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SEPARATING PLANE CONVEX SETS

RAFAEL HOPE and MEIR KATCHALSKI

Abstract.

We prove that given an integer $k \geqq 2$ and any family of $m = 12(k-1)$ pairwise disjoint nonempty convex sets in the plane there exists a closed halfplane which contains at least k of the sets while its complementary closed halfplane contains at least one of the remaining $m - k$ sets.

This improves for $k \geqq 3$ a result of Tverberg.

1. Introduction.

For an integer $k \geqq 1$ let $f = f(k)$ denote the smallest integer $f \geqq k + 1$ such that for any finite family of nonempty convex sets in the plane with pairwise disjoint relative interiors, there is a closed halfplane which contains at least k of the sets while its complementary closed halfplane contains at lest one of the remaining $f - k$ sets.

A standard separation theorem is that $f(1) = 2$. Tverberg has proved in [5] that $f(2) = 5$ and that $f(k)$ exists. He showed that $f(k) \leqq R(k) + k - 1$ where $R = R(k)$ is an integer such that whenever the 3-subsets of an R-set S are split into 3 families F_1, F_2, F_3, then for some $i \in \{1, 2, 3\}$ there is a k_i-subset S_i, $S_i \subset S$ such that all the 3-subsets of S_i belong to the family F_i with $k_1 = k_2 = k + 1$, $k_3 = 5$. Such an integer $R(k)$ exists by Ramsey's theorem, see [2].

Since very little is known about Ramsey numbers it is desirable to obtain more explicit and better upper bounds for $f(k)$. A step in this direction is:

THEOREM. *For any integer $k \geqq 2$: $f(k) \leqq 12(k-1)$.*

This theorem is an easy consequence of a result of Edelsbrunner, Robinson and Shen [1], who improved on a result of Wenger [6], proving:

E.R.S. THEOREM. *A collection of $n \geqq 3$ compact, convex and pairwise disjoint convex sets in the plane may be covered with n non-overlapping convex polygons with a total of not more than $6n - 9$ sides.*

It is interesting to note that, more than thirty years ago, essentially the same results as in the E.R.S. theorem have been obtained by L. Fejes Tòth in [4].

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2. Proof of Theorem.

It has to be proved that for any family of $n = 12(k-1)$ nonempty convex sets with pairwise disjoint relative interiors there exists a straight line separating k of the sets from one of the remaining $n - k$ sets. As in Tverberg's paper it is sufficient to assume that the sets are also compact, with non-empty interiors and pairwise disjoint. Due to the E.R.S. Theorem it suffices to assume that the n convex sets are non-overlapping polygons with a total of at most $6n - 9$ sides.

The following observation shall be used:

OBSERVATION. For any two non-overlapping convex polygons there is a line containing one of the sides of one of them and which separates the two polygons.

For each of the $\binom{n}{2}$ pairs $\{A, B\}$ of polygons choose, using the observation, a side from either A or B which is contained in a line separating A from B. Then some side must have been chosen at least $\binom{n}{2}\Big/(6n-9) = n(n-1)/(12n-18) > k-1$ times. But then it is clear that the polygon having that side is separated by the line through that side from at least k other polygons

The best known lower bound for $f(k)$ has been obtained in [3]. Using a construction of Villanger, described in [5], it is proved that $f(k) \geqq 3k - 1$ by constructing for $k \geqq 2$ a family of $3k - 2$ convex sets with disjoint relative interiors such that there is no line separating one of the sets from k others. For

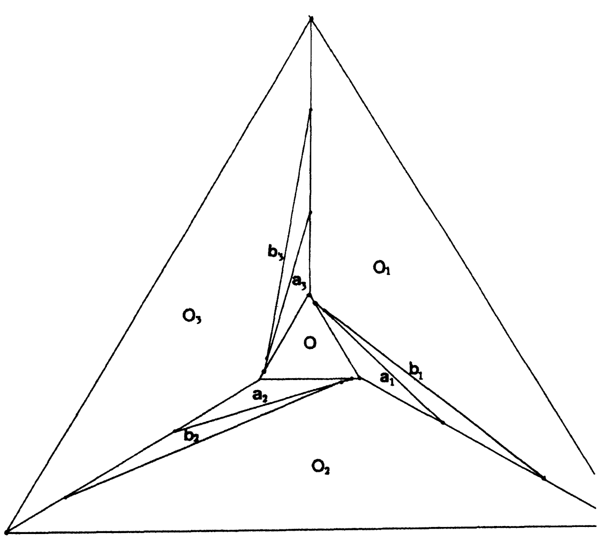

Figure 1.

$k = 4$ the construction is illustrated in Figure 1. It contains six segments a_i, b_i, $i = 1, 2, 3$, an equilateral triangle O and three hexagons O_1, O_2, O_3. Note that the configuration has a 120° rotational symmetry about the center of O. Note that both the vertex of a_1 lying on the side of O and the vertex of b_1 lying on a_1 are close to the top vertex of O.

In the construction for general k the segments a_i, b_i, for $i = 1, 2, 3$, are replaced by $k - 2$ segments, similar to the ones described in Villanger's construction and the three hexagons are replaced by three $(k + 2)$-gons. The 120° rotational symmetry is preserved.

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