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LIMIT SETS OF PLURISUBHARMONIC FUNCTIONS

LARS HÖRMANDER and RAGNAR SIGURDSSON[1]

0. Introduction.

In this paper we give a complete description of limit sets of plurisubharmonic functions in $\mathbf{C}^n$ of finite order and finite type. The limit sets, $L_\infty(p)$, were first introduced by Azarin [1] as a growth characteristic of subharmonic functions p in $\mathbf{R}^N$. They are defined for $p \in \mathrm{SH}(\mathbf{R}^N)$, of order ϱ and finite type, as the set of all $q \in \mathrm{SH}(\mathbf{R}^N)$ that are limits in $\mathscr{D}'(\mathbf{R}^N)$ of sequences of the form $\{t_j^{-\varrho} p(t_j \cdot)\}$, where $t_j \to \infty$. Azarin investigated the basic properties of the limit sets, and gave a partial solution of the problem of characterizing those subsets of $\mathrm{SH}(\mathbf{R}^N)$ that are limit sets of some function. He proved, that if σ denotes the type of p, then $L_\infty(p)$ is a compact subset of $\{q \in \mathrm{SH}(\mathbf{R}^N); q(0) = 0, q(x) \leqq \sigma |x|^\varrho\}$, given the distribution topology, and that $L_\infty(p)$ is invariant under the action of the one parameter group

$$(0.1) \qquad T_t \colon L^1_{\mathrm{loc}}(\mathbf{R}^N) \to L^1_{\mathrm{loc}}(\mathbf{R}^N), \quad (T_t q)(x) = \frac{1}{t^\varrho} q(tx), \quad t > 0,$$

of continuous linear operators. Conversely, given $\varrho > 0$ not an integer, and a compact T invariant subset M of $\mathrm{SH}(\mathbf{R}^N)$, then there exists a function $p \in \mathrm{SH}(\mathbf{R}^N)$ of order ϱ and finite type, such that M is contained in $L_\infty(p)$, which in turn is contained in the union of all line segments with end points in M.

Later, Azarin and Giner [2] characterized limit sets that are the closure of one orbit, $M = \overline{\{T_t q; t > 0\}}$ for some $q \in \mathrm{SH}(\mathbf{R}^N)$, and gave examples of compact, connected, T invariant subsets of $\mathrm{SH}(\mathbf{R}^N)$ that are not limit sets. In his thesis, Sigurdsson [6, section 1.2] improved Azarin's results to hold for every $\varrho > 0$, and generalized them to plurisubharmonic functions in $\mathbf{C}^n$. He also proved, that if $M \subset \mathrm{SH}(\mathbf{R}^N)$ is compact, connected and its elements are homogeneous of degree ϱ, then M is a limit set.

In this paper we only deal with plurisubharmonic functions in $\mathbf{C}^n$. Our proofs are immediately modified to hold for subharmonic functions. We let $\mathrm{PSH}(\mathbf{C}^n)$

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denote the set of all plurisubharmonic functions in $\mathbf{C}^n$. The topologies of $\mathscr{D}'(\mathbf{C}^n)$ and $L^1_{\mathrm{loc}}(\mathbf{C}^n)$ coincide on $\mathrm{PSH}(\mathbf{C}^n)$ and $\mathrm{PSH}(\mathbf{C}^n)$ is closed in this topology, hence a complete metrizable space. For a proof see Theorems 4.1.8 and 4.1.9 in Hörmander [5]. We let $T = \{T_t\}_{t>0}$ denote the one parameter group defined by (0.1), where $\mathbf{R}^N$ is replaced by $\mathbf{C}^n$.

Many of our results even hold for a flow Φ on a general metric space (X, d), that is, for a continuous mapping $\Phi: X \times \mathbf{R}_+ \to X$ satisfying $\Phi(\Phi(p, s), t) = \Phi(p, st)$ and $\Phi(p, 1) = p$ for all s and t in $\mathbf{R}_+$ and all p in X. We always write $\Phi_t p$ instead of $\Phi(p, t)$ and view $\Phi = \{\Phi_t\}_{t>0}$ as a one parameter group of continuous mappings on X. For any q in X we define the limit set of q at infinity $L_\infty(q)$ as the set of all limits of sequences of the form $\{\Phi_{t_j} q\}$, where $t_j \to \infty$. We define the limit set of q at the origin $L_0(q)$ similarly with $t_j \to 0$. The continuity of Φ_t implies that the sets $L_\infty(q)$ and $L_0(q)$ are Φ invariant, that is, $\Phi_t L_\infty(q) = L_\infty(q)$ and $\Phi_t L_0(q)$ for all $t > 0$. If p and q are in X, $\varepsilon > 0$ and $\omega > 1$, then an (ε, ω)-chain from p to q is a finite sequence $p_0 = p, p_1 \ldots, p_N = q$, satisfying $d(\Phi_{t_j} p_j, p_{j+1}) < \varepsilon$ for some $t_j \geqq \omega$.

DEFINITION 0.1. *Let Φ be a flow on a metric space (X, d), and let M be a Φ invariant subset of X. We say that Φ is chain recurrent on M, if for every q in M, $\varepsilon > 0$ and $\omega > 1$, there exists an (ε, ω)-chain in M from q to q.*

In Theorem 1.1 we shall give a number of properties equivalent to chain recurrence which are really more illuminating in connection with the main result of this paper:

THEOREM 0.2. *Let ϱ be a positive real number and let M be a compact, connected and Tinvariant subset of* $\mathrm{PSH}(\mathbf{C}^n)$. *Then $M = L_\infty(p)$ for some $p \in \mathrm{PSH}(\mathbf{C}^n)$, of order ϱ and finite type, if and only if T is chain recurrent on M.*

By Sigurdsson [6, Theorem 1.3.1] the function p in the theorem can be chosen of the form $p = \log|f|$, where f is an entire analytic function.

In section 1 we begin by studying the topological properties of forward orbits and limit sets of functions in $\mathrm{PSH}(\mathbf{C}^n)$ of order ϱ. Then we study chain recurrent flows and prove the necessity in Theorem 0.2. We end the section by proving that the theorem contains the results of Azarin [1] and Azarin and Giner [2]. In section 2 we study regularizations of plurisubharmonic functions in $\mathbf{C}^n$, and finally, in section 3 we complete the proof of the theorem.

1. Chain recurrent flows.

Let us begin by reviewing the definitions of order and type. A function $p \in \mathrm{PSH}(\mathbf{C}^n)$ is said to be of finite order if there exist positive real numbers τ, σ and ϱ such that

$$p(z) \leqq \tau + \sigma|z|^\varrho, \quad z \in \mathbf{C}^n. \tag{1.1}$$

The order ϱ_p of p is then defined as the infimum over ϱ such that (1.1) holds for some τ and σ. The function p is said to be of finite type if it is of finite order and (1.1) holds for some τ and σ with $\varrho = \varrho_p$. The type σ_p is then defined as the infimum over σ.

Let $p \in \mathrm{PSH}(\mathbf{C}^n)$ be of order ϱ and finite type. Then the mean value property implies

$$\int_{|z| \leqq \varepsilon} T_t p(z) d\lambda(z) \geqq \frac{1}{t^\varrho} \int_{|z| \leqq \varepsilon} p(z)\, d\lambda(z), \quad t \geqq 1. \tag{1.2}$$

Now Theorem 4.1.9 in Hörmander [5] gives that the forward orbit $\{T_t p; t \geqq 1\}$ is relatively compact in $\mathrm{PSH}(\mathbf{C}^n)$. Thus

$$L_\infty(p) = \bigcap_{N=1}^{\infty} \overline{\{T_t p; t \geqq N\}} \tag{1.3}$$

is compact and connected. If the backward orbit $\{T_t p; 0 < t \leqq 1\}$ is relatively compact, then $L_0(p)$ is also compact and connected, for

$$L_0(p) = \bigcap_{N=1}^{\infty} \overline{\{T_t p; 0 < t \leqq 1/N\}}. \tag{1.4}$$

If M is a compact T invariant subset of $\mathrm{PSH}(\mathbf{C}^n)$, then the semi-continuity and the invariance imply that $q(0) \geqq 0$ for all q in M. The mean value property gives

$$q(z) \leqq (\mathcal{M}_{\varepsilon|z|} q)(z) = |z|^\varrho (\mathcal{M}_\varepsilon(T_t q))(z/|z|), \quad q \in M,$$

where $t = |z|$ and $(\mathcal{M}_r q)(z)$ denotes the mean value of q over the ball with center z and radius r. Now the compactness implies that there exists $\sigma > 0$ such that $M \subset \{q \in \mathrm{PSH}(\mathbf{C}^n); q(0) = 0, q(z) \leqq \sigma |z|^\varrho\}$.

For the proof of Theorem 0.2 we need equivalent descriptions of chain recurrence:

THEOREM 1.1. *Let Φ be a flow on a metric space (X, d) and let M be a compact and Φ invariant subset of X. Then the following conditions are equivalent:*

(i) *M is connected and Φ is chain recurrent on M.*

(ii) *For every open proper subset U of M satisfying*

$$\Phi_t U \subset U, \quad 0 < t < 1, \tag{1.5}$$

the boundary ∂U of U in M contains a non-empty Φ invariant subset of M.

(iii) *For every closed proper subset K of M satisfying*

$$\Phi_t K \subset K, \quad t \geqq 1, \tag{1.6}$$

the boundary ∂K of K in M contains a non-empty Φ invariant subset of M.

(iv) *There does not exist any open proper subset V of M satisfying $\Phi_\tau \overline{V} \subset V$ for some $\tau > 1$.*

(v) *For every* $\varepsilon > 0, \omega > 1$ *and every pair of points* p *and* q *in* M, *there exists an* (ε, ω)*-chain in* M *from* p *to* q.

PROOF. (ii)$\Leftrightarrow$(iii) is obvious since (1.5) is equivalent to (1.6) if $K = M\backslash U$.

(i)$\Rightarrow$(iii). Let K be a closed proper subset of M satisfying (1.6) and assume that ∂K does not contain any non-empty Φ invariant set. Since M is connected ∂K is non-empty. Let W denote the interior of K in M. The continuity of Φ_t and (1.6) imply that $\Phi_t W \subset W$ for all $t > 1$, and if $\Phi_{\tau_p} p \in W$ for some $p \in \partial K$, and $\tau_p > 0$, then $\Phi_t q \in W$ for all $q \in \partial K$ in some neighborhood of p and all $t \geqq \tau_p$. By our assumption such a τ_p exists for all $p \in \partial K$, so the Borel-Lesbesgue lemma implies that there exists $\omega > 1$, such that $\Phi_t \mathrm{K} \subset \Phi_\omega \mathrm{K} \subset \mathrm{W}$ for all $t \geqq \omega$. We choose $\varepsilon > 0$ as the distance between ∂K and $\Phi_\omega K$. Then there does not exist any (ε, ω)-chain from a point p in ∂K to itself.

(iii)$\Rightarrow$(iv). Assume that there exists an open proper subset V of M satisfying $\Phi_\tau \overline{V} \subset V$ for some $\tau > 1$. Set $W = \bigcup_{1 \leqq \sigma \leqq \tau} \Phi_\sigma V$ and $K = \overline{W}$. Then W and K satisfy (1.6), W is open, and the compactness of the interval $[1, \tau]$ gives that $K = \bigcup_{1 \leqq \sigma \leqq \tau} \Phi_\sigma \overline{V}$. Hence $\Phi_\tau K \subset W$. If we can show that K is a proper subset of M, then this implies that ∂K does not contain any non-empty Φ invariant set, contradicting (iii).

Since V is a neighborhood of the compact set $\Phi_\tau \overline{V}$ we can find $\alpha > 1$ such that

$$\Phi_t \Phi_\tau \overline{V} \subset V \quad \text{if} \quad 1 \leqq t \leqq \alpha.$$

Hence

$$\Phi_{t^j} \overline{V} \subset \Phi_{\tau^{-j}} V \quad \text{if} \quad 1 \leqq t \leqq \alpha.$$

When j is so large that $\alpha^j > \tau$ it follows that $K \subset \Phi_{\tau^{-j}} V$, hence K is not equal to M.

(iv)$\Rightarrow$(v). Let $\varepsilon > 0, \omega > 1$ and let $p \in M$. Let V denote the set of all $q \in M$, such that there exists an (ε, ω)-chain from p to q. Then V is open, $\Phi_\omega p \in V$ and $\Phi_\omega \overline{V} \subset V$. By (iv) $V = M$.

(v)$\Rightarrow$(i). Assume that M is not connected. Then it can be written as the union of two non-empty disjoint sets A and B that are both open and closed. Since M is compact, the distance ε between A and B is positive. Then every (ε, ω)-chain starting at a point in A remains in A, contradicting (v). Hence M is connected. The last statement of (i) is contained in (v), so the theorem is proved.

REMARK. (i) The equivalence of (i) and (iv) in Theorem 1.1 was first proved by Franke and Selgrade [4]. See also Bowen [3] for similar results on discrete flows.

(ii) If K is a subset of M satisfying (1.6), then K always contains an invariant subset F, for we can choose F as $L_\infty(q)$ for any $q \in K$. The point in condition (iii) is thus that the invariant set is contained in the *boundary* of K.

The following proposition implies the necessity of chain recurrence in Theorem 0.2:

PROPOSITION 1.2. *Let Φ be a flow on a metric space X and let $M = L_\infty(p)$, for some p in X with relatively compact forward orbit. Then M is compact and Φ is chain recurrent on M.*

PROOF. Let U be an open proper subset of M satisfying (1.5) and let F be a Φ invariant subset of $K = M\setminus U$. If F intersects ∂U at r, then $L_0(r) \subset F \cap \partial U$, so ∂U contains an invariant set in this case. Now assume that F does not intersect ∂U. Let U_0 be an open subset of X such that $U_0 \cap M = U$ and $\overline{U_0} \cap M = \overline{U}$. We can for example choose U_0 as

$$U_0 = \bigcup_{q\in U} \{r \in X; d(q,r) < d(q,K)/2\}.$$

Thus $\overline{U_0}$ does not intersect F, and we can take a sequence of open neighborhoods $U_1, U_2, \dots$ of F in X, such that $\overline{U_j}$ are all disjoint with $\overline{U_0}$ and U_j decrease to F. Since $M = L_\infty(p)$ we can find intervals $a_j \leqq t \leqq b_j$ with $a_j \to \infty$, such that

$$\Phi_{a_j} p \in \partial U_j, \quad \Phi_{b_j} p \in \partial U_0, \quad \Phi_t p \notin \overline{U_0} \cup \overline{U_j}, \quad a_j < t < b_j.$$

Since the forward orbit of p is relatively compact, we may pass to a subsequence and assume that $\Phi_{a_j} p \to r \in F$, which implies that $\Phi_{ta_j} p \to \Phi_t r \in F$, uniformly for bounded t. This shows that $b_j/a_j \to \infty$. By passing again to a subsequence we may assume that $\Phi_{b_j} p \to q \in M \cap \partial U_0 = \partial U$. Since $\Phi_{tb_j} p \to \Phi_t q$ and $\Phi_{tb_j} p \notin U_0$ when $a_j/b_j < t < 1$, we obtain $\Phi_t q \notin U$ when $t \leqq 1$. Hence the whole backward orbit $\{\Phi_t q; 0 < t \leqq 1\}$ lies in ∂U, which must therefore contain the Φ invariant set $L_0(q)$. This completes the proof.

The following proposition shows that Theorem 0.2 is a generalization of the results of Azarin and Giner [2]:

PROPOSITION 1.3 *Let Φ be a flow on a metric space X, and assume that $M = \overline{\{\Phi_t q; t > 0\}} = L_0(q) \cup \{\Phi_t q; t > 0\} \cup L_\infty(q)$ is compact. Then Φ is chain recurrent on M if and only if*

$$L_0(q) \cap L_\infty(q) \neq \emptyset. \tag{1.7}$$

PROOF. Assume that (1.7) holds and let U be an open proper subset of M satisfying (1.5). Set $K = M\setminus U$.

i) If $L_0(q) \cap L_\infty(q)$ contains a point of U, then $\Phi_t q \in U$ for all $t > 0$ since U is open and (1.5) holds. Hence $\overline{U} = M$, so $K \subset \partial U$ and ∂U contains a non-empty Φ invariant subset.

ii) If $L_0(q) \cap L_\infty(q)$ contains no point of U, then this is a Φ invariant subset of ∂U, for $L_0(q) \subset \overline{U}$ by (1.5).

Assume now that $L_0(q) \cap L_\infty(q) = \emptyset$, then $\Phi_\tau q \neq \Phi_\sigma q$ if $\tau \neq \sigma$, and $\Phi_\tau q \notin L_0(q) \cup L_\infty(q)$ for $\tau > 0$. In fact, if $\Phi_\tau q = \Phi_\sigma q$, then $\lambda \mapsto \Phi_{\exp \lambda} q$ is periodic with period $\log(\tau/\sigma)$, and $L_0(q) = L_\infty(q) = \{\Phi_t q; t > 0\}$. Moreover, M is a disjoint union of $L_0(q)$, $\{\Phi_t q; t > 0\}$, and $L_\infty(q)$. In fact, if $\Phi_\tau q \in L_0(q)$, say, then $\Phi_t q \in L_0(q)$ for $t > 0$, so $L_\infty(q) \subset L_0(q)$, which is a contradiction. Hence

$$U = L_0(q) \cup \{\Phi_t q; t < 1\}$$

is an open proper subset of M satisfying (1.5). But $\partial U = \{q\}$, which is not invariant. By Theorem 1.1, Φ is not chain recurrent on M, and the proof is complete.

Next we shall prove that Theorem 0.2 also contains the results of Azarin [1].

PROPOSITION 1.4. *Let X be a compact T invariant subset of* $\mathrm{PSH}(\mathbf{C}^n)$, *and let M denote the union of all line segments with endpoints in X. Then M is connected and T is chain recurrent on M.*

PROOF. Let U be an open proper subset of M satisfying (1.5). We choose $p \in U$ and q in a T invariant subset of $K = M \setminus U$. Since M is the union of all line segments with endpoints in X, we can choose q_1 and q_2 in X, such that the line segments with endpoints $\{p, q_1\}$ and $\{q_2, q\}$ are contained in M. Set $q_0 = p$ and $q_3 = q$ and define a continuous path $[0,3] \ni \theta \mapsto q_\theta \in M$, consisting of the three line segments

$$q_\theta = (j + 1 - \theta)q_j + (\theta - j)q_{j+1}, \qquad \theta \in [j, j+1], \quad j = 0, 1, 2.$$

Now M is T invariant, so for each t the continuous path $[0,3] \ni \theta \mapsto T_t q_\theta$ lies in M. If $t \in (0,1)$, then its initial point $T_t p$ is in U and its endpoint $T_t q$ is in K. For each $t \in (0,1)$ we set

$$\theta(t) = \min\{\theta \in [0,3]; T_t q_\theta \in K\}.$$

Then $\theta(t) > 0$, $T_t q_{\theta(t)} \in \partial U$, and (1.5) implies that $(0,1) \ni t \mapsto \theta(t)$ is a decreasing function. Hence the limit $\theta(0) = \lim_{t\to 0} \theta(t)$ exists and is positive. Set $r = q_{\theta(0)}$.

We claim that $L_0(r) \subset \partial U$. If $\theta(0) \in (j, j+1]$, where $j = 0, 1$ or 2, then $\theta(t) \in (j, j+1]$ if t is sufficiently small, and

$$T_t r = T_t q_{\theta(t)} + (\theta(t) - \theta(0))T_t q_j + (\theta(0) - \theta(t))T_t q_{j+1}.$$

The first term in the right hand side lies in ∂U. The set X is compact and T invariant so the other terms tend to 0, in the sense of distributions, as t tends to 0. Hence $L_0(r) \subset \partial U$. The proof is complete.

REMARK. With a similar proof it follows, that if M is a compact T invariant subset of $\mathrm{PSH}(\mathbf{C}^n)$, and each pair of its points can be joined by a polygonal path, then T is chain recurrent on M.

We conclude this section by preparing for the proof of sufficiency in Theorem 0.2:

LEMMA 1.5. *Let Φ be a flow on a metric space (X,d). Let M be a compact, connected and Φ invariant subset of X and assume that Φ is chain recurrent on M. Let $\{q_j\}$ be a sequence in M. Then there exist sequences $\{\alpha_\nu\}$ and $\{\omega_\nu\}$ of positive real numbers and a sequence $\{p_\nu\}$ in M having $\{q_j\}$ as a subsequence, such that*

$$\alpha_\nu \leqq 1, \quad \alpha_\nu \to 0, \quad \omega_\nu > 2, \quad \omega_\nu \to \infty \tag{1.8}$$

and

$$d(\Phi_{\omega_\nu} p_\nu, \Phi_{\alpha_{\nu+1}} p_{\nu+1}) \to 0 \quad as \quad \nu \to \infty. \tag{1.9}$$

PROOF. In addition to $\{\alpha_\nu\}$, $\{\omega_\nu\}$ and $\{p_\nu\}$ we define, by induction, a sequence $\{\varepsilon_\nu\}$ of positive real numbers, tending to zero, and an increasing sequence $\{\nu_j\}$ of positive integers, such that $p_{\nu_j} = q_j$ and $d(\Phi_{\omega_\nu} p_\nu, \Phi_{\alpha_{\nu+1}} p_{\nu+1}) \leqq \varepsilon_\nu$, $\nu = 1, 2, \ldots$. We begin by setting $\alpha_1 = \varepsilon_1 = \nu_1 = 1$, $\omega_1 = 5$ and $p_1 = q_1$. Assume now that α_ν, ε_ν, ω_ν and p_ν have been chosen for $\nu = 1, 2, \ldots, \nu_j$. Set $\alpha = \alpha_{\nu_j}/2$, $\varepsilon = \varepsilon_{\nu_j}/2$ and $\omega = \omega_{\nu_j}$. By Theorem 1.1 there exists a sequence $r_0 = \Phi_\omega q_j, r_1, \ldots, r_m = \Phi_\alpha q_{j+1}$ such that $d(\Phi_{t_k} r_k, r_{k+1}) < \varepsilon$ for $k = 0, \ldots, m-1$, where $t_k \geqq \omega$. Now we set $\nu_{j+1} = \nu_j + m + 1$. For $\nu = \nu_j + k + 1$, $k = 0, \ldots, m-1$, we set $\alpha_\nu = 1/\sqrt{t_k}$, $\varepsilon_\nu = \varepsilon, \omega_\nu = \sqrt{t_k}, p_\nu = \Phi_{\sqrt{t_k}} r_k$, and finally, for $\nu = \nu_{j+1}$ we set $\alpha_\nu = \alpha$, $\varepsilon_\nu = \varepsilon$, $\omega_\nu = 2\omega$, $p_\nu = q_{j+1}$. With this definition, (1.8) and (1.9) follow. The proof is complete.

2. Regularization of plurisubharmonic functions.

In our construction of plurisubharmonic functions with prescribed limit sets we need to be able to approximate general plurisubharmonic functions by smooth ones. In this section we introduce a certain regularization operator for that purpose and study its properties. The results are improvements of those of Sigurdsson [6, Section 1.2].

Let $\mathrm{GL}(n, \mathbf{C})$ be the group of invertible $n \times n$ matrices with complex entries. We regard $\mathrm{GL}(n, \mathbf{C})$ as a subset of the space $\mathbf{C}^{n \times n}$ of all $n \times n$ matrices and shall use the Lebesgue measure $d\lambda$ in $\mathbf{C}^{n \times n}$ when we integrate over $\mathrm{GL}(n, \mathbf{C})$ although the Haar measure might be more natural. By $d\lambda$ we shall also denote the Lebesgue measure in $\mathbf{C}^n$.

LEMMA 2.1. *Let $f \in L^1_{\mathrm{loc}}(\mathbf{C}^n \setminus \{0\})$ and let $\psi \in C_0^\infty(\mathrm{GL}(n, \mathbf{C}))$. Define $R_\psi f$ by*

$$R_\psi f(z) = \int_{\mathrm{GL}(n,\mathbf{C})} f(Az)\psi(A)\, d\lambda(A), \quad z \in \mathbf{C}^n \setminus \{0\}. \tag{2.1}$$

i) *If $\psi \geqq 0$ and $\int \psi\, d\lambda = 1$, then*

$$R_\psi f - f \to 0 \quad in \quad L^1_{\mathrm{loc}}(\mathbf{C}^n \setminus \{0\}) \quad \text{as} \quad \operatorname{supp} \psi \to \{I\}, \tag{2.2}$$

where I denotes the identity matrix. The convergence is uniform for f in a compact subset of $L^1_{\mathrm{loc}}(\mathbf{C}^n\setminus\{0\})$.

ii) *$R_\psi f\in C^\infty(\mathbf{C}^n\setminus\{0\})$. If K and U are compact subsets of* $\mathbf{C}^n\setminus\{0\}$ and $\mathrm{GL}(n,\mathbf{C})$ *respectively, with* $\operatorname{supp}\psi\subset U$, *and k is a positive integer, then*

$$\sum_{|\alpha|\leqq k}\sup_K|\partial^\alpha R_\psi f|\leqq C_{k,K,U}\sum_{|\alpha|\leqq k}\sup|\partial^\alpha\psi|\int_{UK}|f(z)|\,d\lambda(z). \tag{2.3}$$

iii) *If $f\in L^1_{\mathrm{loc}}(\mathbf{C}^n)$ then $R_\psi f\in L^1_{\mathrm{loc}}(\mathbf{C}^n)$,* and (2.2) *holds in* $\mathbf{C}^n$ *if $\psi\geqq 0$ and $\int\psi\,d\lambda=1$; if $f\in\mathrm{PSH}(\mathbf{C}^n)$ then $R_\psi f\in\mathrm{PSH}(\mathbf{C}^n)$.*

PROOF. If $z\in\mathbf{C}^n\setminus\{0\}$ then the map $\mathrm{GL}(n,\mathbf{C})\ \ A\mapsto Az\in\mathbf{C}^n\setminus\{0\}$ has surjective differential, so the pullback $A\mapsto f(Az)$ is in $L^1_{\mathrm{loc}}(\mathrm{GL}(n,\mathbf{C}))$, which makes the integral (2.1) well defined. For any compact set $K\subset\mathbf{C}^n$ we have

$$\int_K|R_\psi f(z)|\,d\lambda(z)\leqq\int_K\int|f(Az)\psi(A)|\,d\lambda(A)\,d\lambda(z)$$

$$\leqq\int|\det A|^{-2}|\psi(A)|\,d\lambda(A)\int_{UK}|f(z)|\,d\lambda(z).$$

If $0\notin K$ then $0\notin UK$. Now i) is obvious with uniform convergence in $\mathbf{C}^n$ if $f\in C_0^\infty(\mathbf{C}^n\setminus\{0\})$. Since this is a dense set in $L^1_{\mathrm{loc}}(\mathbf{C}^n\setminus\{0\})$ and in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$, we obtain i) and the analogue for $\mathbf{C}^n$ stated in iii).

Next we shall prove that for given $z_0\in\mathbf{C}^n\setminus\{0\}$ the estimate (2.3) is valid for some neighborhood K of z_0 when the compact set $U\subset\mathrm{GL}(n,\mathbf{C})$ is given, with $\operatorname{supp}\psi\subset U$, and $f\in C_0^\infty(\mathbf{C}^n\setminus\{0\})$. Since the map $A\mapsto Az_0$ is surjective, we can choose an (affine) right inverse B from a neighborhood K of z_0 to a neighborhood of the identity in $\mathrm{GL}(n,\mathbf{C})$. Thus $z=B(z)z_0$ if $z\in K$, so

$$R_\psi f(z)=\int f(AB(z)z_0)\psi(A)\,d\lambda(A)$$

$$=\int f(Az_0)\psi(AB(z)^{-1})|\det B(z)|^{-2n}\,d\lambda(A).$$

Here we have used that the map $A\mapsto AT=\{\Sigma_j a_{ij}t_{jk}\}$ in $\mathbf{C}^{n\times n}$ has determinant $(\det T)^n$ since it acts as the transpose of T on each row of A. For $z\in K$ we can now estimate the left-hand side of (2.3) by

$$C_{k,K}\sum_{|\alpha|\leqq k}\sup|\partial^\alpha\psi|\int_{UB(K)}|f(Az_0)|\,d\lambda(A).$$

The integral can be estimated by the L^1 norm of f in $UB(K)z_0$ since the linear map $\mathbf{C}^{n\times n}\ \ A\mapsto Az_0$ is surjective. This proves (2.3) when K is sufficiently small. By the Borel-Lebesgue lemma we conclude that (2.3) holds for any compact set

$K \subset \mathbf{C}^n \setminus \{0\}$, which completes the proof of the lemma apart from obvious statements.

In what follows we shall denote by ψ a fixed function in $C_0^\infty(\mathbf{C}^{n\times n})$ satisfying $\psi \geqq 0$ and $\int \psi \, d\lambda = 1$. If $\|A\| \leqq 1/2$ in $\operatorname{supp}\psi$ then the support of

$$\psi_\delta(A) = \delta^{-N}\psi((A-I)/\delta), \quad N = 2n^2, \tag{2.4}$$

is contained in a fixed compact subset of $\mathrm{GL}(n, \mathbf{C})$ when $0 < \delta \leqq 1$, and

$$\sum_{|\alpha| \leqq k} \sup |\partial^\alpha \psi_\delta| \leqq C_k \delta^{-N-k}, \quad 0 < \delta \leqq 1 \tag{2.5}$$

$$\sum_{|\alpha| \leqq k} \sup |\partial^\alpha(\psi_{\delta_1} - \psi_{\delta_2})| \leqq C_k \delta_2^{-N-k-1}(\delta_1 - \delta_2), \quad 0 < \delta_2 \leqq \delta_1 \leqq 1. \tag{2.6}$$

(2.5) is obvious, and (2.6) follows if we note that the derivative of ψ_δ with respect to δ is $\delta^{-1}(-N\psi_\delta - \tilde{\psi}_\delta)$ where $\tilde{\psi}$ is the radial derivative of ψ.

We are now ready to prove the main lemma of this section:

LEMMA 2.2. *Let $q \in \mathrm{PSH}(\mathbf{C}^n)$ and define $R(\delta)q$ by* (2.1) *with ψ replaced by ψ_δ, $0 < \delta \leqq 1$. Then*

i) *$R(\delta)q \in \mathrm{PSH}(\mathbf{C}^n) \cap C^\infty(\mathbf{C}^n \setminus \{0\})$; if q is of order $\leqq \varrho$ and finite type, then $R(\delta)q$ is also of order $\leqq \varrho$ and finite type. We have*

$$R(\delta)q - q \to 0 \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbf{C}^n) \quad \text{as} \quad \delta \to 0.$$

The convergence is uniform for q in a compact subset of $PSH(\mathbf{C}^n)$.

ii) *Let $\{p'_\nu\}$ and $\{p''_\nu\}$ be bounded sequences in $PSH(\mathbf{C}^n)$ such that $p'_\nu - p''_\nu \to 0$ in $L^1_{\mathrm{loc}}(\mathbf{C}^n \setminus \{0\})$. Then there exists a sequence $\{\delta_\nu\}$ in* $(0, 1]$ *such that $\delta_\nu \to 0$ and*

$$R(\delta_\nu)p'_\nu - R(\delta_{\nu+1})p''_\nu \to 0 \quad \text{in} \quad C^\infty(\mathbf{C}^n \setminus \{0\}) \quad \text{as} \quad \nu \to \infty.$$

PROOF. The first statement is contained in Lemma 2.1. To prove the second we choose an increasing sequence K_ν of compact subsets of $\mathbf{C}^n \setminus \{0\}$ containing any such set for large ν, and a sequence $\varepsilon_\nu \downarrow 0$ such that the L^1 norm of $p'_\nu - p''_\nu$ over K_ν is $\leqq \varepsilon_\nu$. Writing

$$R(\delta_\nu)p'_\nu - R(\delta_{\nu+1})p''_\nu = R(\delta_\nu)(p'_\nu - p''_\nu) + (R(\delta_\nu) - R(\delta_{\nu+1}))p''_\nu$$

we conclude from (2.3), (2.5) and (2.6) that ii) holds if $\delta_\nu \downarrow 0$ and for every integer $k \geqq 0$

$$\delta_\nu^{-k-N}\varepsilon_\nu \to 0 \quad \text{as } \nu \to \infty, \tag{2.7}$$

$$\delta_{\nu+1}^{-k-N-1}(\delta_\nu - \delta_{\nu+1}) \to 0 \quad \text{as } \nu \to \infty. \tag{2.8}$$

To satisfy these conditions we first introduce another sequence ε_ν^* by

$$1/\varepsilon_\nu^* = \min_{1 \leqq \mu \leqq \nu} (\nu - \mu + 1/\varepsilon_\mu).$$

Then

$$1/\varepsilon_\nu^* \leqq 1/\varepsilon_\nu, \quad 1/\varepsilon_\nu^* \leqq 1/\varepsilon_{\nu+1}^* \leqq 1 + 1/\varepsilon_\nu^*,$$

and $\varepsilon_\nu^* \to 0$ as $\nu \to \infty$. If we choose $\delta_\nu^{-1} = \log(2\varepsilon_0^*/\varepsilon_\nu^*)$ then

$$1/\delta_{\nu+1} - 1/\delta_\nu = \log(\varepsilon_\nu^*/\varepsilon_{\nu+1}^*) \leqq \log(1 + \varepsilon_\nu^*) \leqq \varepsilon_\nu^*,$$

so (2.7), (2.8) follow at once even with ε_ν replaced by ε_ν^*. This completes the proof.

The final result of this section is:

LEMMA 2.3. *Let ϱ be a positive real number and γ be a positive continuous function on $\mathbf{R}_+$, such that $\gamma(r) \to 0$ as $r \to \infty$. Then one can find $\Phi \in \mathrm{PSH}(\mathbf{C}^n) \cap C^\infty(\mathbf{C}^n \backslash \{0\})$, such that*

$$\sum_{jk} \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k \geqq \gamma(|z|)|z|^{\varrho - 2}|w|^2, \quad w \in \mathbf{C}^n, |z| \geqq 1,$$

and $T_t\Phi(z)$ decreases to zero as $t \to \infty$ for all $z \in \mathbf{C}^n$.

For a proof see Sigurdsson [6, Lemma 1.2.3].

3. Plurisubharmonic functions with prescribed limit sets.

In this section we complete the proof of Theorem 0.2 by constructing a function p in $\mathrm{PSH}(\mathbf{C}^n)$, of order ϱ and finite type, with $L_\infty(p)$ equal to a given subset M of $\mathrm{PSH}(\mathbf{C}^n)$. It will be of the form

$$p = \sum_{\nu=0}^{\infty} \phi_\nu(T_{1/\tau_\nu} r_\nu) + \Phi, \tag{3.1}$$

where $\{\phi_\nu\}$ is a partition of unity in $\mathbf{C}^n$, $\{r_\nu\}$ is a regularization of a certain sequence $\{p_\nu\}$ in M, $\{\tau_\nu\}$ is a sequence of positive real numbers, and Φ is a strictly plurisubharmonic function, which does not affect the asymptotic behavior of the sum, but ensures the plurisubharmonicity.

We begin by defining the partition of unity $\{\phi_\nu\}$. Let $\{\alpha_\nu\}$ and $\{\omega_\nu\}$ be sequences of positive real numbers satisfying (1.8). We then define the sequences $\{\tau_\nu\}$ and $\{\sigma_\nu\}$ by

$$\tau_0 = 1, \quad \sigma_\nu = \omega_\nu \tau_\nu, \quad \tau_{\nu+1} = \sigma_\nu/\alpha_{\nu+1}. \tag{3.2}$$

Choose $\chi \in C^\infty(\mathbf{R})$ satisfying

$$0 \leqq \chi \leqq 1, \quad \chi(x) = 1 \text{ if } x \leqq 1, \quad \chi(x) = 0 \text{ if } x \geqq 2, \tag{3.3}$$

and define $\chi_\nu \in C^\infty(\mathbf{R})$ by

(3.4) $$\chi_\nu(x) = \chi(x/\sigma_\nu), \quad x \in \mathbf{R}.$$

We then define $\{\phi_\nu\}$ by

(3.5) $$\phi_0(z) = \chi_0(|z|), \quad \phi_\nu(z) = \chi_\nu(|z|) - \chi_{\nu-1}(|z|), \quad \nu = 1, 2, 3, \ldots$$

By (3.3) and (3.4) we have

(3.6) $$\phi_\nu \in C_0^\infty(\mathbf{C}^n), \quad 0 \leqq \phi_\nu \leqq 1, \quad \sum \phi_\nu = 1, \quad |D^\beta \phi_\nu| \leqq C_\beta |z|^{-|\beta|},$$

where C_β is a positive constant and β is any multi-index. Since $\alpha_\nu \leqq 1$ and $\omega_\nu > 2$ we have

(3.7) $$\operatorname{supp} \phi_\nu \subset \{z \in \mathbf{C}^n; \sigma_{\nu-1} \leqq |z| \leqq 2\sigma_\nu\},$$

(3.8) $$\phi_\nu(z) = 1 \text{ if } 2\sigma_{\nu-1} \leqq |z| \leqq \sigma_\nu$$

and

(3.9) $$\operatorname{supp} \phi_\nu \cap \operatorname{supp} \phi_\mu = \emptyset \text{ if } |\nu - \mu| > 1.$$

Let $t_j \to \infty$, $\tau_{\nu_j} \leqq t_j < \tau_{\nu_j+1}$, and let K be any compact subset of $\mathbf{C}^n \setminus \{0\}$. Since $\alpha_\nu \to 0$ and $\omega_\nu \to \infty$ we have

(3.10) $$\phi_{\nu_j}(t_j z) + \phi_{\nu_j+1}(t_j z) = 1 \quad \text{for } z \in K \text{ and } j \text{ large}.$$

If $t_j = \tau_{\nu_j}$, or even more generally $t_j/\sigma_{\nu_j} \to 0$, then

(3.11) $$\phi_{\nu_j}(t_j z) = 1, \quad \text{for } z \in K \text{ and } j \text{ large}.$$

If, on the other hand, $t_j/\sigma_{\nu_j} \to +\infty$, then

(3.12) $$\phi_{\nu_j+1}(t_j z) = 1, \quad \text{for } z \in K \text{ and } j \text{ large}.$$

PROOF OF THE SUFFICIENCY IN THEOREM 0.2. We let d be some metric defining the topology on PSH$(\mathbf{C}^n)$. Let $\{q_j\}$ be a sequence in M with every element repeated infinitely often and forming a dense subset of M. We choose the sequences $\{\alpha_\nu\}$, $\{\omega_\nu\}$ and $\{p_\nu\}$ as in Lemma 1.5 and define the sequences $\{\tau_\nu\}$, $\{\sigma_\nu\}$ and $\{\phi_\nu\}$ by (3.2) and (3.5). We set $p'_\nu = T_{\omega_\nu} p_\nu$, $p''_\nu = T_{\alpha_{\nu+1}} p_{\nu+1}$, choose a corresponding sequence δ_ν according to Lemma 2.2 (ii), and set $r_\nu = R(\delta_\nu) p_\nu$. The function p will be of the form (3.1). In order to estimate the Levi form of the sum in (3.1), we set $s_\nu = T_{1/\tau_\nu} r_\nu$ and $s = \sum \phi_\nu s_\nu$. By (3.8) the Levi form of s is non-negative in $\{z \in \mathbf{C}^n; 2\sigma_{\nu-1} \leqq |z| \leqq \sigma_\nu\}$. In the set $\{z \in \mathbf{C}^n; \sigma_\nu < |z| < 2\sigma_\nu\}$ we have $\phi_\nu + \phi_{\nu+1} = 1$, so the Levi form of s is given there by the formula

(3.13) $$\sum_{jk} \frac{\partial^2 s}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k = \phi_\nu \sum_{jk} \frac{\partial^2 s_\nu}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k + \phi_{\nu+1} \sum_{jk} \frac{\partial^2 s_{\nu+1}}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k$$

$$+ 2\,\mathrm{Re}\,(\langle\frac{\partial\phi_\nu}{\partial z}, w\rangle\langle\frac{\partial}{\partial\bar z}(s_\nu - s_{\nu+1}), \bar w\rangle)$$

$$+ \sum_{jk}\frac{\partial^2\phi_\nu}{\partial z_j\partial\bar z_k} w_j\bar w_k(s_\nu - s_{\nu+1}).$$

The first two terms in the right hand side are non-negative. If $\sigma_\nu < |z| < 2\sigma_\nu$, then $z = \sigma_\nu\zeta$ for some ζ with $1 < |\zeta| < 2$, so

$$s_\nu(z) = \sigma_\nu^\varrho(T_{\sigma_\nu} T_{1/\tau_\nu} r_\nu)(\zeta) = \sigma_\nu^\varrho(T_{\omega_\nu} r_\nu)(\zeta)$$

and

$$s_{\nu+1}(z) = \sigma_\nu^\varrho(T_{\sigma_\nu} T_{1/\tau_{\nu+1}} r_{\nu+1})(\zeta) = \sigma_\nu^\varrho(T_{\alpha_{\nu+1}} r_{\nu+1})(\zeta).$$

This gives

$$(s_\nu - s_{\nu+1})(z) = \sigma_\nu^\varrho((R(\delta_\nu)T_{\omega_\nu}p_\nu) - (R(\delta_{\nu+1})T_{\alpha_{\nu+1}}p_{\nu+1}))(\zeta)$$

and

$$\langle\frac{\partial}{\partial\bar z}(s_\nu - s_{\nu+1})(z), \bar w\rangle = \sigma_\nu^{\varrho-1}(\langle\frac{\partial}{\partial\bar\zeta}((R(\delta_\nu)T_{\omega_\nu}p_\nu) - (R(\delta_{\nu+1})T_{\alpha_{\nu+1}}p_{\nu+1}))(\zeta), \bar w\rangle).$$

Now Lemma 2.2 ii) and (3.6) imply that the last two terms in the right hand side of (3.13) can be estimated by

$$\gamma_\nu|z|^{\varrho-2}|w|^2 \quad \text{if} \quad \sigma_\nu < |z| < 2\sigma_\nu,$$

where $\{\gamma_\nu\}$ is a sequence of positive real numbers tending to 0. We let γ be a positive continuous function on $\bar{\mathbf{R}}_+$, such that $\gamma(r) \to 0$ as $r \to +\infty$, and

$$\gamma(r) \geqq \gamma_\nu \quad \text{if} \quad \sigma_\nu < r < 2\sigma_\nu.$$

Then we choose Φ satisfying the conditions in Lemma 2.3. If p is defined by (3.1), then $p \in \mathrm{PSH}(\mathbf{C}^n)$, is of order ϱ and finite type.

It now remains to prove that $L_\infty(p) = M$. For proving $M \subset L_\infty(p)$, it is sufficient to show that every element q in the sequence $\{q_j\}$ lies in $L_\infty(p)$. In fact, $\{q_j\}$ is dense in M and $L_\infty(p)$ is closed. Since $\{q_j\}$ is a subsequence of $\{p_\nu\}$ and q appears infinitely often in $\{q_j\}$, we have $q = p_{\nu_j}$ for some subsequence. We set $t_j = \tau_{\nu_j}$. If K is a compact subset of $\mathbf{C}^n\setminus\{0\}$, then (3.11) gives

$$(T_{t_j}p)(z) = (T_{t_j}R(\delta_{\nu_j})T_{1/\tau_{\nu_j}}p_{\nu_j})(z) + (T_{t_j}\Phi)(z)$$
$$= (R(\delta_{\nu_j})q)(z) + (T_{t_j}\Phi)(z) \quad \text{for } z \in K \text{ and } j \text{ large.}$$

Now Lemma 2.2 i) and Lemma 2.3 give that $T_{t_j}p \to q$ in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$, so $q \in L_\infty(p)$.

For proving $L_\infty(p) \subset M$, we let $t_j \to \infty$ and assume that $T_{t_j}p \to q$ in $L^1_{\mathrm{loc}}(\mathbf{C}^n)$. Let $\tau_{\nu_j} \leqq t_j < \tau_{\nu_j+1}$. We treat separately each of the three cases where $t_j/\sigma_{\nu_j} \to 0$,

$t_j/\sigma_{\nu_j} \to \infty$ and that $\{t_j/\sigma_{\nu_j}\}$ converges to a positive limit s. One of these must occur after passage to a subsequence.

Assume that $t_j/\sigma_{\nu_j} \to 0$. Then (3.11) gives that for any compact subset K of $\mathbf{C}^n \setminus \{0\}$

$$(3.15) \quad (T_{t_j}p)(z) = (R(\delta_{\nu_j})T_{t_j/\tau_{\nu_j}}p_{\nu_j})(z) + (T_{t_j}\Phi)(z) \quad \text{for } z \in K \text{ and } j \text{ large.}$$

Since M is compact and T invariant, $\{T_{t_j/\tau_{\nu_j}}p_{\nu_j}\}$ has a convergent subsequence with limit in M. We conclude from Lemma 2.2 and Lemma 2.3 that this limit is equal to q. Hence $q \in M$. If $t_j/\sigma_{\nu_j} \to +\infty$, then (3.12) gives that (3.15) holds with ν_j replaced by $\nu_j + 1$, and we conclude as in the previous case that $q \in M$.

Assume now that $s_j := t_j/\sigma_{\nu_j} \to s \in \mathbf{R}_+$. By (3.10) we have

$$(T_{t_j}p)(z) = (R(\delta_{\nu_j})T_{t_j/\tau_{\nu_j}}p_{\nu_j})(z) + (T_{t_j}\Phi)(z) -$$
$$- \phi_{\nu_j+1}(t_jz)\,[(T_{s_j}((R(\delta_{\nu_j})T_{\omega_{\nu_j}}p_{\nu_j}) - (R(\delta_{\nu_j+1})T_{\alpha_{\nu_j+1}}p_{\nu_j+1})))(z)]$$

for $z \in K$ and j large. The continuity of $L^1_{\mathrm{loc}}(\mathbf{C}^n) \times \mathbf{R}_+ \ni (f,t) \mapsto T_tf$, and Lemma 2.2 ii) now give that the last term tends to zero in $L^1(K)$. The same reasoning as in the previous cases then gives $q \in M$. The proof is complete.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LUND
LUND
SWEDEN

SCIENCE INSTITUTE
UNIVERSITY OF ICELAND
REYKJAVIK
ICELAND