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# CANCELLATION AND NON-CANCELLATION AMONGST PRODUCTS OF SPHERICAL FIBRATIONS

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**Abstract.**

Let $B$, $D$ and $F$ be given spaces, and $p: X \to B$, $q: Y \to B$ and $r: Z \to B$ be Hurewicz fibrations whose fibres have the homotopy types of $D$, $F$ and $F$ respectively. We investigate some circumstances under which the existence of a fibre homotopy equivalence between the fibred product fibrations $p \times_B q$: $X \times_B Y \to B$ and $p \times_B r$: $X \times_B Z \to B$ implies, or fails to imply, that $q$ and $r$ are fibre homotopy equivalent. For the particular situation where $B = S^{k+1}$, $D = S^m$, $F = S^n$, and $n$ is relatively large we show that this cancellation property holds in most situations with $0 \leqq k \leqq 16$ and $0 \leqq m \leqq 16$, but can fail for $(k, m) = (0, 0), (1, 1), (3, 3)$ and $(7, 7)$; a few cases remain undecided. This follows from results which specify sufficient conditions for cancellation when $B$ is a sphere or when $p$ has a section, and from a necessary condition for cancellation in a more general situation.

## 1. Introduction

The literature of mathematics contains numerous investigations into cancellation questions, both in algebra and topology. Research has focussed on two types of topological cancellation: if $X$, $Y$ and $Z$ are pointed spaces then *wedge cancellation* concerns the question of whether or not $X \vee Y \simeq X \vee Z$ (i.e. $X \vee Y$ is homotopy equivalent to $X \vee Z$) ensures that $Y \simeq Z$, if $X$, $Y$ and $Z$ are spaces then *Cartesian product cancellation* concerns the corresponding question for $X \times Y \simeq X \times Z$ and $Y \simeq Z$. In both situations the topic has usually been non-cancellation; the wedge case is discussed in [6, 7, 8, 9, 10, 11, 15, 23, 24 and 33], the Cartesian product case in [4, 12, 13, 14, 15, 22, 23, 26, 33 and 34].

The best known amonst many examples of the latter situation is the result of [15] that there is an $H$-manifold $E_{7\omega}$ with $S^3 \times E_{7\omega}$ diffeomorphic to $S^3 \times \mathrm{Sp}(2)$ yet $E_{7\omega} \not\simeq \mathrm{Sp}(2)$. Now $E_{7\omega}$ and $\mathrm{Sp}(2)$ have the same *genus*, i.e. their $p$-localizations are homotopy equivalent for all primes $p$; the above example illustrates the strong connection that exists between non-cancellation and genus. Aspects of this relationship are examined in many of the listed papers.

We introduce a third type of topological cancellation: *fibred product cancellation*. If $p: X \to B$ and $q: Y \to B$ are Hurewicz fibrations then the *fibred product* or *pullback space* of $X$ and $Y$ is the subspace $X \times_B Y = \{(x, y) \mid p(x) = q(y)\}$ of $X \times Y$

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and *the fibred product of $p$ and $q$*, $p \times_B q\colon X \times_B Y \to B$, the fibration that takes $(x, y)$ to $p(x) = q(y)$. We recall that fibrations over $B$, e.g. $p\colon X \to B$ and $q\colon Y \to B$, and maps over $B$, e.g. $f\colon X \to Y$ such that $qf = p$, constitute a category; then $p \times_B q$ is the product of $p$ and $q$ in this category. Then $p$ will be said to *cancel relative to all $F$-fibrations* (= fibrations whose fibres all have the homotopy type of $F$) if for all choices of $F$-fibrations $q\colon Y \to B$ and $r\colon Z \to B$, $p \times_B q$ is FHE to $p \times_B r$ implies that $q$ is FHE to $r$, where FHE abbreviates *fibre homotopy equivalent.*

We focus on both cancellation and non-cancellation, establishing sufficient conditions for cancellation in the case where $p$ has a section (theorem 3.4), where $B$ is a sphere (theorem 4.6), and a necessary condition for cancellation in a very general situation (theorem 6.3). A *spherical fibration* is a fibration whose fibres all have the homotopy type of a given sphere; applying the theorems mentioned above to spherical fibrations over spheres enables us to establish (in section 6):

MAIN EXAMPLE 1.1 Given that $k$, $m$ and $n$ are non-negative integers with $n$ relatively large (i.e. $n \geqq k + 2$ and $n \neq m$), then $S^m$-fibrations over $S^{k+1}$ always cancel relative to $S^n$-fibrations over $S^{k+1}$ in 277 of the 289 cases that occur with $0 \leqq k \leqq 16$ and $0 \leqq m \leqq 16$, but cancellation sometimes fails for each of the cases $(k, m) = (0, 0), (1, 1), (3, 3)$ and $(7, 7)$. Our results do not enable us to reach either conclusion in the eight remaining cases: $(k, m) = (3, 2), (8, 7), (9, 7), (11, 11), (13, 13), (15, 15), (16, 7)$ *and* $(16, 15)$.

The proof of our cancellation theorem 4.6, the main result behind the above example, depends on the author's fibred mapping space construction: the following is an indication of the method used. Any FHE from $p \times_B q$ to $p \times_B r$ determines, by composition with the projection $X \times_B Z \to Z$, a map $X \times_B Y \to Z$ over $B$; this corresponds by a "fibred" exponential law – a convenient category and "over $B$" extension of the "ordinary" exponential law [27, p. 6] – to a map from $X$ into the fibred mapping space $(YZ)$. Now this is a map between fibrations over their base space $B$, and we are able to use the associated exact homotopy ladder in an argument that determines sufficient conditions for the characteristic element of the fibration $(YZ) \to B$ to be zero; hence we can then show that $q$ is FHE to $r$.

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## 2. Preliminaries.

(2.1) We work in the context of the category of *compactly generated spaces* [21]; these are defined as having the final topology relative to all incoming maps from compact Hausdorff spaces. Any space can be cg-*ified*, i.e. retopologized as a compactly generated space, by giving it this final toplogy. We use $\mathscr{W}$ to denote the class of spaces (this now means compactly generated spaces) having the homotopy type of a CW-complex.

(2.2) If $X$ and $Y$ are spaces then $\mathcal{M}(X, Y)$ and $\mathcal{M}_0(X, Y)$ will denote the spaces of maps of $X$ into $Y$ in the unbased and based senses respectively, with the (of course cg-ifications of the) compact-open topologies. Further $\mathcal{H}(X)$ will denote the space of all unbased self-homotopy equivalences of $X$ (its base point is normally the identity map) and $[X, Y]$ the set of unbased homotopy classes of maps from $X$ to $Y$.

(2.3) There is an *evaluation map* $e: \mathcal{M}(X, Y) \to Y$ defined by $e(f) = f(*)$, $f \in \mathcal{M}(X, Y)$. If $*$ is a non-degenerate base point in $X$ ($\{*\} \subset X$ is a cofibration) then $e$ is a fibration (by the cg-version of [27, Theorem 2.8.2]); the distinguished fibre is $\mathcal{M}_0(X, Y)$.

Taking $X = Y$ and restricting $e$ to the path components of $\mathcal{M}(X, X)$ that consist of homotopy equivalences we again use $e$ denote an evaluation map, in this case $e: \mathcal{H}(X) \to X$; further if the base point is non-degenerate then this $e$ is also a fibration. If there is a binary operation $m: X \times X \to X$ such that $m(x, *) = x$ and $m(x, -): X \to X$ is a homotopy equivalence, for all $x \in X$, then the adjoint map $m': X \to \mathcal{H}(X)$ defined by $m'(x_1)(x_2) = m(x_1, x_2)$, for $x_1, x_2 \in X$, is a section to the fibration $e: \mathcal{H}(X) \to X$. In particular such sections exist for $X = S^m$, where $m = 0$, 1, 3, or 7.

(2.4) If $A$ is any pointed space then $\mathcal{M}_0(S^m, A)$ is an $H$-group [27, p. 35], so its path components all have the same homotopy type and, taking $c$ to denote the constant map of $S^m$ to the base point of $A$, their $k$-th homotopy groups are isomorphic to $\pi_k(\mathcal{M}_0(S^m, A), c) = \pi_k(\Omega^m A) \approx \pi_{k+m}(A)$.

(2.5) The *characteristic element* $\omega_p$ of the fibration $p: X \to S^{k+1}$ is defined to be $\delta(\iota_{k+1}) \in \pi_k(p^{-1}(*))$, where $*$ denotes the distinguished point of $S^{k+1}$, $\iota_{k+1}$ is the homotopy class of the identity on $S^{k+1}$ and $\delta: \pi_{k+1}(S^{k+1}) \to \pi_k(p^{-1}(*))$ the homomorphism that appears in the homotopy sequence of $p$.

(2.6.) The fibration $p$ has a base point preserving section if and only if $\omega_p = 0$ (for $\iota_{k+1}$ is in the image of $p_\#: \pi_{k+1}(X) \to \pi_{k+1}(S^{k+1})$ if and only if $\delta(\iota_{k+1}) = 0$.

(2.7) $\omega_p$ is in the image of the homomorphism $e_\#: \pi_k(\mathcal{H}(p^{-1}(*)) \to \pi_k(p^{-1}(*))$.

PROOF. Let Prin$(X)$ denote the space of homotopy equivalences from $p^{-1}(*)$ into individual fibres of $p$, and prin$(p)$: Prin$(X) \to S^{k+1}$ the associated projection and principal fibration. Then evaluation at a fixed point of $p^{-1}(*)$ defines a map Prin$(X) \to X$ over $S^{k+1}$ and the result follows from the associated homotopy ladder.

## 3. The Cancellation of Fibrations with Sections.

DEFINITION 3.1. The space $F$ will be said to have the *self-equivalence property* relative to the space $D$ if for all homotopy equivalences (in the unbased sense) $h$: $D \times F \to D \times F$ and all $d \in D$ the composites

$$F \underset{i(d)}{\to} D \times F \underset{h}{\to} D \times F \underset{w}{\to} F,$$

where $i(d)(x) = (d, x)$, $x \in F$, and $w$ denotes the projection, are self-homotopy equivalences of $F$.

(3.2) We notice that if, for such a $D$, $F$ and $h$, $k$: $D \to \mathscr{M}(F, F)$ is defined by $k(d)(x) = wh(d, x)$, $d \in D$, $x \in D$, $x \in F$, then $k(D) \subset \mathscr{H}(F)$.

EXAMPLE 3.3 If $m$ and $n$ are non-negative integers, then $S^m$ has the self-equivalence property relative to $S^n$ if and only if $m \neq n$.

PROOF: The $m \neq n$ case follows easily from the Künneth and Hurewicz theorems; for $m = n$ there is a switch map $(x, y) \to (y, x)$, $x \in S^m$, $y \in S^n$.

THEOREM 3.4. *If $B$ is numerably contractible* (e.g. $B \in \mathscr{W}$ [5, Theorem 6.3]), *$F$ has the self-equivalence property relative to the fibres of $p$: $X \to B$, and $p$ has a section then $p$ cancels relative to all $F$-fibrations over $B$.*

PROOF. Let $s$ be a section to $p$: $X \to B$, $q$: $Y \to B$ and $r$: $Z \to B$ $F$-fibrations and $h$: $X \times_B Y \to X \times_B Z$ a FHE. The composition of $h$ with the projection $X \times_B Z \to Z$ and the map $Y \to X \times_B Y$, $y \to (sq(y), y)$ determines a map $g$: $Y \to Z$ over $B$ such that for each $b \in B$, $g \mid q^{-1}(b)$ is the composite map

$$q^{-1}(b) \xrightarrow[i(s(b))]{} p^{-1}(b) \times q^{-1}(b) \xrightarrow[h|p^{-1}(b) \times q^{-1}(b)]{} p^{-1}(b) \times r^{-1}(b) \underset{w}{\to} r^{-1}(b),$$

so $g$ is an FHE [5, Theorem 6.3].

LEMMA 3.5. *Let $p$: $X \to S^{k+1}$ be any $S^m$-fibration. If $e_\#$: $\pi_k(\mathscr{H}(S^m)) \to \pi_k(S^m)$ is zero then $p$ has a section. If $k \geqq 1, m \geqq 1$ and a homomorphism $\pi_k(S^m) \to \pi_{k+m-1}(S^m)$, $\alpha \to$ the Whitehead product $[\iota_m, \alpha]$, is a monomorphism, then $p$ has a section.*

PROOF. The first part is a consequence of 2.6 and 2.7; the second follows because the image of $e_\#$ is the kernel of $[\iota_m, \ ]$ (see [32, Theorem 3.2]).

EXAMPLE 3.6. All $S^m$-fibrations over $S^{k+1}$ have sections, and hence cancel relative to all $S^n$-fibrations, where $m \neq n$, in the following cases:
(i) $k \neq 0$ and $m = 0$, (ii) $k \neq 1$ and $m = 1$, (iii) $k < m$, (iv) $k = m > 0$ and $m$ is even, (v) $k = m + 1$ and $m \equiv 0, 1$ or $2 \bmod 4, m \neq 2$ or $6$, (vi) $k = m + 2$ and $m \equiv 0$ or $1 \bmod 4$, but $m \neq 5$, (vii) $k = m + 4$ and $m \geqq 6$, (viii) $k = m + 5$ and

$m \geqq 7$, (ix) $k = m + 6$ with $m \geqq 6$, $m \equiv 0, 1, 2, 3$ or $6 \bmod 8$ and $m + 9 \notin N$, where $N$ is the set of integers described on p. 304 and p. 305 of [18], and (x) $k = m + 12$ with $m = 7, 8, 9$ or $m \geqq 14$.

PROOF. For (i), (ii), (iii), (vii), (viii) and (x) $\pi_k(S^m) = 0$ and the result follows from 2.6. Cases (iv), (v), (vi) and (ix) are consequences of 3.5; the required information on $[\iota_m, \ ]$ is given in [28, 2.15] for (iv); [16, Theorem 4.16], [17, Lemma 5.1] and [30, p. 80] for (v), [16, 4.20] and [17, Lemma 5.1] for (vi) and [18, Theorem 1.3] for (ix).

## 4. Main Cancellation Results.

We first review some of the theory of fibred mapping spaces (for more details see [3, Section 7] and also [1]), assuming throughout that $B$ is Hausdorff.

(4.1) If $q: Y \to B$ and $r: Z \to B$ are maps then the *fibred mapping space* $(YZ)$ has underlying set $\bigcup_{b \in B} \mathcal{M}(q^{-1}(b), r^{-1}(b))$ and the function $(qr): (YZ) \to B$ is defined by $(qr)(f) = b$, where $f \in \mathcal{M}(q^{-1}(b), r^{-1}(b))$. We topologize $(YZ)$ with the cg-ification (see 2.1) of the topology that has subbasic open sets of the forms (i) $(qr)^{-1}(U)$ for all $U$ that are open in $B$, and (ii) $W(A, V) = \{f \in (YZ) \mid f(A) \subset V\}$, for all compact $A$ in $Y$ and open $V$ in $Z$, $f(A)$ being the set of all meaningful $f(a)$, with $a \in A$. Then $(qr)$ is continuous, and the fibre of $(qr)$ over $b \in B$ is the space $\mathcal{M}(q^{-1}(b), r^{-1}(b))$ [1, Proposition 3.2].

(4.2) If $p: X \to B$ is a map, then the fibred product space $X \times_B Y$ and the fibred mapping space $(YZ)$ are related by the following *fibred exponential law.* There is a bijective correspondence between: (i) maps $f: X \times_B Y \to Z$ over $B$, and (ii) maps $g: X \to (YZ)$ over $B$, determined by the rule $f(x, y) = g(x)(y)$, $p(x) = q(y)$, [3, Theorem 7.3].

(4.3) In particular if $p = 1_B: B \to B$ then 4.2 implies that there is a bijective correspondence between (i) maps $f: Y \to Z$ over $B$, and (ii) sections $g$ to $(qr)$, determined by $f \mid q^{-1}(b) = g(b)$, $b \in B$.

(4.4) If $q$ and $r$ are Hurewicz fibrations then so is $(qr)$ [1, Theorem 3.4].

(4.5) Given fibrations $q: Y \to S^{k+1}$, $r: Z \to S^{k+1}$, then $q$ and $r$ are FHE if and only if there is a homotopy equivalence $q^{-1}(*) \to r^{-1}(*)$ with the property that when it is taken as the base point for $\mathcal{M}(q^{-1}(*), r^{-1}(*))$ and $(YZ)$, then $\omega_{(qr)} = 0$. The proof is immediate from (2.6), (4.3), (4.4) and [5, Theorem 6.3].

THEOREM 4.6. *Let $D$ and $F$ be spaces, $F$ having the self-equivalence property relative to $D$ and such that, for $e: \mathcal{H}(D) \to D$ and all $h \in \mathcal{M}_0(D, \mathcal{H}(F))$, the homomorphisms $h_\# e_\#: \pi_k(\mathcal{H}(D)) \to \pi_k(\mathcal{H}(F))$ are zero. Then all $D$-fibrations cancel relative to all $F$-fibrations over $S^{k+1}$.*

PROOF. If $p: X \to B$, $q: Y \to B$ and $r: Z \to B$ are fibrations such that there is a FHE $X \times_B Y \to X \times_B Z$, then the projection $X \times_B Z \to Z$ composed with this FHE determines a map $X \times_B Y \to Z$ over $B$. Applying the fibred exponential law (4.2) we obtain a map $g: X \to (YZ)$ that is between fibrations and over $B$. Selecting base points $*$ in $X$ and $B$ such that $p(*) = *$ and defining the homotopy equivalence $g(*): q^{-1}(*) \to r^{-1}(*)$ (see 3.2) to be the base point in $\mathcal{M}(q^{-1}(*), r^{-1}(*))$ and $(YZ)$, there is an associated ladder of exact homotopy sequences including:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_{k+1}(X) & \xrightarrow[p_\#]{} & \pi_{k+1}(B) & \xrightarrow[\delta_p]{} & \pi_k(p^{-1}(*) & \longrightarrow & \cdots \\
 & & \downarrow g_\# & & \downarrow 1 & & \downarrow (g \mid p^{-1}(*))_\# & & \\
\cdots & \longrightarrow & \pi_{k+1}(YZ) & \xrightarrow[(qr)_\#]{} & \pi_{k+1}(B) & \xrightarrow[\delta_{(qr)}]{} & \pi_k(\mathcal{M}(q^{-1}(*), r^{-1}(*)) & \longrightarrow & \cdots
\end{array}$$

Taking $B = S^{k+1}$ we notice that $(g \mid p^{-1}(*))_\#: \pi_k(p^{-1}(*)) \to \pi_k(\mathcal{M}(q^{-1}(*), r^{-1}(*))$ takes the characteristic element $\omega_p$ for $p$ (2.5) to $\omega_{(qr)}$. Now $\omega_p$ is in the image of the homomorphism $e_\#: \pi_k(\mathcal{H}(p^{-1}(*)) \to \pi_k(p^{-1}(*))$ (2.7) induced by the map $e$: $\mathcal{H}(p^{-1}(*)) \to p^{-1}(*)$ (2.3), so if (as we assume) the homomorphisms of the form $(g \mid p^{-1}(*))_\# \, e_\#$ are all zero then $\omega_{(qr)} = 0$ and the result follows from 4.5

COROLLARY 4.7. *Let $k$ and $m$ be non-negative integers (the case $k = m = 0$ excluded), $n\,(\neq m)$ be an integer $\geqq k + 2$ and $E: \pi_k(S^m) \to \pi_{k+n}(S^{m+n})$ the homomorphism that suspends $n$ times. If the composite function $\Phi = C(1 \times Ee_\#)$,*

$$G_m \times \pi_k(\mathcal{H}(S^m)) \xrightarrow[1 \times e_\#]{} G_m \times \pi_k(S^m) \xrightarrow[1 \times E]{} G_m \times G_{k-m} \xrightarrow[C]{} G_k,$$

*is zero, where $G_m$ denotes the stable group $\lim_{n \to \infty} \pi_{m+n}(S^n)$ and $C$ composition, then $S^m$-fibrations always cancel relative to $S^n$-fibrations over $S^{k+1}$.*

PROOF. If $m = 0$ and $k \neq 0$, or $k < m$, then $\pi_k(S^m) = 0$ and it is clear that both $\Phi = 0$ and the condition of either 3.6(i) or 3.6(iii) is satisfied, so we will just consider the situation with $k \geqq m > 0$.

We see from 2.3 and 2.4 that, for $j = k$ or $m$, $\pi_j(\mathcal{H}(S^n)) = \pi_j(\mathcal{M}(S^n, S^n), 1) \approx \pi_j(\mathcal{M}_0(S^n, S^n), c) \approx \pi_{j+n}(S^n)$, so there are isomorphisms $Q: \pi_k(\mathcal{H}(S^n)) \to \pi_{k+n}(S^n)$ and $R: \pi_m(\mathcal{H}(S^n)) \to \pi_{m+n}(S^n)$. It is routine to verify that the following diagram is commutative:

$$\begin{array}{ccccc}
\pi_m(\mathcal{H}(S^n)) \times \pi_k(\mathcal{H}(S^m)) & \xrightarrow[1 \times e_\#]{} & \pi_m(\mathcal{H}(S^m)) \times \pi_k(S^m) & \xrightarrow[C]{} & \pi_k(\mathcal{H}(S^n)) \\
R \times 1 \downarrow \approx & & R \times E \downarrow & & Q \downarrow \approx \\
\pi_{m+n}(S^n)) \times \pi_k(\mathcal{H}(S^m)) & \xrightarrow[1 \times Ee_\#]{} & \pi_{m+n}(S^n) \times \pi_{k+n}(S^{m+1}) & \xrightarrow[C]{} & \pi_{k+n}(S^n)
\end{array}$$

Now $n \geqq k + 2$, $n \geqq m + 2$ and $m + n \geqq k - m + 2$ so the groups $\pi_{k+n}(S^n)$, $\pi_{m+n}(S^n)$ and $\pi_{k+n}(S^{m+n})$ can be identified with $G_k$, $G_m$ and $G_{k-m}$ respectively. The image of the composite function along the top line consists of the images of all homomorphisms $h_{\#} e_{\#}$ as described in 4.6, that along the bottom line is $\Phi$ and so the result follows.

## 5. Computation of $\Phi$ for Spherical Fibrations over Spheres.

EXAMPLE 5.1. Let $k$ and $m$ be integers with $0 \leqq k \leqq 16$ and $0 \leqq m \leqq 16$; the associated $\Phi$ is non-zero only in the cases: $(k, m) = (1,1), (3,2), (3,3), (7,7), (8,7), (9,7), (11,11), (13,13), (15,15), (16,7)$ and $(16,15)$.

PROOF. (i). *The cases listed in* 3.6 *have* $e_{\#} = 0$ *and hence* $\Phi = 0$.

(ii) *If* $k = m$ *then* $\Phi \neq 0$ *only when* $m = 1$ *or when m is odd and* $G_m$ *contains terms of order* $> 2$, i.e. only in the cases $(k, m) = (1,1), (3,3), (7,7), (11,11), (13,13), (15,15), (19,19), \ldots$. To see this we notice via 3.5 and 3.6 (iv) that $e_{\#} = 0$, and so $\Phi = 0$, when $m > 0$ is even. When $k = m = 0$ we have $E$ defined as a homomorphism from a cyclic group of order 2 to an infinite group, so $E = 0$ and $\Phi = 0$. When $m$ is odd ($m \neq 1, 3$ or $7$) then it is well known that $[\iota_m, \iota_m]$ is of order 2 [28, 2.15], so it follows from the exact homotopy sequence of the fibration $e$: $\mathscr{H}(S^m) \to S^m$ that the image of $e_{\#}$: $\pi_m(\mathscr{H}(S^m)) \to \pi_m(S^m)$ consists of the "even" elements of $\pi_m(S^m)$. Now $1 \times E$: $G_m \times \pi_m(S^m) \to G_m \times G_0$ is an isomorphism and, if $\alpha \in G_m$ and $2\beta \in G_0$, then $\alpha \circ 2\beta \neq 0$ occurs and can only occur if the order of $\alpha$ is greater than two. In the cases $m = 1, 3$ or $7$ we know that $e_{\#}$ is an epimorphism (2.3), $E$ is an isomorphism and $C$ is surjective; hence $\Phi$ is surjective.

(iii) *If* $k = m + 1$ *then* $\Phi \neq 0$ *only when* m $= 2$, or $m \equiv 3 \bmod 4$ *and* $C$: $G_m \times G_1 \to G_{m+1}$ *is non-zero.* i.e. only in cases $(k, m) = (3,2), (8,7), (16,15), \ldots$. It follows via 3.6(v) that $e_{\#} = 0$ and so $\Phi = 0$ when $m \neq 2, m \neq 6$ or $m \not\equiv 3 \bmod 4$, whereas $e_{\#}$ is an epimorphism in the remaining cases. If $m \geqq 2$ then $E$ is an epimorphism; when $e_{\#}$ and $E$ are both epimorphisms then $\Phi \neq 0$ if and only if the corresponding $C \neq 0$. For $m = 2$ $C \neq 0$ and so $\Phi \neq 0$, for $m = 6$ $C = 0$ so $\Phi = 0$ [31, p. 190]. It is easily seen that $\Phi = 0$ for $m = 0$ and $m = 1$.

(iv) *If* $k = m + 2$ *then* $\Phi \neq 0$ *only when* $m \equiv 2$ or $3 \bmod 4$, *and* $C$: $G_m \times G_2 \to G_{m+2}$ *is non-zero*, i.e. only in the cases $(k, m) = (9,7), (17,15), \ldots$. The proof is similar to that for 5.4.

(v) *When any of* $k$, $m$ or $k - m$ *is in the set* $\{4, 5, 12\}$ *then* $\Phi = 0$; for either $G_k = 0$, $G_m = 0$ or $G_{k-m} = 0$ and hence $C = 0$.

(vi) *In cases where* $G_k$, $G_m$ *and* $G_{k-m}$ *are all non-zero it frequently happens that* $C = 0$ *and so* $\Phi = 0$. The function $C$ may be determined using information from [31, p. 189 and p. 190]: such examples include the cases $(k, m) = (8,2), (10,2), (10,3), (10,7), (11,3), (11,8), (13,2), (13,6), (13,7), (14,3), (14,6), (14,8), (14,11), (15,2), (15,6), (15,7), (15,8), (15,9), (16,2), (16,3), (16,6), (16,8), (16,10), (16,13)$.

(vii) *The remaining cases are* $(k, m) = (6,3), (9,2), (9,3), (9,6), (11,2), (13,3), (13,10), (14,7), (16,7)$ *and* $(16,9)$.

Considering the case $(k,m) = (9,6)$, $\Phi$ is the composite:

$$G_6 \times \pi_9(\mathscr{H}(S^6)) \xrightarrow[1 \times e_\#]{} G_6 \times \pi_9(S^6) \xrightarrow[1 \times E]{} G_6 \times G_3 \xrightarrow[C]{} G_9$$

Now $G_6 = Z_2$, $G_3 = Z_8 \oplus Z_3$ with respective generators $\nu^2$, and $\nu$ and $\alpha_1$, in the terminology of [31]. Now $\nu^2 \circ \nu = \nu^3 \neq 0$ [31, p. 190] and $\nu^2 \circ \alpha_1 = 0$ (because of the orders of $\nu^2$ and $\nu^2$ and $\alpha_1$), hence we only need to determine whether of not $\nu$ is in the image of $Ee_\#$. Now $E: \pi_9(S^6) = G_3$ and $[\nu, \iota_6] \neq 0$ [18, p. 307], so $\nu \in \pi_9(S^6)$ is not in the image of $e_\#$ (see 3.5) and so $\Phi = 0$.

When $(k,m) = (14,7)$, $\Phi$ is the composite

$$G_7 \times \pi_{14}(\mathscr{H}(S^7)) \xrightarrow[1 \times e_\#]{} G_7 \times \pi_{14}(S^7) \xrightarrow[1 \times E]{} G_7 \times G_7 \xrightarrow[C]{} G_{14}$$

where $G_7 \approx Z_{16} \oplus Z_3 \oplus Z_5$, $G_{14} = Z_2 \oplus Z_2$. Now $\sigma \in G_7$ is a term of order 16, $\sigma^2$ is a term of order 2 in $G_{14}$ [31, p. 189] so for $\Phi$ to be non-zero we require a non-zero homomorphism $\pi_{14}(S^7) \to G_{14}, \alpha \to \sigma \circ E(\alpha)$; however there can be no non-zero homomorphism $Z_8 \to Z_2 \oplus Z_2$ that factors through $z_{16} \oplus Z_3 \oplus Z_5$ so the homomorphism $\pi_{14}(S^7) \to G_{14}$ is zero and $\Phi = 0$.

We find that $\Phi = 0$ in the other cases, with the exception of $(k,m) = (16,7)$, by similar arguments; details are left to the reader. Data on particular Whitehead products that must be considered can obtained from [17, Lemma 5.1], [18], [19], [28] and [29, Theorem 2.1].

## 6. Non-Cancellation of Fibrations.

EXAMPLE 6.1. If $p: S^1 \to S^1$ denotes the double covering $(z \to z^2)$ and $n$ is any non-negative integer then $p$ fails to cancel relative to some $S^n$-fibrations.

PROOF. Let $Z$ be the quotient space of $S^n \times I$ obtained by identifying $S^n \times \{0\}$ with $S^n \times \{1\}$, using the homeomorphism $\mu: S^n \to S^n$ that reverses the first coordinate; then $r: Z \to S^1$ is the projection onto $S^1$ ($= I$ with $\{0\}$ and $\{1\}$ identified). Now $S^1 \times_{S^1} Z$ can be taken to be the quotient space of $[0, \frac{1}{2}] \times S^n$ and $[\frac{1}{2}, 1] \times S^n$ obtained using $\mu$ to identify both $S^n \times \{0\}$ with $S^n \times \{1\}$, and one copy of $S^n \times \{1\}$, and one copy of $S^n \times \{\frac{1}{2}\}$ with the other; then $p \times_{S^1} r$ is defined by $(p \times_{S^1} r)(t,z) = (2t) \bmod 1$, for all $t \in I$, $z \in S^n$. The homomorphism $f$: $S^1 \times S^n \to S^1 \times_{S^1} Z$ given by $f(t,x) = (t,x)$ for $t \leqq \frac{1}{2}$ and $(t, \mu(x))$ for $t \geqq \frac{1}{2}$, $t \in [0,1]$, $z \in S^n$, is a FHE from $p \times_{S^1} q$ to $p \times_{S^1} r$, where $q: S^1 \times S^n \to S^1$ is the usual projection.

Considering the maps $g: S^n \to Z$, $g(x) = (\frac{1}{2}, x)$ and $h: S^n \to Z$, $h(x) = (\frac{1}{2}, \mu(x))$, $x \in S^n$: they are clearly homotopic and so any pair of homotopy equivalences of $S^n \to \{\frac{1}{2}\} \times S^n$ are homotopic when viewed as maps into $Z$. Yet the homotopy equivalences $S^n \to \{\frac{1}{2}\} \times S^n$ defined by the above formulae are not homotopic when regarded as maps into $S^1 \times S^n$, so $q$ and $r$ cannot be FHE

EXAMPLE 6.2. If there is a lifting of the fibration $p: X \to B$ over the non-trivial principal $G$-fibration $q: Y \to B$, i.e. a map $f: X \to Y$ such that $qf = p$, then $p$ fails to cancel relative to some $G$-fibrations. (e.g. take $p = q$ and $f = 1_X$).

PROOF. The induced principal $G$-fibration $q_p: X \times_B Y \to X$ has a section and hence is trivial, so if $r$ denotes the projection $B \times G \to B$ we have $q_p$ FHE to $r_p$ and $p \times_B q = p(q_p)$ is FHE to $p(r_p) = p \times_B r$.

THEOREM 6.3. *If the fibration $p: X \to B$ cancels relative to all $F$-fibrations and both $B$ and $X$ are in $\mathscr{W}$ then the induced function*

$$p^{\#}: [B, B_{\mathscr{H}(F)}] \to [X, B_{\mathscr{H}(F)}], p^{\#}[k] = [kp], [k] \in [B, B_{\mathscr{H}(F)}]$$

*is injective.*

PROOF. The sets of FHE classes of $F$-fibrations over $B$ and $X$ are classified [20, Cor. 9.5(ii)] by $[B, B_{\mathscr{H}(F)}]$ and $[X, B_{\mathscr{H}(F)}]$ respectively, hence the result is equivalent to the assertion that if $p$ cancels then the function $q \to q_p$ is injective on sets of FHE classes of $F$-fibrations; this holds because if $q_p$ is FHE to $r_p$ then $p \times_B q = p(q_p)$ is FHE to $p \times_B r = p(r_p)$ and so $q$ is FHE to $r$.

REMARKS 6.4. The result of [20] quoted requires that $F$ is compact and in $\mathscr{W}$; the account in [25] establishes that a universal $F$-fibration $p_\infty: E_\infty \to B_\infty$ exists without restriction on $F$, but not that $B_\infty = B_{\mathscr{H}(F)}$. However a proof that $B_\infty = B_{\mathscr{H}(F)}$, without restriction on $F$, will appear in [2].

COROLLARY 6.5. *Let $p: X \to B$ be a fibration, where $X$ and $B$ have the homotopy types of the spheres $S^{j+1}$ and $S^{k+1}$, respectively, for some non-negative integers $j$ and $k$. If $\#$ denotes cardinal number and $n$ is a given positive integer such that $\#\pi_k(\mathscr{H}(S^n)) > 2\#\pi_j(\mathscr{H}(S^n))$ then $p$ fails to cancel relative to certain $S^n$-fibrations.*

PROOF. It follows from the homotopy sequence for $e: \mathscr{M}(S^{k+1}, B_{\mathscr{H}(S^n)}) \to B_{\mathscr{H}(S^n)}$ that $[S^{k+1}, B_{\mathscr{H}(S^n)}]$ can be identified with the quotient of $\pi_{k+1}(B_{(H(S^n)}) \approx \pi_k(\mathscr{H}(S^n))$ under an action of $\pi_1(B_{\mathscr{H}(S^n)}) \approx \pi_0(\mathscr{H}(S^n)) \approx Z_2$; hence $\#[S^{k+1}, B_{\mathscr{H}(S^n)}] \geqq \frac{1}{2}\#(\pi_k(\mathscr{H}(S^n))) > \#(\pi_j(\mathscr{H}(S^n))) = \#(\pi_{j+1}(B_{\mathscr{H}(S^n)})) \geqq \#[S^{j+1}, B_{\mathscr{H}(S^n)}]$, $p^{\#}$ as described in Theorem 6.3 cannot be an injection, and so $p$ sometimes fails to cancel relative to $S^n$- fibrations.

EXAMPLE 6.6. The Hopf fibrations $S^7 \to S^4$ and $S^{15} \to S^8$ fail to cancel relative to certain $S^n$-fibrations, for all choices of $n \geqq 8$ and $n \geqq 16$ respectively.

PROOF. We see from 2.3 and 2.4 that $\pi_k(\mathscr{H}(S^n)) \approx \pi_k(\mathscr{H}_0(S^n)) \approx \pi_{k+n}(S^n)$ for $k =$ both 3 and 6, where $n \geqq 8$, so $\pi_3(\mathscr{H}(S^n)) \approx Z_{24}$, $\pi_6(\mathscr{H}(S^n)) \approx Z_2$ and the non-cancellation of $S^7 \to S^4$ follows; a similar argument applies to $S^{15} \to S^8$.

REMARKS 6.7 (i). Theorem 6.3 and Corollary 6.5 make it easy to generate many non-cancellation examples, by factoring either null homotopic maps or maps $S^{j+1} \to S^{k+1}$ into composites of homotopy equivalences and fibrations $p$.

(ii). An argument similar to that of 6.6 for the Hopf fibration $p: S^3 \to S^2$ does not yield any conclusion about cancellation; however, P. Selick has shown the author, by a homology argument, that the image of $p^{\#}: \pi_2(B_{\mathscr{H}(S^n)}) \to \pi_3(B_{\mathscr{H}(S^n)})$ is 0, hence the $p^{\#}$ of 6.3 is not injective and cancellation fails for $S^3 \to S^2$ relative to $S^n$-fibrations.

PROOF OF MAIN EXAMPLE 1.1. This is an immediate consequence of 4.7, 5.1, and examples 6.1, 6.6, and 6.7(ii).

REMARK 6.8. The assumption concerning $n$ in example 1.1 ensures that $\pi_k(S^n) = 0$; hence the characteristic elements of $q$ and $r$ located in this group are zero and $q$ and $r$ both have sections (2.6). So it is not reasonable to speculate, on the basis of this example, that cancellation predominates for spherical fibrations over spheres in general.

## REFERENCES


1. P. Booth, *The section problem and the lifting problem*, Math. Z. 121 (1971), 273–287.
2. P. Booth, *Classifying spaces for a general theory of fibrations*, to appear.
3. P. Booth and R. Brown, *Spaces of partial maps, fibred mapping spaces and the compact-open topology*, Gen. Top. and its Applics. 8 (1978), 181–195.
4. L. S. Charlap, *Compact flat Rieemannian manifolds*, Ann. of Math. 81 (1965), 15–30.
5. A. Dold, *Partitions of unity in the theory of fibrations*, Ann. of Math. 78 (1963), 223–255.
6. P. Freyd, Proc. Conf. *Cat. Algebra* (La Jolla, Calif. 1965), 121–172, Springer-Verlag, New York, 1966.
7. P. Freyd, *The Grothendieck group for stable homotopy is free*, Bull. Amer. Math. Soc. 73 (1967), 84–86.
8. P. Hilton, *On the Grothendieck group of compact polyhedra*, Fund. Math. 61 (1967), 199–214.
9. P. Hilton, *On the homotopy type of compact polyhedra*, Fund. Math. 61 (1967), 105–109.
10. P. Hilton, *Some remarks concerning the semiring of polyhedra*, Bull. Soc. Math. Belg. 19 (1967), 277–288.
11. P. Hilton, *Note on the homotopy type of mapping cones*, Comm. Pure Appl. Math. 21 (1968), 515–519.
12. P. Hilton, *Groups with isomorphic homology groups and non-cancellation in homotopy theory*, J. Pure Appl. Alg. 44 (1987), 221–226.
13. P. Hilton, G. Mislin and J. Roitberg, *Sphere bundles over spheres and non-cancellation phenomena*, Springer LN 249 (1971), 34–46.
14. P. Hilton, G. Mislin and J. Roitberg, *H-spaces of rank 2 and non-cancellation phenomena*, Invent. Match, 16 (1972), 325–334
15. P. Hilton and J. Roitberg, *On principal $S^3$-bundles over spheres*, Ann. of Math. 90 (1969), 91–107.
16. P. Hilton and J. H. C. Whitehead, *Note on the Whitehead product*, Ann. of Math. 58(3) 1953, 429–442.
17. W. C. Hsiang, J. Levine and R. J. Szczarba, *On the normal bundle of a homotopy sphere*, Topology 3 (1965), 173–181.
18. L. Kristensen and I. Madsen, *Note on Whitehead products in spheres*, Math. Scand. 21 (1967), 301–314.

19. M. Mahowald, *Some Whitehead products in $S^n$*, Topology 4 (1965), 17–26.
20. J. P. May, *Classifying spaces and fibrations*, Mem. Amer. Math. Soc. 155 (1975).
21. M. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. 146 (1969), 273–298.
22. G. Mislin, *The genus of an H-space*, Springer LN in Math. 249, 75–83, Springer-Verlag, New York, 1971.
23. G. Mislin, *Cancellation properties of H-spaces*, Comment. Math. Helv. 49 (1974), 195–200.
24. E. Molnar, *Relation between wedge cancellation and localization for complexes with two cells*, J. Pure Appl. Algebra 3 (1973), 141–158.
25. R. Schön, *The Brownian classification of fibre spaces*, Arch. Math. 39 (1982), 359–365.
26. A. Sieradski, *Non-cancellation and a related phenomenon for lens spaces*, Topology 17 (1978), 85–93.
27. E. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966.
28. S. Thomeier, *Einige Ergebnisse über Homotopiegruppen von Sphären*, Math. Ann. 164 (1966), 225–250.
29. S. Thomeier, *Whitehead products and homotopy groups of spheres*, Proc. 13th Bien. Conf. Can. Math. Congress 1971, Canadian Math. Congress, Montreal 1972, (ed. J. R. Vanstone), 144–155.
30. H. Toda, *Generalized Whitehead products and homotopy groups of spheres*, J. of the Inst. Polytech, Osaka City Univ. Ser. A. Math., 3 (1952), 43–82.
31. H. Toda, *Composition methods in homotopy groups of spheres*, Ann. of Math. Studies 49, Princeton Iniversity Press, Princeton, New Jersey, 1962.
32. G. W. Whitehead, *On products in homotopy groups*, Ann. of Math. 47 (1946), 460–475.
33. C. Wilkerson, *Genus and cancellation*, Topology 14 (1975), 29–36.
34. A. Zabrodsky, *On the genus of finite* CW-*H-spaces*, Comment. Math. Helv. 49 (1974), 48–64.

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