MATH. SCAND. 66 (1990), 17–20

ON THE HOMOLOGY OF SL_2, A COMPLEMENT

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The sequel is a short complement to my paper [1]; the main result (théorème 2) about homology of SL_2, was stated under unnecessary restriction; we assumed for the field k, the hypothesis: $x \mapsto x^2$ is surjective. Actually we are able to prove

THEOREM. *Let k be any zero characteristic field, then*

$$H_1(SL_2(k), sl_2(k)) \simeq \Omega_k{}^1$$

$$H_2(SL_2(k), sl_2(k)) = 0$$

$\Omega_k{}^1$ is the space of absolute Kähler differentials and the homology is taken with twisted adjoint action coefficients. As a particular case, the result is true for $k = \mathbb{R}$ and for a number field we have $H_1(\ldots) = H_2(\ldots) = 0$.

To prove the theorem we start from the well known diagram of groups

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \{\pm 1\} & \to & k^* & \to & (k^*)^2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & SL_2(k) & \to & GL_2(k) & \to & k^* & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & PSL_2(k) & \to & PGL_2(k) & \to & k^*/(k^*)^2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

each element of $k^*/(k^*)^2$ has order two, so

$$H_i(k^*/(k^*))^2, M) = 0, \qquad i > 0,$$

Received January 26, 1989

for any Q-module M; by Hochschild-Serre spectral sequence of the lower line of the diagram, we obtain

$$H_i(\mathrm{PSL}_2(k), \mathrm{sl}_2(k)) \xrightarrow{\sim} H_i(\mathrm{PGL}_2(k), \mathrm{sl}_2(k)), \qquad i \geqq 0.$$

An analogous argument, applied to the left hand side of the diagram, shows that

$$H_i(\mathrm{SL}_2(k), \mathrm{sl}_2(k)) \xrightarrow{\sim} H_i(\mathrm{PSL}_2(k), \mathrm{sl}_2(k)), \qquad i \geqq 0.$$

We are now left to prove

$$H_1(\mathrm{PGL}_2(k), \mathrm{sl}_2(k)) \simeq \Omega_k{}^1$$

$$H_2(\mathrm{PGL}_2(k), \mathrm{sl}_2(k)) = 0;$$

the proof of [1] works quite well in that setting with one slight remark added. Take $G = \mathrm{PGL}_2(k)$, $\underline{G} = \mathrm{sl}_2(k)$, and consider the sequence of G-modules as in [1]

$$\coprod_{\substack{(a,b)\in \mathsf{P}_1(k)^2 \\ a \neq b}} \underline{B}_a \cap \underline{B}_b \xrightarrow{\partial} \coprod_{a \in \mathsf{P}_1(k)} \underline{B}_a \xrightarrow{\omega} \underline{G} \to 0; \tag{1}$$

we recall the notations for convenience: $\underline{B}_a$ is the Lie algebra of the Borel subgroup B_a which stabilizes a in the action of $\mathrm{PGL}_2(k)$ in $\mathsf{P}_1(k)$,

$$\partial(u|a, b|) = u|b| - u|a|,$$

$$\omega(u|a|) = u$$

The remark we need is an improvement of proposition 3 in [1].

PROPOSITION. *The sequence* (1) *is exact for any field.*

Here is the argument; an element of $\operatorname{Ker}\omega$, $\sum_{i\in I} u_i |a_i|$ is said to be of length n if $\#\{a_i : i \in I\} = n$; if $k^* = (k^*)^2$, we proved in [1] that $\operatorname{Ker}\omega$ is generated by elements of length $\leqq 3$. Actually, in the general case, $\operatorname{Ker}\omega$ is generated by elements of length $\leqq 4$; to this end, take

$$\sum_{i=1}^{n} u_i |a_i| \in \operatorname{Ker}\omega, \text{ of length } n > 4;$$

we can write

$$u_1 + u_2 + u_3 = v_1 + v_2$$

where

$$v_1 \in \underline{B}_{a_1}, \quad v_2 \in \underline{B}_b$$

for a well chosen b (by conjugation think of the case $a_1 = 0, b = \infty$, where $\underline{B}_0$ and

$\underline{B}_\infty$ are given by lower and upper triangular matrices); then, we put

$$\sum_{i=1}^{n} u_i |a_i| = \alpha + \beta$$

with

$$\alpha = (u_1 - v_1)|a_1| + u_2|a_2| + u_3|a_3| - v_2|b|$$

$$\beta = v_1|a_1| + v_2|b| + \sum_{i \geqq 4} u_i|a_i|$$

α and β are in $\operatorname{Ker}\omega$, of length $<n$.

Now let $\gamma = u_1|a_1| + u_2|a_2| + u_3|a_3| + u_4|a_4|$, with a_1, a_2, a_3, a_4 distinct and $u_1 + u_2 + u_3 + u_4 = 0$; we try to find

$$\lambda = v_{12}|a_1, a_2| + v_{13}|a_1, a_3| + v_{14}|a_1, a_4| + v_{23}|a_2, a_3| + v_{24}|a_2, a_4| + v_{34}|a_3, a_4|$$

such that

$$\partial\lambda = \gamma;$$

this is equivalent to solve the system

$$\begin{cases} -v_{12} - v_{13} - v_{14} = u_1 \\ v_{12} - v_{23} - v_{24} = u_2 \\ v_{13} + v_{23} - v_{34} = u_3 \\ v_{14} + v_{24} + v_{34} = u_4, \quad \text{with } v_{ij} \in \underline{B}_{a_i} \cap \underline{B}_{a_j}, i < j; \end{cases}$$

it is a simple exercise, using the fact that $(\underline{B}_{a_i})_{i=1,2,3,4}$ are planes in general position in sl_2, to check that the set of solutions is an affine line. This ends the proof of the proposition.

The sequel of the proof of the theorem works the same way as in [1], without any change.

Remarks on [1]. There is now a direct homological proof of Sydler's theorem (see J. Dupont and C. H. Sah, Homology of euclidean groups of motions made discrete and euclidean scissors congruences, Acta Math. 164 (1990) 1–27.

The relations introduced in [1], to define the space $\mathscr{B}_k$, are actually well known in information theory, on the level of functional equations (see J. Aczél, The state of the second part of Hilbert's fifth problem, Bull. Amer. Math. Soc. 20 (1989), p. 159).

I take the opportunity to correct a few misprints in [1]:

p. 64, line -8, read: "$P/Z \oplus \mathscr{B}_{\mathbb{R}}$" instead of "$P/Z \oplus$".

p. 76, in the right vertical arrow of the diagram, read "s" instead of "p".

p. 81, in the middle of the diagram, read "$\otimes_{ZB}$" instead of "$\otimes_{ZG}$".

p. 84, in the proposition, relation 2) must be read:

$$\langle b,c\rangle - \langle ab,c\rangle + \langle a,bc\rangle - \langle a,b\rangle = 0.$$

REFERENCE

1. J. L. Cathelineau, *Sur l'homologie de* SL_2 *à coefficients dans l'action adjointe,* Math. Scand. 63 (1988), 51–86.

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