows from (i).

The automorphism group. In the next two lemmas we show that the automorphisms fixing the ideal lattice are precisely the pointwise-inner automorphisms. We write γ_π for the canonical permutation automorphism of $T((n_k))$ associated with a permutation π such that $n_k = n_{\pi(k)}$ for all k.

LEMMA 5.2. *Let $\gamma \in \operatorname{Aut} T((n_k))$. Then $\gamma = \beta \circ \gamma_\pi$ where γ_π is a permutation automorphism and β is an automorphism with $\beta(J) = J$ for every two-sided ideal J.*

PROOF. Since γ preserves meet-irreducibility of ideals, γ induces an isomorphism $\gamma: \delta((n_k)) \to \delta((n_k))$. But such an automorphism is a composition of a permutation automorphism $\hat{\gamma}_\pi$ and an automorphism, $\hat{\beta}$ say, which acts locally. In fact each $\delta(n_i)$ supports a flip automorphism (exchanging coordinates in $\delta(n_i)$), and $\hat{\beta}$ must either fix or flip each coordinate. Since $\hat{\beta}$ derives from the *algebra* automorphism $\beta = \alpha \circ \alpha_\pi^{-1}$, it is easy to check that in fact $\hat{\beta}$ has no flip action, and hence that $\beta(J) = J$ for every meet-irreducible ideal, and hence for all ideals.

The hypothesis in the next lemma cannot be relaxed too much as can be seen the following example. Let $\mathscr{A}$ be the subalgebra of $T(4)$ spanned by the matrix units e_{ij} of $T(4)$ other than e_{12} and e_{34}. It can be seen that $\mathscr{A}$ admits automorphisms that preserve ideals but which are not inner. For example consider the automorphism α such that $\alpha(e_{14}) = -e_{14}$ and $\alpha(e_{ij}) = e_{ij}$ for all other matrix units in $\mathscr{A}$. This fails to be inner because α fails to preserve the rank of some elements. (See [6] for related matters).

LEMMA 5.3. *Let $\mathscr{A}$ be a subalgebra of the algebra $T(n)$ which contains the matrix units e_{ii}, e_{in}, for $1 \leqq i \leqq n$. If α is an automorphism of $\mathscr{A}$ such that $\alpha(J) = J$ for every two sided ideal J then α is an inner automorphism. Moreover the same holds true for ideal preserving automorphisms if the algebraic tensor product $\mathscr{A} \otimes \mathscr{B}$, where $\mathscr{B}$ is a commutative unital C^*-algebra.*

PROOF. By the ideal invariance of α we see that $\alpha(e_{11}) = e_{11} + \sum_{j=2}^{n} a_{1j}e_{1j}$. Let a_{1r} be a nonzero coefficient with $r \geqq 2$ and let $S_{1r}(\lambda) = I + \lambda e_{1r}$. Then $S_{1r}(\lambda)^{-1} = S_{1r}(-\lambda)$ and we see that $S_{1r}(\lambda)$ is an invertible element of $\mathscr{A}$ such that

$$(S_{1r}(\lambda)^{-1}\alpha(e_{11})S_{1r}(\lambda))_{1r} = a_{1r} - \lambda.$$

It follows that we may construct an invertible element S in A such that $S^{-1}\alpha(e_{11}) S = e_{11}$.

Since α is an automorphism we observe that for $1 < i \leqq j$

$$\begin{aligned}(S^{-1}\alpha(e_{ij})S)_{1r} &= (e_{11}S^{-1}\alpha(e_{ij})S)_{ir}\\ &= (S^{-1}\alpha(e_{11}e_{ij})S)_{1r}\\ &= 0.\end{aligned}$$

Thus α leaves invariant the subalgebra, $\mathscr{A}_1$ say, spanned by $\{e_{ij}: e_{ij} \in \mathscr{A}, 2 \leqq i\}$. In particular, with respect to the associated decomposition $\mathbf{C}^n = \mathbf{C} \oplus \mathbf{C}^{n-1}$, we can assume α has the form

$$\alpha: \begin{pmatrix} a_{11} & \underline{a} \\ 0 & A_1 \end{pmatrix} \to \alpha(A) = \begin{pmatrix} a_{11} & \delta(\underline{a}) \\ 0 & \alpha_1(A_1) \end{pmatrix}$$

where α_1 is the restriction of α to $\mathscr{A}_1$, and δ is a linear map on the linear space of row vectors $\underline{a}$.

We shall show that α is inner by induction on n. By the induction hypothesis α_1 is implemented by an invertible element T_1 of the algebra $\mathscr{A}_1$. Conjugating by $T = e_{11} \oplus T_1$ obtain a new ideal preserving automorphism which is the identity map on $\mathscr{A}_1$. Without loss then, we assume that α already has this form. In particular $\delta(\underline{a}A_1) = \delta(\underline{a})A_1$ for all operators A_1 in $\mathscr{A}_1$, from which it follows that $\delta(e_{1j}) = d_j e_{1j}$ for some scalars d_j (associated with indices $j \geqq 2$ for which e_{1j} is in $\mathscr{A}$). Suppose e_{1j} lies in $\mathscr{A}$. Then $d_n e_{1n} = \delta(e_{1n}) = \delta(e_{1j}e_{jn}) = \delta(e_{1j})\delta(e_{jn}) = d_j e_{1j}e_{jn} = d_j e_{1n}$. Thus all the d_j coincide with a single scalar d say. Thus $\alpha(\cdot) = D^{-1}\cdot D$ where D is the diagonal matrix with entries $1, d, d, \ldots, d$, and the first assertion is proven.

Note that $\mathscr{A} \otimes \mathscr{B}$ can be considered as the algebra of matrices from $\mathscr{A}$ whose entries are operators in B. Replacing the role of the scalar field by $\mathscr{B}$ in the argument above leads to an almost identical proof for the second assertion of the proposition.

The next lemma characterizes the ideal fixing automorphes as the pointwise inner automorphisms.

LEMMA 5.4. *Let α be an automorphisms of $T((n_k))$ such that $\alpha(J) = J$ for every closed two sided ideal J. Then there exist invertible operators S_r, with S_r and S_r^{-1} in*

$T((n_k))$, for $r = 1,2,\ldots$, such that $S_r^{-1}XS_r \to \alpha(X)$ as $r \to \infty$ for every element X of $T((n_k))$.

PROOF. Let $A_r = \bigotimes_{k=1}^{r} T(n_k)$, $A^r = \bigotimes_{k=r+1}^{\infty} T(n_k)$, $C_r = \bigotimes_{k=1}^{r} C(n_k)$ $C^r = \bigotimes_{k=r+1}^{\infty} C(n_k)$, regarded as the usual subalgebras of $T((n_k))$. The Jacobson radical $\operatorname{rad} A^r$ of the subalgebra A^r is the strictly upper triangular part of A^r and we have $A^r = C^r + \operatorname{rad} A^r$. Moreover $J = \mathscr{A}_r \otimes \operatorname{rad} A^r$ is an ideal such that the quotient $T((n_k))/J$ is canonically isomorphic to $\mathscr{A}_r \otimes C^r$. To see this observe that $\mathscr{A}_r \otimes \operatorname{rad} A^r$ is the kernel of the natural contractive homomorphism from $T((n_k))$ to $\mathscr{A}_r \otimes C^r$). In particular, since J is invariant, α induces an automorphism α_r of $\mathscr{A}_r \otimes C^r$, and moreover α_r leaves invariant the ideals of $\mathscr{A}_r \otimes C^r$. The ascending subalgebras $\mathscr{A}_r \otimes C^r$ have dense union in $T((n_k))$, and so it will be sufficient to show that each automorphism α_r is inner. This follows from the second part of Lemma 5.3, since the algebras A_r are subalgebras of $T(n_1 n_2 \ldots n_r)$ of the required form.

The results above are summarized in the next theorem. We write $\operatorname{Out}(T((n_k)))$ for the quotient group determined by the normal subgroups of pointwise inner automorphisms.

THEOREM 5.5. *Let $\Pi((n_k))$ be the discrete group of permutations π such that $n_k = n_{\pi(k)}$, $k = 1,2,\ldots$. Then each automorphism α in* Aut *$T((n_k))$ admits a decomposition $\alpha = \beta \circ \alpha_\pi$ with β a pointwise inner automorphism and π in $\Pi((n_k))$. In particular* Out *$T((n_k))$ is the discrete group $\Pi((n_k))$.*

REMARK. The classification in Theorem 5.1(i) has been substantially generalized in the author's paper "Classifications of tensor products of triangular operator algebras" (to appear in Proc. London Math. Soc.).

6. The K_0 group.

Let A be the algebra $T((n_k))$ with diagonal subalgebra $C = C((n_k))$, associated as usual with integers $n_k \geqq 2$, $k = 1,2,\ldots$. We show that $K_0(A) = K_0(C)$. In particular K_0 does not distinguish the isomorphism type.

Recall that A decomposes as a direct sum $A = C + \operatorname{rad} A$, where $\operatorname{rad} A$ is the Jacobson radical. Let $p = (p_{ij})$ be an idempotent in $M_n(A)$ and let $p_{ij} = c_{ij} + r_{ij}$ in C and r_{ij} in $\operatorname{rad} A$, so that $c = (c_{ij})$ is an idempotent in $M_n(C)$. We show that there is continuous path of idempotents p^t, $0 \leqq t \leqq 1$, in $M_n(A)$, such that $p^2 = p$ and $p^0 = c$. From this it will follow that the natural map $K_0(A) \to K_0(C)$ induced by the quotient mapping, is an isomorphism.

Let $d_{t,k}$ be the invertible element of C given by

$$d_{t,k} = 1 \otimes \ldots \otimes 1 \otimes D_{t,k} \otimes 1 \otimes \ldots,$$

$$D_{t,k} = \begin{bmatrix} 1 & & & & \\ & t & & & \\ & & t^2 & & \\ & & & \ddots & \\ & & & & t^n \end{bmatrix}, 0 < t < 1.$$

Then the inner automorphism $\alpha_{t,k}$: $a \to d_{t,k}^{-1} a d_{t,k}$ is a contractive on A. It follows that we can define the pointwise inner homomorphism α_t by

$$\alpha_t(a) = \lim_{k \to \infty} \alpha_{t,1} \circ \alpha_{t,2} \circ \ldots \circ \alpha_{t,k}(a).$$

Indeed, this limit exists on a dense subspace, and the composed automorphisms are contractive. Note that α_t is a homomorphism and $\alpha_t(a)$, $0 < t \leqq 1$, is a continuous path in A. A simple approximation argument shows that if $a = c + r$ with c in C and r in $\operatorname{rad} A$, then $\alpha_0(a) = \lim_{t \to 0} \alpha_t(a) = c$. Thus the idempotents $p^t = (\alpha_t(p_{ij}))$ form a path with the desired properties.

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