MATH. SCAND. 66 (1990), 5–11

THE KRULL-SCHMIDT THEOREM FOR CATEGORIES OF FINITELY GENERATED MODULES OVER VALUATION DOMAINS

PAOLO ZANARDO*

A central result in the theory of abelian groups is the existence, first proved by B. Jónsson in [4] (see especially [5]), of finite rank torsion-free abelian groups which admit non-isomorphic decompositions into indecomposable summands. This fact led Jónsson to introduce in [6] the notion of *quasi-isomorphism* (originally called almost isomorphism), proving that a finite rank torsion-free abelian group decomposes in a unique way, *up to quasi-isomorphism*, into strongly indecomposable summands ([6], Theorem 2.6.).

The categorical point of view is the following (see Ch. 7 of [1]; see also [11]): one considers the category A, whose objects are finite rank torsion-free abelian groups, and the morphisms (called *quasi-homomorphisms*) are defined, for M, $N \in A$, by $\mathrm{Hom}_A(M, N) = \mathbf{Q} \otimes \mathrm{Hom}_{\mathbf{Z}}(M, N)$. If M is *strongly indecomposable* (i.e. indecomposable in the category A), then $\mathrm{End}_A(M)$ *is a local ring*. Thus we can apply the Krull-Schmidt theorem for additive categories, obtaining Theorem 2.6. of [6].

In the present paper we show that a similar idea can be applied to finitely generated modules over a valuation domain R. For every prime ideal H of R, we consider a category $C(H)$; the objects are finitely generated R-modules *whose annihilators either contain H or are H-primary*, and the morphisms are defined, for X, $Y \in C(H)$, by $\mathrm{Hom}_{C(H)}(X, Y) = R_H \otimes \mathrm{Hom}_R(X, Y)$ (R_H denotes the localization of R at H). $C(H)$ turns out to be an additive category such that idempotents split in it. We prove that, if X is indecomposable in $C(H)$, then $\mathrm{End}_{C(H)}(X)$ is a local ring (Theorem 4); this yields the uniqueness of decomposition of objects in $C(H)$ into indecomposable summands, up to isomorphism in $C(H)$ (Theorem 7).

We note that our investigation is strongly motivated by an important result by P. Vámos [10], who first showed the existence of a finitely generated module over a valuation domain which admits two non-isomorphic decompositions into

* Lavoro eseguito con il contributo del M.P.I.
Received December 19, 1988

indecomposable summands; he actually gives conditions on valuation domains R for the existence of finitely generated R-modules with the above pathology (Theorem 20 and Corollary 21 of [10]). Vámos' results was conjectured mainly because of many similarities between the theories of finitely generated torsion modules and of finite rank torsion-free modules over valuation domains; these analogies are emphasized in [8], [9].

Results on uniqueness of decomposition into indecomposable summands for special classes of finitely generated R-modules can be found in [7], Theorem 12 and in [13], Theorems 18 and 19.

It is also worthy of note that, in Remark 9, we explain why we must consider R-modules with the above condition on annihilators instead of arbitrary finitely generated modules; this limitation is, in fact, necessary to ensure that $\mathrm{End}_{C(H)}(X)$ is local for each X indecomposable in $C(H)$.

In the sequel, R will denote a valuation domain, P its maximal ideal. For general references about valuation domains and their modules, in particular finitely generated modules, we refer to the book by Fuchs and Salce [3].

For a complete exposition on quasi-homomorphisms of finite rank torsion-free abelian groups we refer to the book by Arnold [1], Chapter 7.

Before confining ourselves to finitely generated modules, we begin our discussion in a more general context.

Let H be a prime ideal of R; let $C(H)$ be any subclass of R-mod, containing $\{0\}$; let us define, for X, $Y \in C(H)$,

$$\mathrm{Hom}_{C(H)}(X, Y) = R_H \otimes \mathrm{Hom}_R(X, Y);$$

$C(H)$ is a category with the obvious definition of composition of morphisms.

It is immediate to check that $\mathrm{Hom}_{C(H)}(X, Y) = \{1/r \otimes f\colon r \in R \setminus H, f \in \mathrm{Hom}_R(X, Y)\}$. The morphisms in $C(H)$ are called *$C(H)$-homomorphisms*. If X, $Y \in C(H)$ are isomorphic in $C(H)$, X and Y are said to be *$C(H)$-isomorphic*; we shall write $X \cong_{C(H)} Y$.

We shall need a characterization of $C(H)$-homomorphisms which are different from zero.

LEMMA 1. *Let $1/r \otimes f \in \mathrm{Hom}_{C(H)}(X, Y)$. Then $1/r \otimes f = 0$ if and only if there exists $a \in R \setminus H$ such that $af = 0$.*

PROOF. It is enough to prove that $af \neq 0$ for all $a \in R \setminus H$ implies $r(1/r \otimes f) = 1 \otimes f \neq 0$. Let us consider the R-module $Rf \leqq \mathrm{Hom}_R(X, Y)$. Since $af \neq 0$ for all $a \in R \setminus H$, we deduce that $H \geqq \mathrm{Ann}(f) = B$. Let $S = R \setminus H$, and let us consider the R_H-module $S^{-1}(R/B)$ (see Chapter 3 of [2]). Since $Rf \cong R/B$, from Prop. 3.5.

of [2] it follows

$$S^{-1}(R/B) \cong R_H \otimes Rf \leqq \mathrm{Hom}_{C(H)}(X, Y).$$

We conclude that $1 \otimes f \neq 0$ if and only if $1 + B \in S^{-1}(R/B) \setminus \{0\}$, and this last fact follows from $B \leqq H$.

The next Lemma 2 corresponds to Corollary 7.7. (a) of [1].

LEMMA 2. *Let X, $Y \in C(H)$; then X is $C(H)$-isomorphic to Y if and only if there exist $f \in \mathrm{Hom}_R(X, Y)$, $g \in \mathrm{Hom}_R(Y, X)$ and $a \in R \setminus H$ such that $fg = a \cdot \mathrm{id}_Y$, $gf = a \cdot \mathrm{id}_X$. In particular, $X \cong_{C(H)} \{0\}$ if and only if the annihilator of X properly contains H.*

PROOF. $X \cong_{C(H)} Y$ if and only if there exist $1/r \otimes f \in \mathrm{Hom}_{C(H)}(X, Y)$ and $1/s \otimes g \in \mathrm{Hom}_{C(H)}(Y, X)$ such that $(1/r \otimes f)(1/s \otimes g) = 1 \otimes \mathrm{id}_Y$ and $(1/s \otimes g)(1/r \otimes f) = 1 \otimes \mathrm{id}_X$, if and only if, in view of Lemma 1, $t(fg - rs \cdot \mathrm{id}_Y) = 0$ and $t(gf - rs \cdot \mathrm{id}_X) = 0$, for a suitable $t \in R \setminus H$; if we set $a = rst \in R \setminus \mathrm{H}$, we get the first assertion. The second assertion follows easily from the first one.

If every $X \in R$-mod is in $C(H)$, using Lemmas 1 and 2 and following "verbatim" the proof of Theorem 3.1. of [11], we see that $C(H)$ is the *quotient category* $(R\text{-mod})/A$ (see [11] for the definitions), where A is the subclass of R-mod consisting of R-modules whose annihilators properly contain H.

From now on $C(H)$ will denote *the class of finitely generated modules X, such that either* Ann X *contain H, or it is H-primary.* Note that, since R is a valuation domain, a finitely generated module Z is not in $C(H)$ if and only if there exists a prime ideal K of R such that $\mathrm{Ann}\, Z \leqq K < H$. If there is not possibility of confusion, we shall denote $C(H)$ simply by C.

The class C is closed for *pure* submodules and homomorphic images: if Y is a pure submodule of $X \in C$, then Y is finitely generated and $\mathrm{Ann}\, Y \geqq \mathrm{Ann}\, X$; moreover for all $f \in \mathrm{Hom}_R(X, Z)$, fX is finitely generated and $\mathrm{Ann}\,(fX) \geqq \mathrm{Ann}\, X$.

It is easy to verify that *C is an additive category* (C contains finite direct sums in view of Lemmas 1 and 2; cf. Example 7.6. (iii) of [1]).

We recall the definition of splitting of idempotents (see [1]): if A is an additive category, we say that *idempotents split in A* if for each idempotent $e \in \mathrm{End}_A(X)$, with $X \in A$, there exist $Y \in A$, $p \in \mathrm{Hom}_A(X, Y)$ and $q \in \mathrm{Hom}_A(Y, X)$ such that $e = qp$ and pq is the identity of $\mathrm{End}_A(Y)$.

LEMMA 3. *Idempotents split in C.*

PROOF. Let $X \in C$ and let $e = 1/r \otimes f$ be an idempotent of $\mathrm{End}_C(X)$. From $e^2 - e = 0$ it follows $1/r^2 \otimes (f^2 - rf) = 0$. In view of Lemma 1 we deduce that

$a(f^2 - rf) = 0$ for a suitable $a \in R \setminus H$. If we write e in the form $e = 1/ra \otimes af$, we get $e^2 - e = 1/r^2a^2 \otimes (a^2f^2 - ra^2f)$. Thus we can assume, without loss of generality, that $e = 1/r \otimes f$ and $f^2 = rf$. Let us consider the submodule fX of X; we have observed above that $fX \in C$; let $j: fX \to X$ be the canonical injection. Let us observe that $fj \in \mathrm{End}_R(fX)$ coincides with $r(\mathrm{id}_{fX})$. In fact, for all $x = f(x') \in fX$, we have: $fj(x) = f^2(x') = rf(x') = rx$. Moreover, trivially, $jf = f$. Set now $p = 1/r \otimes f \in \mathrm{Hom}_C(X, fX)$, $q = 1 \otimes j \in \mathrm{Hom}_C(fX, X)$. We have:

$$qp = (1 \otimes j)(1/r \otimes f) = 1/r \otimes jf = 1/r \otimes f = e,$$
$$pq = 1/r \otimes fj = 1/r \otimes r \cdot \mathrm{id}_{fX} = 1 \otimes \mathrm{id}_{fX}.$$

The desired conclusion follows.

An object $X \in C$ is said to be *C-indecomposable* if X is indecomposable in the category C.

THEOREM 4. *If $X \in C$ is C-indecomposable, then* $\mathrm{End}_C(X)$ *is a local ring.*

PROOF. It is enough to prove that, for an arbitrary element $1/r \otimes f \in \mathrm{End}_C(X)$, either $1/r \otimes f$ or $1 \otimes \mathrm{id}_X - 1/r \otimes f$ is a unit of $\mathrm{End}_C(X)$. We shall actually prove that either $1 \otimes f$ is a unit of $\mathrm{End}_C(X)$, or $1 \otimes f$ is *nilpotent*; hence the same will be true for $1/r \otimes f$, and the desired conclusion will follow. Since X is a finitely generated R-module and $f \in \mathrm{End}_R(X)$, by Prop. 2.4. of [2], there exist $a_0, a_1, \ldots, a_{n-1} \in R$ such that

$$f^n + a_{n-1}f^{n-1} + \ldots + a_1 f + a_0 = 0, \tag{1}$$

from which we also get the following relation in $\mathrm{End}_C(X)$:

$$(1 \otimes f)^n + a_{n-1}(1 \otimes f)^{n-1} + \ldots + a_1(1 \otimes f) + a_0(1 \otimes \mathrm{id}_X) = 0. \tag{2}$$

If now $a_0 \in R \setminus H$, since $\mathrm{End}_C(X)$ is an R_H-module, from (2) we get:

$$(1 \otimes f)(-1/a_0)((1 \otimes f)^{n-1} + \ldots + a_1) = 1 \otimes \mathrm{id}_X; \tag{3}$$

we conclude that $1 \otimes f$ is invertible, as desired. Thus we can assume that $a_0, a_1, \ldots, a_{r-1} \in H$ and $a_r \notin H$ for a suitable $r \leqq n$ (here we set $a_n = 1$). From (1) we get

$$f^n + \ldots + a_r f^r = a(b_{r-1}f^{r-1} + \ldots + b_1 f + b_0), \tag{4}$$

for a suitable $a \in H$ such that $-ab_i = a_i (i = 0, \ldots, r-1)$. Since $a \in H$ and $\cap_{n>0} a^n R$ is a prime ideal properly contained in H, the condition on $\mathrm{Ann}\, X$ implies that $a^m \in \mathrm{Ann}\, X$, for a suitable $m > 0$. Raising both members of (4) to the m-th power, we obtain:

$$f^{rm}(f^{(n-r)m} + \ldots + a_r^m) = 0. \tag{5}$$

Since $a_r^m \notin H$, the relation (5) can be written in the form

$$bf^h = f^{h+1}\, G(f), \tag{6}$$

where $G(x)$ is a suitable *monic* polynomial of $R[x]$, $h > 0$ and $b \notin H$. From (6) we at once get: $b^h f^h = f^{2h} G(f)^h$. Set $g = f^h G(f)^h$ and $\varphi = b^h \mathrm{id}_X - g$; let us consider the two submodules $f^h X$ and φX of X. We want to prove that X is the direct sum in C of $f^h X$ and φX. For this purpose, by Lemma 7.1. of [1]. it is enough to find $q_1 \in \mathrm{Hom}_C(f^h X, X)$, $p_1 \in \mathrm{Hom}_C(X, f^h X)$, $q_2 \in \mathrm{Hom}_C(\varphi X, X)$, $p_2 \in \mathrm{Hom}_C(X, \varphi X)$, such that: $p_1 q_1 = 1 \otimes \mathrm{id}_{f^h X}, p_2 q_2 = 1 \otimes \mathrm{id}_{\varphi X}, q_1 p_2 = 0, q_2 p_1 = 0, q_1 p_1 + q_2 p_2 = 1 \otimes \mathrm{id}_X$. Let j_1: $f^h X \to X$ and j_2: $\varphi X \to X$ be the canonical injections. Let us observe that $g j_1 = b^h(\mathrm{id}_{f^h X})$. In fact for all $x = f^h(x') \in f^h X$ we have

$$g j_1(x) = f^h G(f)^h f^h(x') = b^h f^h(x') = b^h x.$$

Moreover, $\varphi X \leqq \mathrm{Ker}\,(f^h)$, because of $z = b^h x - g(x) \in \varphi X$, then

$$f^h(z) = b^h f^h(x) - f^{2h} G(f)^h(x) = 0;$$

this fact implies that $\varphi j_2 = b^h \mathrm{id}_{\varphi X}$, since for all $z \in \varphi X \leqq \mathrm{Ker}\,(f^h)$ we have $\varphi j_2(z) = b^h z - G(f)^h f^h(z) = b^h z$ (note that $\mathrm{Ker}\,(f^h)$ is not necessarily in C, because it can be not finitely generated). Finally, we have $g j_2 = 0$, since $\varphi X \leqq \mathrm{Ker}\,(f^h)$, and $\varphi j_1 = b^h \mathrm{id}_{f^h X} - g j_1 = 0$. Now we set:

$$p_1 = 1/b^h \otimes g;\ q_1 = 1 \otimes j_1;\ p_2 = 1/b^h \otimes \varphi;\ q_2 = 1 \otimes j_2.$$

We have:

$$\begin{aligned}
&p_1 q_1 = 1/b^h \otimes g j_1 = 1/b^h \otimes b^h \mathrm{id}_{f^h X} = 1 \otimes \mathrm{id}_{f^h X},\\
&p_2 q_2 = 1/b^h \otimes \varphi j_2 = 1 \otimes \mathrm{id}_{\varphi X},\ p_1 q_2 = 1/b^h \otimes g j_2 = 0,\\
&p_2 q_1 = 1/b^h \otimes \varphi j_1 = 0,\ q_1 p_1 + q_2 p_2 = 1/b^h \otimes (j_1 g + j_2 \varphi) =\\
&1/b^h \otimes (g + \varphi) = 1/b^h \otimes b^h \mathrm{id}_X = 1 \otimes \mathrm{id}_X,
\end{aligned}$$

as desired. By hypothesis, X is C-indecomposable, hence either $f^h X$ or φX is C-isomorphic to $\{0\}$. If $f^h X \cong_C \{0\}$, by Lemma 2 there exists $c \in R \setminus H$ such that $c f^h X = \{0\}$; therefore $c f^h = 0$ and $1/c \otimes c f^h = (1 \otimes f)^h = 0$, so that $1 \otimes f$ is nilpotent, as desired. Suppose now that $\varphi X \cong_C \{0\}$. By Lemma 2, we get that $c \varphi X = \{0\}$ for a suitable $c \in R \setminus H$; hence $0 = c\varphi = c b^h - c G(f)^h f^h$. We deduce that in $\mathrm{End}_C(X)$ the following relation holds:

$$c G(1 \otimes f)^h (1 \otimes f)^h - c b^h (1 \otimes \mathrm{id}_X) = 0. \tag{7}$$

Since $\mathrm{End}_C(X)$ is an R_H-module, multiplying (7) by $1/c$ and recalling that G is a monic polynomial, we get a relation of the form (2), with $a_0 \in R \setminus H$; hence in this case $1 \otimes f$ is a unit. This concludes the proof.

Recall that the *length* $l(X)$ of a finitely generated module X is *the minimal number of generators of* X (see Ch. IX of [3]). If $X \in C$, we see at once that, in general, $l(X)$ is not invariant for C-isomorphism; however, since $X \leqq Y$ implies $l(X) \leqq l(Y)$, for X, Y finitely generated (Prop. 3 of [13]), we deduce that there exists $a \in R \setminus H$ such that $l(aX) = l(abX)$ for all $b \in R \setminus H$. The length of such aX is said to be the *C-length* of X; it coincides with the minimal length of submodules of the form cX, with $c \in R \setminus H$, and it is denoted by $l_C(X)$.

PROPOSITION 5. *Let* $X, Y \in C$; *if* $X \cong_C Y$, *then* $l_C(X) = l_C(Y)$.

PROOF. In view of Lemma 2, there exist $f \in \mathrm{Hom}_R(X, Y)$ and $g \in \mathrm{Hom}_R(Y, X)$ such that $fg = a \cdot \mathrm{id}_Y$ and $gf = a \cdot \mathrm{id}_X$, with $a \in R \setminus \mathrm{H}$. We can assume, without loss of generality, that $l_C(X) = l(aX) = l(abX)$ and $l_C(Y) = l(aY) = l(abY)$ for all $b \in R \setminus H$. We have:

$$l_C(X) = l(aX) = l(a^2X) = l(g(af)X) \leqq l(afX) \leqq l(aY) = l_C(Y);$$

analogously $l_C(Y) \leqq l_C(X)$, from which the assertion.

COROLLARY 6. *Let* $X \in C$ *be a direct sum in* C *of* $X_1, \ldots, X_m$. *Then*

$$l_C(X) = l_C(X_1) + \ldots + l_C(X_m).$$

PROOF. Let us consider $Z = X_1 \oplus \ldots \oplus X_m$; Z is a direct sum in C of $X_1, \ldots, X_m$, too; since C is an additive category, we have $X \cong_C Z$; since, obviously, $l_C(Z) = l_C(X_1) + \ldots + l_C(X_m)$, it suffices to invoke Prop. 5.

We can now prove that the Krull-Schmidt theorem holds in C.

THEOREM 7. *Every object* X *in* C *is a finite direct sum in* C *of* C*-indecomposable summands; if* $X \cong_C \oplus_{i=1}^{n} X_i \cong_C \oplus_{j=1}^{m} Y_j$, *where* X_i, Y_j *are* C*-indecomposable for all* i, j, *then* $n = m$ *and* $X_i \cong_C Y_{\sigma i}$, *for a suitable permutation* σ *of* $\{1, \ldots, n\}$.

PROOF. In view of Cor. 6, we deduce that $l_C(X)$ is an upper bound for the number of summands (not C-isomorphic to $\{0\}$) in a direct decomposition of X in C; this yields the first statement of the theorem. The second statement is a consequence of the Krull-Schmidt theorem for additive categories (see Theorem 7.4. of [1]), which can be applied because idempotents split in C and $\mathrm{End}_C(X_i)$, $\mathrm{End}_C(Y_j)$ are local rings for all i, j, in view of Theorem 4.

Let us note that, if R is an *archimedean valuation domain*, i.e. if P is the unique nonzero prime ideal of R, then every finitely generated R-module is in $C(P)$, and $\mathrm{Hom}_{C(P)}(X, Y) = \mathrm{Hom}_R(X, Y)$ for all X, Y. Hence from Theorem 7 we reobtain Theorem 10 and Corollary 11 of [12].

REMARK 8. It could seem natural that C-homomorphisms and C-isomorphisms were called quasi-homomorphisms and quasi-isomorphisms. We avoid this terminology for two reasons. The first is that the class C depends by the choice of the prime ideal H. The second reason is that it seems more appropriate to say that two R-modules X and Y are *quasi-isomorphic* if each one is isomorphic to a submodule of the other (see [13]); this definition agrees with the behaviour of finite rank torsion-free abelian groups (see Cor. 7.7.(b) of [1]), but, in general, C-isomorphic objects in C are not quasi-isomorphic in this sense.

REMARK 9. It is convenient to motivate why we confine ourselves to modules X such that either $\mathrm{Ann}\, X \geqq H$, or it is H-primary. Actually, if we allow that not all the objects in $C(H)$ have the above property, we cannot be sure that $\mathrm{End}_{C(H)}(X)$ is local, for every X $C(H)$-indecomposable, hence Theorems 4 and 7 fail. For example, we can choose a suitable valuation domain R, in such a way that, for every prime ideal H of R, we can construct a two-generated R-module X which is $C(H)$-indecomposable and such that $R_H \otimes \mathrm{End}_R(X)$ is not a local ring (for the construction we use the results in [7], mainly Theorem 7; we omit the proof, which would involve techniques extraneous to the present paper).

REFERENCES

1. D. M. Arnold, *Finite Rank Torsion Free Abelian Groups and Rings*, Lecture Notes n. 931, Springer Verlag, Berlin Heidelberg New York, 1982.
2. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley 1969.
3. L. Fuchs and L. Salce, *Modules over Valuation Domains*, Lecture Notes in Pure Appl. Math., Marcel Dekker, New York and Basel, 1985.
4. B. Jónsson, *On unique factorization problems for groups and other algebraic systems*, Bull. Amer. Math. Soc. 51 (1945), p. 364.
5. B. Jónsson, *On direct decompositions of torsion-free abelian groups*, Math. Scand. 5 (1957), 230–235.
6. B. Jónsson, *On direct decomposition of torsion-free abelian groups*, Math. Scand. 7 (1959), 361–371.
7. L. Salce and P. Zanardo, *On two-generated modules over valuation domains*, Arch. Math. (Basel) 46 (1986), 408–418.
8. L. Salce and P. Zanardo, *A duality for finitely generated modules over valuation domains*, in "*Abelian Group Theory*, Proc. Oberwolfach Conf. 1985", Gordon and Breach, New York 1987, 433–450.
9. L. Salce and P. Zanardo, *Rank-two torsion-free modules over a valuation domain*, to appear in Rend. Sem. Mat. Univ. Padova.
10. P. Vámos, *Decomposition problems for modules over valuation domains*, to appear.
11. E. A. Walker, *Quotient categories and quasi-isomorphism of abelian groups*, Proc. Colloquium on Abelian Groups, Budapest 1964, 147–162.
12. P. Zanardo, *Indecomposable finitely generated modules over valuation domains*, Annali Univ. Ferrara – Sc. Mat. 31 (1985) 71–89.
13. P. Zanardo, *Quasi-isomorphisms of finitely generated modules over valuation domains*, Annali Mat. Pura Appl. 151(IV) (1988) 109–123.

DIPARTIMENTO DI MATEMATICA PURA E APPLICATA
UNIVERSITÀ DELL'AQUILA
VIA ROMA
67100 L'AQUILA
ITALY