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# ALGEBRAIC AND GEOMETRIC CONVERGENCE OF KLEINIAN GROUPS

To the memory of Werner Fenchel

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## 0. Introduction.

A kleinian group $G$ has an algebraic structure when viewed as an abstract group and a geometric structure when viewed as a discrete group of hyperbolic 3-space $\mathsf{H}^3$. Geometrically, $G$ is associated with the hyperbolic 3-manifold (orbifold) $\mathsf{H}^3/G$. The analysis of a sequence $\{G_n\}$ of kleinian groups likewise has an algebraic aspect, relating to the behavior of the sequence of group generators, and a geometric aspect, relating to the behavior of the sequence of associated 3-manifolds. The purpose of this paper is to examine questions of convergence of $\{G_n\}$ from these two points of view. A special goal is to understand the convergence of $\{G_n\}$ in terms of the Carathéodory convergence of the ordinary sets $\{\Omega(G_n)\}$ in the sphere of infinity $\partial\mathsf{H}^3$. This is of particular importance when doing function theory (Poincaré series, etc.) on the ordinary sets.

This joint work began in Copenhagen in 1972. To think about boundaries of spaces of kleinian groups in the associated representation varieties is to think about sequences of Dirichlet (Poincaré) fundamental polyhedra. What can happen? A great deal of the basic theory of kleinian groups can be seen in this context. The studies [5] of cyclic loxodromic groups and [7] of "reopening" cusps, which were done at about this time, are of fundamental importance in understanding what happens in general. From a different direction, joint work [3] with C. Earle introducing geometric, global complex coordinates for Teichmüller space, also begun in 1972 but in Djursholm, made it essential to understand the convergence of ordinary sets on approaching the boundary in a Bers slice of quasifuchsian space.

Our work is organized as follows. Chapter 2 introduces terminology and provides background information. Chapter 3 contains the definitions of algebraic and geometric convergence and proves a number of basic properties.

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Particularly important is the relation to the convergence of fundamental polyhedra (which we will define as "polyhedral convergence"). In our work, it is the fundamental polyhedra that represent the quotient manifolds and the fact of polyhedral convergence justifies the adjective in the term "geometric convergence".

In Chapter 4, the focus is on the convergence of ordinary sets. Our main result, Proposition 4.7. describes the relation of fundamental polyhedra of the approximants $\{G_n\}$ to the fundamental polyhedron of their geometric limit $H$, under the assumption that $H$ is geometrically finite and that the groups $G_n$ are torsion free. This allows us to also describe the algebraic relation of $G_n$ to $H$. The two theorems of this paper, Theorems 4.8 and 4.9, give a summary of our principal results. The first asserts that if $\varphi_n$: $\Gamma \to G_n$ is a sequence of isomorphisms converging algebraically to $\varphi$: $\Gamma \to G$, then a necessary and sufficient condition for Carathéodory convergence $\Omega(G_n) \to \Omega(G)$ is that $\{G_n\}$ convergence geometrically to $G$, provided $G$ is geometrically finite and $\Omega(G) \neq \varnothing$. The second describes in the torsion free case the more general situation that the geometric limit $H$ of $\{G_n\}$, while still required to be geometrically finite, is larger than $G$.

Geometric limits can be dramatically different from their approximants. For instance, examples are known [8] of algebraic convergent sequences $\{G_n\}$ of kleinian groups which are free of rank two, converging geometrically to groups which are not even finitely generated (in fact, are infinite, free amalgamated products with each factor a free group of rank two).

In our work here, we have tried to develop the insight that the "classical" methods reveal. Thus we have limited ourselves to the situations which do not require the powerful techniques of Thurston for dealing with geometrically infinite ends [16].

The final Chapter 5 consists of an example of the geometric convergence of particular cyclic loxodromic groups to rank two parabolic groups. The reader will find explicitly displayed many of the phenomena described in much greater generality elsewhere in the paper. (From a different point of view, the example is very helpful in understanding phenomena on the Bers boundary of Teichmüller space.)

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## 2. Terminology and background.

2.1. We work with hyperbolic space $\mathsf{H}^3$ and its visual boundary $\partial\mathsf{H}^3 = S^2$. In the upper half space model our coordinates will be,

$$\mathsf{H}^3 = \{(z, t): z \in \mathsf{C}, t > 0\}.$$

A kleinian group is a non-elementary discrete subgroup $G$ of PSL(2, C). The limit set is denoted by $\Lambda(G)$, the ordinary set by $\Omega(G)$, possibly $\Omega(G) = \varnothing$. Elementary descrete means that card $\Lambda(G) \leqq 2$.

The notation we use for the associated 3-manifold (orbifold) is,

$$\mathscr{M}(G) = \mathsf{H}^3 \cup \Omega(G)/G \quad \partial\mathscr{M}(G) = \Omega(G)/G.$$

A geometric model for $\mathscr{M}(G)$ is provided by a fundamental polyhedron $\mathscr{P}(G)$. Given a point $O \in \mathsf{H}^3$, not a fixed point of $G$, the (Poincaré or Dirichlet) fundamental polyhedron with center at $O$ is,

$$\mathscr{P}(G) = \{x \in \mathsf{H}^3: d(x, O) \leqq d(x, TO), \forall\, T \in G\}.$$

Its euclidean closure in $\mathsf{H}^3 \cup \partial\mathsf{H}^3$ is denoted by $\mathscr{P}(G)^-$. The intersection with $\Omega(G)$, $\mathscr{P}(G)^- \cap \Omega(G)$, which is a union of circular polygons and perhaps isolated points, contains a fundamental set for the action of $G$ on $\Omega(G)$.

The Ahlfors finiteness theorem [1] says that if $G$ is finitely generated, then $\partial\mathscr{M}(G)$ has a finite number of components, and each component is a compact, closed surface with at most a finite number of punctures, and with at most a finite number of points over which $\Omega(G)$ is branched. It also implies that $\mathscr{P}(G)^- \cap \Omega(G)$ has a finite number of components, and each component is either a point or a finite sided circular polygon [2].

Selberg's lemma [15] says that if $G$ is finitely generated, there is a normal subgroup $G_0$ of finite index in $G$ which contains no elliptic transformations. Thus, $\mathscr{M}(G_0)$ is a finite sheeted cover of $\mathscr{M}(G)$.

2.2. A group $G$ is said to be geometrically finite (the term was coined by Leon Greenberg) if $G$ has a finite sided fundamental polyhedron $\mathscr{P}(G)$. It turns out [11] that if $\mathscr{P}(G)$ is finite sided for one choice of center $O \in \mathsf{H}^3$, it is finite sided for any choice of center. An equivalent definition, emphasized in [16], is that $G$ is geometrically finite if and only if $\mathscr{C}(G)/G$ has finite volume, where $\mathscr{C}(G)$ is the hyperbolic hull of $\Lambda(G)$ (if $G$ preserves a plane, then $\mathscr{C}(G)$ must be taken as the planar convex hull).

The importance of geometric finiteness lies in the fact that $G$ is geometrically finite if and only if $\mathscr{M}(G)$ is "essentially" compact, that is, compact once standard neighborhoods of this "cusps" are removed. We will give a brief description of these neighborhoods.

Because of Selberg's lemma, it is sufficient to consider the case that $G$ has no elliptic elements. If $G$ is to be geometrically finite, a parabolic fixed point $\xi$ must fall into one of two types. The first holds in any kleinian group, but the second is special.

The first possibility is that the parabolic subgroup $\mathrm{Stab}(\xi) = \{T \in G: T\xi = \xi\}$ has rank two. There exists a horoball $\mathscr{H}' \subset \mathsf{H}^3$ at $\xi$ for which $T\mathscr{H}' \cap \mathscr{H}' \neq \emptyset$, $T \in G$, if and only if $T \in \mathrm{Stab}(\xi)$ (in which case $T\mathscr{H}' = \mathscr{H}'$). The quotient,

$$\mathscr{T} = \mathscr{H}'/\mathrm{Stab}(\xi) \cong \{z \in \mathsf{C}; 0 < |z| < 1\} \times S^1,$$

is called a *cusp torus*. It is naturally embedded in $\mathrm{Int}\,\mathscr{M}(G)$.

The other possibility is that $\mathrm{Stab}(\xi)$ is cyclic. Here arises a key criterion for geometric finiteness. Namely that each such subgroup must correspond to a *pair of punctures $p$, $q$* on $\partial\mathscr{M}(G)$. Two punctures $p$, $q$ of $\partial\mathscr{M}(G)$ are said to be paired if there are disjoint neighborhoods $N_1$ of $p$, $N_2$ of $q$ in $\partial\mathscr{M}(G)$, and an associated *pairing tube* $\mathscr{T}$ in $\mathscr{M}(G)$,

$$\mathscr{T} \cong \{z: 0 < |z| < 1\} \times [0, 1],$$

such that $\mathscr{T} \cap \partial\mathscr{M}(G) = N_1 \cup N_2$, and a generator of $\pi_1(\mathscr{T})$ corresponds to a generator of $\mathrm{Stab}(\xi)$. Canonical constructions of pairing tubes as well as an alternate construction of cusp tori will be given in §4.3.

The group $G$ is geometrically finite if and only if there are at most a finite number of (mutually disjoint) cusp tori and pairing tubes in $\mathscr{M}(G)$ such that their complement is compact.

2.3. Although we will be referring to other universal properties of kleinian groups in §3.5, the following result is basic.

*Universal Ball.* There exists $\varepsilon > 0$ such that if $G$ is any kleinian group, there exists a point $O \in \mathsf{H}^3$ such that for the hyperbolic ball $B_\varepsilon(O)$ of radius $\varepsilon$ about $O$,

$$\{T \in G: TB_\varepsilon(O) \cap B_\varepsilon(O) \neq \emptyset\} = \mathrm{id}.$$

In other words, the ball $B_\varepsilon$ of hyperbolic radius $\varepsilon$ can be imbedded in any quotient manifold (orbifold) $\mathscr{M}(G)$, independently of $G$.

2.4. In this paper we will be dealing with sequences of kleinian groups and will require the following two fundamental results in the subject.

THEOREM A (Jørgensen [6]). *Let $\Gamma$ be a group and $\varphi_n: \Gamma \to G_n$ a sequence of isomorphisms onto kleinian groups $G_n$. If* $\lim \varphi_n(\gamma)$ *exists as a Möbius transformation for all $\gamma \in \Gamma$, then the set $G = \{\varphi(\gamma): \gamma \in \Gamma\}$ is a kleinian group and $\varphi: \gamma \mapsto \varphi(\gamma)$ an isomorphism of $\Gamma$ onto $G$.*

THEOREM B (Jørgensen-Klein) [9]). *Let $G_n = \langle g_{1n}, \dots, g_{rn}\rangle$ be a sequence of r-generator kleinian groups such that* $\lim g_{in} = g_i$ *exists as a Möbius transform-*

*ation, $1 \leqq i \leqq r < \infty$. Then $G = \langle g_1, \ldots, g_r \rangle$ is also a kleinian group and the correspondence $\psi_n$: $g_i \mapsto g_{in}$ extends to a homomorphism $\psi_n$: $G \to G_n$ for all large n.*

## 3. Definitions and basic properties.

3.1. Let $\Gamma$ be an abstract group and $\varphi_n$: $\Gamma \to G_n$ a sequence of representations (homomorphisms) into PSL(2, C).

DEFINITION. The sequence of representations $\{\varphi_n\}$ converges *algebraically* to $G$ if $\lim \varphi_n(\gamma) = g$ exists as a Möbius transformation for all $\gamma \in \Gamma$ and

$$G = \{g: g = \lim \varphi_n(\gamma), \ \gamma \in \Gamma\}.$$

If $\Gamma = \langle \gamma_1, \ldots, \gamma_r \rangle$ is finitely generated, and $\{G_n = \varphi_n(\Gamma)\}$ are kleinian, then by the Jørgensen-Klein theorem, the algebraic limit $G$ is kleinian as well and the correspondence $\psi_n$: $g_i \to \varphi_n(\gamma_i)$, where $g_i = \lim \varphi_n(\gamma_i)$, extends to a homomorphism $\psi_n$: $G \to G_n$ for all large $n$.

REMARK. Sometimes we will forget the representation and consider only a sequence of groups $G_n = \langle g_{1n}, g_{2n}, \ldots \rangle$ given in terms of presentations. We say that $\{G_n\}$ converges algebraically to $G$ if $\lim g_{in} = g_i$ exists for all $i$ and $G = \langle g_1, g_2, \ldots \rangle$.

3.2. If $\{G_n\}$ is a sequence of subgroups of PSL(2, C), define

$$\text{Env}\{G_n\} = \{g \in \text{PSL}(2, \text{C}): g = \lim g_n, \ g_n \in G_n\}.$$

It is clear that Env $\{G_n\}$ is itself a group.

LEMMA. If each $G_n$ is discrete, then $H = \text{Env}\{G_n\}$ is either a kleinian group, or it is elementary.

PROOF. According to [6]. $H$ is discrete if and only if every two generator subgroup is discrete. Here the term elementary is used in the extended sense (which applies when $H$ is not discrete) that $H$ elementary means that every two elements of infinite order have a common fixed point on the sphere at infinity. Thus if $H$ is not elementary, given any element $h_1$ of infinite order there is another $h_2$ with distinct fixed points. If $\langle h_1, h_2 \rangle$ were not discrete, we could find $h'_1$, $h'_2 \in \langle h_1, h_2 \rangle$ with $h'_2$ close to id such that $\langle h'_1, h'_2 \rangle$ is non-elementary while $h'_1, h'_2$ violate Jørgensen's inequality [6]. When applied to the approximates $g_{1n}$ and $g_{2n}$ for large indices $n$, $h'_1 = \lim g_{1n}$, $h'_2 = \lim g_{2n}$, this gives a contradiction. A similar argument rules out any two generator, non-discrete subgroup.

DEFINITION. The sequence $\{G_n\}$ of subgroups of PSL(2, C) converges *geometrically* (to $H = \text{Env}\{G_n\}$) if and only if for every subsequence $\{G_{n_j}\}$ of $\{G_n\}$, $\text{Env}\{G_{n_j}\} = \text{Env}\{G_n\}$.

3.3. REMARK Suppose that the group $H$ and sequence of groups $\{G_n\}$ have the properties (a) for every $h \in H$, $h = \lim g_n$, $g_n \in G_n$, and (b) if $g_{n_j} \in G_{n_j}$ is such that $\lim g_{n_j} = h$ exists, then $h \in H$. Then $H = \text{Env}\{G_n\}$ and $\{G_n\}$ converges geometrically. Conversely, if $\{G_n\}$ converges geometrically, then $H = \text{Env}\{G_n\}$ satisfies (a) and (b).

PROOF. This is a restatement of the definitions.

In practice, we will only be interested in geometric convergence to a kleinian group $H$.

3.4. As a step towards justification of the use of the adjective "geometric", we introduce a seemingly different notion of convergence.

DEFINITION. The sequence of discrete groups $\{G_n\}$ converges *polyhedrally* to the group $H$ if $H$ is discrete and for some point $O \in \mathsf{H}^3$, the fundamental polyhedra $\{\mathscr{P}(G_n)\}$ centered at $O$ for $G_n$ converge to that $\mathscr{P}(H)$ at $O$ for $H$, uniformly on compact subsets of $\mathsf{H}^3$.

More precisely, the criterion for convergence is this: Given any (large) $r > 0$ set

$$B_r = \{x \in \mathsf{H}^3 \colon d(O,x) < r\}$$

and let $\mathscr{P}_r = \mathscr{P} \cap B_r$ denote the truncated polyhedron. We refer to the possibly truncated faces $f \cap B_r$, where $f$ is a face of $\mathscr{P}_r$. The faces of $\mathscr{P}_r$ are congruent in pairs, under elements of the group.

For each $r > 0$, the requirement for polyhedral convergence is that there exist $N = N(r)$ with the following properties: (a) for each face pairing transformation $h$ of $\mathscr{P}(H)_r$ there exists one $g_n$ of $\mathscr{P}(G_n)_r$ for all $n \geqq N$ such that $\lim g_n = h$, and (b) if $g_n$ is a face pairing transformation of $\mathscr{P}(G_n)_r$ then the limit $h$ of any convergent subsequence of $\{g_n\}$ is a face, edge or vertex pairing transformation of $\mathscr{P}(H)_r$ (in particular, $h \neq \text{id.}$).

3.5. PROPOSITION. *To any infinite sequence of discrete groups $\{G_n\}$ corresponds a sequence of conjugates $\{A_n G_n A_n^{-1}\}$ which contains a polyhedrally convergent subsequence.*

PROOF. Given $O \in \mathsf{H}^3$, choose Möbius transformation $\{A_n\}$ so that the conjugate groups $\{G'_n = A_n G_n A_n^{-1}\}$ have the following property (Universal Ball Property). For each $n$, the polyhedron $\mathscr{P}(G'_n)$ centered at $O$ contains the ball $B_\delta$ of radius $\delta > 0$ about $O$.

For each fixed $r > \delta$, the number of faces of $\mathscr{P}(G'_n)_r$ is uniformly bounded above as $n \to \infty$. The reason for this is that there is an upper bound on the number of mutually disjoint balls of radius $\delta$ contained in the ball $B_{3r}$ of radius $3r$ about $O$. Therefore there is an upper bound on the number of points in the orbit $\{G'_n(O)\}$ which lie in $B_{3r}$, and number of corresponding planes that might contain a face of

$\mathscr{P}(G_n')_r$. A face pairing transformation $g_n$ of $\mathscr{P}(G_n')_r$ satisfies $d(O, T_n(O)) < 2r$. Therefore each sequence $\{g_n\}$ of them has a convergent subsequence. Since also $d(O, g_n(O)) > 2\delta$, the limit is not the identity.

For fixed $r$ and each $n$ list the face pairing transformations (including inverses) of $\mathscr{P}(G_n')_r$. There are at most $M$ of these for some $M < \infty$ and by repetition we may assume that there are exactly $M$, $\{g_{in}\}$, $1 \leqq i \leqq M$. Take a subsequence and relabel so that $h_i = \lim g_{in}$ exists, $1 \leqq i \leqq M$. Corresponding, construct the polyhedron,

$$\mathscr{P}_r = \{x \in \mathsf{H}^3 \colon d(O,x) \leqq d(x, h_i(O)),\ 1 \leqq i \leqq M\}.$$

Thus $\mathscr{P}_r \cap B_r = \lim \mathscr{P}(G_n')_r$ and $\mathscr{P}_r \supset B_\delta$. We must allow the possibility that $\mathscr{P}(G_n')_r = B_r$ for all large $n$ and hence that $\mathscr{P}_r = \mathsf{H}^3$.

Now take a sequence $r = r_k \to \infty$ and go through this process for each $r_k$. We get a sequence $\{h_i\}$, $i = 1, 2, \ldots$, and correspondingly a nested family of polyhedra $\mathscr{P}_{r_1} \supset \mathscr{P}_{r_2} \supset \ldots$ all of which contain $B_\delta$.

Let $H$ be the group generated by all $\{h_i\}$ and let $\mathscr{P}_\infty = \cap \mathscr{P}_{r_n}$. We claim that $H$ is discrete and that $\mathscr{P}_\infty$ is a fundamental polyhedron for it. And furthermore, that $H$ is the geometric limit of $\{G_n'\}$ (now a subsequence of the original sequence). Possibly $H = \{\mathrm{id}\}$ and $\mathscr{P}_\infty = \mathsf{H}^3$, if the groups $\{G_n\}$ blow up completely.

Given $s > 0$ there exists $r = r(s)$ such that the orbit of $\mathscr{P}(G_n')_r$ under $G_n'$ covers the ball $B_s$ for all $n$. The number of polyhedra which meet $B_s$ is uniformly bounded in $n$. Therefore a list of $M$ transformations $W_{1n}, \ldots, W_{Mn}$, for some $M < \infty$, can be made for each $n$ such that $\cup W_{in}(\mathscr{P}(G_n')_r)$ covers $B_s$. Replacing by another subsequence if necessary, we may assume that $\lim W_{in} = W_i$ exists, $1 \leqq i \leqq M(s)$. Necessarily, $W_i \in H$. We conclude that $\cup W_i(\mathscr{P}_\infty \cap B_r)$ covers $B_s$.

Finally, allowing $s \to \infty$ and passing to more subsequences if necessary, we see that the orbit of $\mathscr{P}_\alpha$ under $H$ covers all of $\mathsf{H}^3$.

No two points $x, y \in \operatorname{Int} \mathscr{P}_\infty$ are equivalent under $H$. For suppose to the contrary that $y = Wx$, $W \in H$. The element $W$ is a word in the generators $\{h_i\}$. Let $W_n$ denote the corresponding word in the letters $\{g_{in}\}$ so that $W_n \to W$ and $\lim W_n(x) = y$. That is, if $r > \max(d(O,x), d(O,y))$, then for all large $n$, $x$ and $W_n(x)$ belong to $\mathscr{P}(\mathscr{G}_n')_r$. This is impossible, unless $W_n = W = \mathrm{id}$. We conclude that $\mathscr{P}_\infty$ is a fundamental polyhedron for $H$, so that $H$ necessarily discrete.

3.6. LEMMA. *Suppose $\{G_n\}$ is a sequence of kleinian groups converging algebraically to G. Then there is no sequence $\{S_n\}$, $S_n \in G_n$, $S_n \neq$ id, with* $\lim S_n = \mathrm{id}$, *or with* $\lim S_n = S$ *elliptic of infinite order.*

PROOF. Present $G_n = \langle g_{1n}, g_{2n}, \ldots \rangle$ where $\lim g_{in} = g_i$ and $G = \langle g_1, g_2, \ldots \rangle$. It suffices to rule out the following three individual cases.

*Case* 1. $S_n$ is elliptic for large $n$. We claim that (for all large $n$) no generator $g_{in}$ can share exactly one fixed point with $S_n$. Indeed, if this were false for $g_{1n}$, say, then $g_{1n}$ would be parabolic and $\langle S_n, g_{1n}\rangle$ elementary. But for this to be the case, the order of $S_n$ could not exceed six. Nor is it possible that every generator $g_{in}$ either shares the same fixed points with $S_n$ or is an element of order two and interchanges them. For if this were the case, then $G_n$ would itself be elementary. Consequently, we may assume that $\langle g_{1n}, S_n\rangle$ is not elementary for all large $n$. But now we have a contradiction to the Jørgensen-Klein theorem.

*Case* 2. $S_n$ is parabolic for all large $n$. Not every generator of $G_n$ can share the fixed point of $S_n$; so again, say for $g_{1n}$, $\langle g_{1n}, S_n\rangle$ is not elementary.

*Case* 3. $S_n$ is loxodromic for all large $n$. Once again not every generator $g_{in}$ of $G_n$ can have either the same fixed points as $S_n$ or be an elliptic transformation of order two that interchanges them.

3.7. LEMMA. *Suppose $\{G_n\}$ is an algebraically convergent sequence of kleinian groups. There exists a point $O \in \mathsf{H}^3$ and $\varepsilon > 0$ such that, for a subsequence $\{G_k\}$, no element of $G_k$ for any $k$ has a fixed point in the ball $B_\varepsilon$ of radius $\varepsilon$ about $O$.*

PROOF. The proof relies on two universal properties of kleinian groups:

*Universal Elementary Neighborhoods* [11]. There exists $\theta > 0$ such that for any $x \in \mathsf{H}^3$ and any kleinian group $G$, the subgroup generated by

$$\{g \in G: d(x, gx) < \theta\}$$

is elementary.

*Isolation of rotation axes* [11]. There exists $\delta > 0$ such that for any kleinian group $G$ the distance between any two rotation axes which have no common point in $\mathsf{H}^3 \cup \partial\mathsf{H}^3$ is at least $\delta$, unless both are the axes of rotations of order two.

To prove Lemma 3.7, start with any $x \in \mathsf{H}^3$. We claim that there exists $\varepsilon > 0$ such that any two rotation axes of $G_n$ which intersect $B_\varepsilon(x) = \{y \in \mathsf{H}^3: d(x,y) < \varepsilon\}$, intersect in a point $x_n \in B_\varepsilon(x)$.

Suppose not. Then there is a sequence $\varepsilon_n \to 0$ and rotation axes $\alpha_{1n}, \alpha_{2n}$ of $E_{1n}, E_{2n} \in G_n$ which intersect $B_{\varepsilon_n}(x)$. We may assume $E_{1n} \to E_1$, $E_{2n} \to E_2$ and hence that $x$ is fixed by $E_1$, $E_2$. For all large $n$, $\langle E_{1n}, E_{2n}\rangle$ is elementary. Yet $E_{1n}, E_{2n}$ do not share a common parabolic fixed point on $\partial\mathsf{H}^3$ (their commutator would then be parabolic) nor are they both elliptic of order two with disjoint axes ($E_{1n}E_{2n}$ would then be loxodromic likewise converging to the identity). Consequently $\langle E_{1n}, E_{2n}\rangle$ is a finite non-cyclic group with a common fixed point $x_n \in \mathsf{H}^3$, and $\langle E_{1n}, E_{2n}\rangle$ is isomorphic to $\langle E_1, E_2\rangle$ for all large $n$, by Lemma 3.6. It must also be that $\lim x_n = x$.

The argument shows that there exists $\varepsilon > 0$ and a point $x_n \in B_\varepsilon(x)$ such that any rotation axis of $G_n$ that intersects $B_\varepsilon(x)$ passes through $x_n$, for all large $n$. Moreover, the finite subgroups $\mathrm{Stab}(x_n) \subset G_n$ are all isomorphic to the limit group which we will denote by $\mathrm{Stab}(x)$.

Now there are only a finite number of possibilities for Stab($x$), unless it is cyclic or a $\mathbf{Z}_2$ extension of a cyclic group. We can find $y \in B_\varepsilon(x)$ and $\varepsilon_1 < \varepsilon$ such that $TB_{\varepsilon_1}(y) \cap B_{\varepsilon_1}(y) = \varnothing$ for all $T \neq id. \in \mathrm{Stab}(x)$. This property will persist for Stab($x_n$), for sufficiently large $n$, that is, $TB_{\varepsilon_1}(y) \cap B_{\varepsilon_1}(y) = \varnothing$ for all $T \neq \mathrm{id} \in \mathrm{Stab}(x_n)$. The proof is complete.

3.8. PROPOSITION. *Suppose the kleinian groups* $\{G_n\}$ *converge algebraically to* $G$. *Then there is a polyhedrally convergent subsequence* $\{G_k\}$. *The limit* $H$ *of any subsequence contains* $G$. *If* $H$ *is finitely generated, then there is a homomorphism* $\psi_k$ *of* $H$ *into* $G_k$ *for all large* $k$ *such that* $\lim \psi_k(h) = h$ *for all* $h \in H$. *If in addition* $G$ *is finitely generated, then* $\psi_k(H) = G_k$.

PROOF. Again set $G_n = \langle g_{1n}, g_{2n}, \dots \rangle$ and $G = \langle g_1, g_2, \dots \rangle$ with $g_i = \lim g_{in}$. By Lemma 3.6, there is no sequence $\{S_n\}$ with $S_n \in G_n$, $S_n \neq \mathrm{id}$, yet $\lim S_n = \mathrm{id}$. Consequently, if none of the groups $G_n$ have elliptic elements, there is a small ball $B_\varepsilon$ about $O$ which is contained in every polyhedron $\mathscr{P}(G_n)$. Then $\{G_k\}$ can be found exactly as in the proof of Proposition 3.5.

When there are elliptic elements, we have to know that for some point $O \in \mathrm{H}^3$, such a ball $B_\varepsilon$ exists, at least for a subsequence $\{G_k\}$. This fact is a consequence of Lemma 3.7. From this lemma, we deduce that there exists $\delta > 0$, $\delta < \varepsilon$, such that for all large $k$,

$$\{S \in G_k : SB_\delta \cap B_\delta \neq \varnothing\} = \{\mathrm{id}\}.$$

Here $B_\delta$ is the ball of radius $\delta$ about $O$. For suppose this were false. Then corresponding to a sequence $\delta_n \to 0$ would be a sequence $S_k$, $k = k(n)$, $S_k \in G_k$, $S_k \neq \mathrm{id}$, with $S_k B_{\delta_n} \cap B_{\delta_n} \neq \varnothing$. There is a convergent subsequence, say $\{S_k\}$ again, whose limit $S$ fixes $O$ but whose approximants $S_k$ have no fixed point in $B_{\delta_n}$. The only possibility is that $S_k = \mathrm{id}$ (Lemma 3.6).

So in all cases we can find a subsequence $\{G_k\}$ converging geometrically to a discrete group $H$.

Given a compact set $K$ in $\mathrm{H}^3$, there exists $r > 0$ and $N$ with the following property: $K$ is covered by the images of the truncated polyhedron $\mathscr{P}(G_k)_r$ under all words of length $\leq N$ in the face pairing transformations of $\mathscr{P}(G_k)_r$, for all large $k$.

To see why, choose a larger compact set $K'$ containing $K$ in its interior. For large enough $r$ and $N$, the orbit $\mathscr{P}_N^*$ of $\mathscr{P}(H)_r$ under words of length $\leq N$ in its face pairing transformations covers $K'$. When $k$ is large, $\mathscr{P}(G_k)_r$ is close to $\mathscr{P}(H)_r$ and its orbit $\mathscr{P}_{kN}^*$ under words of length $\leq N$ in its face pairing transformation covers $K$.

This implies that $H$ contains $G$. For given $S \in G$, take $r > d(O, SO)$ and set $K = B_r^-$. We know $S = \lim S_k$, $S_k \in G$, and for large $k$, $S_k(O) \in K$. Therefore $S_k$ is a word of length $\leq N$ in the face pairing transformations of $\mathscr{P}(G_k)_r$. In the limit then, $S \in H$.

Finally, assume that $H$ is finitely generated and hence by the theorem of Scott-Shalen [14], finitely presented. Fix a presentation. Each generator $W$ is a word in the face pairing transformations of $\mathscr{P}(H)$. Thus for sufficiently large $k_0$. let $\psi_k(W)$ designate the element of $G_k$ which is the same word in the corresponding face pairing transformations of $\mathscr{P}(G_k)$, $k \geqq k_0$; $\lim \psi_k(W) = W$.

The correspondence $\psi_k$ determines a homomorphism of $H$ into $G_k$, for $k \geqq k_1 \geqq k_0$. For if $R(W) = 1$ is a relation in $H$, then by Lemma 3.6, $\psi_k(R(W) = 1$ must also hold for all large $k$.

Now return to the correspondence $\varphi_k: g_i \to g_{ik}$ determined by algebraic convergence. The generator $g_1$, say, of $G$ is a word $\Phi_1(W)$ in the generators $\{W\}$ of $H$. Because $\lim \varphi_k(g_1)^{-1}\psi_k\Phi_1(W) = \text{id}$, for all sufficiently large $k$, $\psi_k\Phi_1(W) = \varphi_k(g_1)$. If $G$ has only a finite number of generators, $G = \langle g_1, \ldots . g_r \rangle$, then for sufficiently large $k_2$, we can insure that $\psi_k\Phi_j(W) = \varphi_k(g_j)$, $1 \leqq j \leqq r$, generates $G_k$, $k \geqq k_2$.

REMARK. There are many examples, for instance of fuchsian groups, which show that polyhedral convergence does not imply algebraic convergence.

3.9. Our proof above that $\psi_k: H \to G_k$ is a homomorphism into did not require that $\{G_k\}$ have an algebraic limit. We state this result separately as follows.

COROLLARY. *Suppose that the sequence of kleinian groups $\{G_k\}$ converges polyhedrally to a finitely generated kleinian group $H$. Then there is a homomorphism $\psi_k$ of $H$ into $G_k$ for all large $k$ such that* $\lim \psi_k(h) = h$ *for all $h \in H$.*

3.10. We can now justify use of the term "geometric convergence".

PROPOSITION. (i) *The sequence $\{G_n\}$ of kleinian groups converges geometrically to a kleinian group if and only if $\{G_n\}$ converges polyhedrally to a kleinian group. The geometric and polyhedral limits of $\{G_n\}$ are the same.*

(ii) *Suppose $\varphi_n: \Gamma \to G_n$ is a sequence of isomorphisms of a group $\Gamma$ onto kleinian groups $G_n$ converging algebraically to $\varphi: \Gamma \to G = \varphi(\Gamma)$. Then $\{G_n\}$ also converges geometrically to $G$ if and only if to every subsequence $\{m\}$ such that $h = \lim \varphi_m(\gamma_m)$ exists, $\gamma_m \in \Gamma$, then $h \in G$ and $\gamma_m = \varphi^{-1}(h)$ for all large $n$.*

PROOF OF (i). Assume that $\{G_n\}$ converges polyhedrally to $H$. We refer to the statement of Lemma 3.3. That (b) holds was shown in the course of proving Proposition 3.8. For (a), if $h \in H$ then $h$ is a word in the face pairing transformation of $\mathscr{P}(H)$ and the limit of the corresponding words in the face pairing transformations of $\mathscr{P}(G_n)$.

Conversely, assume (a) and (b) of Lemma 3.3 hold for the kleinian group $H$ with respect to its kleinian approximants $\{G_n\}$. By the Universal Ball Property, for some $\delta > 0$ and $O \in \mathrm{H}^3$, the ball $B_\delta$ about $O$ is mapped disjoint to itself by all elements of $H$. We claim that by (b), the same property holds for $G_n$, for large $n$.

Otherwise, there would be a sequence $g_n \in G_n$ with $g_n \to \mathrm{id}$. Since $H$ is non-elementary, we can find $h_1, h_2, h_3 \in H$ which are loxodromic with mutually distinct fixed points. Then $h_i = \lim S_{in}$, $i = 1, 2, 3$, $S_{in} \in G_n$. For large $n$, at least one of the $S_{in}$ is loxodromic with fixed points distinct from those of $g_n$. Then, $\langle g_n, S_{in} \rangle$ is non-elementary for some $i$, contradicting Jørgensen's inequality.

We have shown that there are polyhedrally convergent subsequences of $\{G_n\}$. By Lemma 3.3, they all converge to the geometric limit $H$ of $\{G_n\}$.

PROOF OF (ii). Suppose $\{G_n\}$ also converges geometrically to $G = \varphi(\Gamma)$. Assume $h = \lim \varphi_m(\varphi_m)$ for some subsequence $\{m\}$ and $\gamma_m \in \Gamma$. Then $h = \varphi(h_0)$ lies in $G$, and $\lim \varphi_m(\gamma_m h_0^{-1}) = \mathrm{id}$. By Lemma 3.6, $\gamma_m = h_0$. The converse follows from (i).

REMARK. Our discussion also applies in the following elementary situation.

A sequence $\{ < T_n > \}$ of cyclic loxodromic groups converges polyhedrally to a discrete parabolic group $P$ if and only if it converges geometrically to $P$ and no subsequence of distinct elements converges to the identity.

3.11. COROLLARY. *If $\{G_n\}$ converges polydedrally to $H$ with respect to one choice of center for the fundamental polyhedra, it converges polyhedrally to $H$ with respect to any other choice of center.*

PROOF. No choice of center is involved in the statement of Proposition 3.10(i).

3.12. PROPOSITION. *Suppose $\varphi_n \colon \Gamma \to G_n$ is a sequence of isomorphisms of a group $\Gamma$ onto kleinian groups $G_n$ converging algebraically to $G = \varphi(\Gamma)$ such that $\{G_n\}$ converges geometrically to $H \supset G$. Suppose $\Gamma_0 \subset \Gamma$ is a subgroup of finite index $r$ and $\{\varphi_n(\Gamma_0)\}$ converges geometrically to $H_0 \supset \varphi(\Gamma_0)$. Then $H_0$ has index $r$ in $H$. Moreover, $H = \varphi(\Gamma)$ if and only if $H_0 = \varphi(\Gamma_0)$.*

PROOF. Let $\Gamma_0 x_1, \ldots, \Gamma_0 x_r$ denote distinct $\Gamma_0$-cosets that fill up $\Gamma$. We start by observing that for $i \neq j$, $\varphi(x_i x_j^{-1}) \in H_0$. For if the contrary were true then $\varphi(x_i x_j^{-1}) = \lim \varphi_n(\gamma_n)$ for some $\gamma_n \in \Gamma_0$. That is, $\lim \varphi_n(\gamma_n x_j x_j^{-1}) = \mathrm{id}$. This implies $x_i = \gamma_n x_j$ for all large $n$ contradicting our assumption that $\Gamma_0 x_i \cap \Gamma_0 x_j = \emptyset$. Consequently the $H_0$-cosets $H_0\varphi(x_1), \ldots, H_0\varphi(x_r)$ are distinct. Their union is $H$. For if $h \in H$ then $h = \lim \varphi_n(\gamma_n)$ for $\gamma_n \in \Gamma$. Write $\gamma_n = \beta_n x_{i(n)}$ where $\beta_n \in \Gamma_0$ and the index $i$ depends on $n$. Choose a subsequence $\{k\}$ of $\{n\}$ for which $i(k) = j$ for all $k$. Then $\lim \varphi_k(\beta_k) = h\varphi(x_j^{-1})$ so that $h\varphi(x_j^{-1}) \in H_0$ and hence $h \in H_0\varphi(x_j)$. This completes the proof of the first statement of the Proposition.

If $\{\varphi_n(\Gamma)\}$ converges geometrically to $\varphi(\Gamma)$ then by Proposition 3.10(ii), $\{\varphi_n(\Gamma_0)\}$ converges geometrically to $\varphi(\Gamma_0)$.

Conversely, suppose $\{\varphi_n(\Gamma_0)\}$ converges geometrically to $\varphi(\Gamma_0)$. For a subsequence $\{k\}$, suppose $\gamma_k \in \Gamma$ is such that $h = \lim \varphi_k(\gamma_k)$ exists.

*Case* 1. $h$ has infinite order. Passing to another subsequence if necessary we may assume that there exists $r$, $1 \leqq r \leqq [\Gamma : \Gamma_0]$ such that $\gamma_k^r \in \Gamma_0$ for all $k$.

Therefore $h^r \in \varphi(\Gamma_0)$ and $h^r = \varphi(T)$ for some $T \in \Gamma_0$. Furthermore for all large $k$, $T = \gamma_k^r$, since $\lim \varphi_k(\gamma_k^r T^{-1}) = \text{id}$.

Thus for all large $k$, we may assume for all $k$, $\gamma_k$ is an $r$th root of $T$. Two $r$th roots differ at most by an elliptic transformation with the same fixed points as $T$ and whose order divides $r$: $\gamma_k = E_k \gamma_1$ where $E_k^r = \text{id}$. There are only a finite number of possible elements $E_k$ so by passing to another subsequence we may assume $E_k = E$ for all $k$. But then $\gamma_k = E\gamma_1$ and $h = \lim \varphi_k(\gamma_k) = \varphi(E\gamma_1) \in \varphi(\Gamma)$.

*Case* 2. $h = \lim \varphi_k(\gamma_k)$ has finite order $q$. Then for all large $k$, we may assume for all $k$, $\gamma_k$ is elliptic of order $q$ as well. We will show that the set of elements $\{\gamma_k\}$ must be finite. This implies that $h \in \varphi(\Gamma)$ thereby completing the proof. Suppose then that $\{\gamma_k\}$ is an infinite set.

First, consider the situation if $q > 2$. Fix $T \in \Gamma$ loxodromic and normalize so that

$$T \sim \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad \gamma_k \sim \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}, \quad a_k d_k - b_k c_k = 1.$$

For $p \in \mathbf{Z}$, since $\operatorname{tr} \gamma_k = 2\cos(\pi q_k/q)$ for some $q_k$ relatively prime to $q$,

$$\operatorname{tr} T^p \gamma_k = (u^p - u^{-p}) a_k + 2u^{-p} \cos(\pi q_k/q).$$

We may assume that $\{a_k\}$ converges in $\mathbf{C} \cup \{\infty\}$. It is then clear that for some integer $p$, after passing to another subsequence if necessary, $T^p \gamma_k$ is not elliptic for all $k$.

Consequently by Case 1, $\lim \varphi_k(T^p \gamma_k) = \varphi(T^p) h \in \varphi(\Gamma)$ and $h = \varphi(T^{-p} T_1)$ for some $T_1 \in \Gamma$. Thus $\gamma_k = T^{-p} T_1$ for all large $k$, a contradiction.

There remains the possibility that $q = 2$. We see from the formula above that $T\gamma_k$ is elliptic of order two only when $\gamma_k$ interchanges the fixed points of $T$ ($a_k = d_k = 0$), that is, only when their axes intersect in a right angle in $\mathbf{H}^3$. By a suitable choice of $T$ we may avoid this occurance. Now replace $\gamma_k$ by $T\gamma_k$ and carry on as above.

3.13. COROLLARY. *Suppose that $\psi_n: G \to G_n$ is a sequence of homomorphisms of a kleinian group $G$ onto kleinian groups $G_n$ such that $\lim \psi_n(g) = g$ for all $g \in G$. Suppose $\{G_n\}$ converges geometrically to $H \supset G$. Let $G_0 \subset G$ be a subgroup of finite index $r$ such that $\{\psi_n(G_0)\}$ converges geometrically to $H_0$. Then $H_0$ has index $\leqq r$ in $H$. Moreover, if $H_0 = G_0$ then $H = G$.*

PROOF. Let $G_0 x_1, \ldots, G_0 x_r$ be a full set of distinct cosets of $G_0$ in $G$. If $h \in H$ then $h = \lim \psi_n(g_n)$ for some $g_n \in G$. Write $g_n = f_n x_{i(n)}$ where $f_n \in G_0$. Choose a subsequence $\{k\}$ for which $i\{k\} = j$ is constant. We find that $h x_j^{-1} = \lim \psi_k(f_k) \in H_0$ and therefore that $h \in H_0 x_j$. This shows that the cosets $H_0 x_1, \ldots, H_0 x_r$ fill up $H$. In particular, if $H_0 = G_0$, then $H = G$.

**4. Convergence of regions of discontinuity.**

4.1. A sequence of open sets $\{\Omega_n\} \subset \partial \mathsf{H}^3$ is said to *converge in the sense of Carathéodory* to the open set $\Omega$ if (i) every compact subset $K$ of $\Omega$ lies in $\Omega_n$ for all suffciently large $n$, and (ii) every open set $U$ which lies in $\Omega_m$ for an infinite subsequence $\{m\}$ of $\{n\}$ also lies in $\Omega$. This is the standard notion of convergence in the theory of conformal mapping.

The concept of Carathéodory convergence $\Omega_n \to \Omega$ is the same as Hausdorff convergence $\Lambda_n \to \Lambda$ of the complement $\Lambda_n$ of $\Omega_n$ to that, $\Lambda$, of $\Omega$.

In this paper we will always be dealing with situations that $\Omega_n = \Omega(G_n)$ and $\Omega = \Omega(G^*)$ for discrete groups $G_n$ and $G^*$ such that each $g \in G^*$ is the limit of elements $g_n \in G_n$: $g = \lim g_n$. For such situations, condition (ii) holds automatically. Therefore $\{\Omega_n\}$ converges to $\Omega$ in the sense of Carathéodory if and only if any given compact subset $K$ of $\Omega$ lies in $\Omega_n$ for all sufficiently large $n$.

4.2. PROPOSITION. *Assume that $\Gamma$ is a finitely generated group and $\varphi_n: \Gamma \to G_n$ a sequence of isomorphisms of $\Gamma$ onto kleinian groups $G_n$ converging algebraically to (an isomorphism) $\varphi: \Gamma \to G$ with $\Omega(G) \neq \varnothing$. Then if $\Omega(G_n) \to \Omega(G)$ in the sense of Carathéodory, $\{G_n\}$ converges geometrically to $G$.*

PROOF. Assume to the contrary that $\{G_n\}$ does not converge geometrically to $G$. Then according to Proposition 3.10(ii), for a subsequence $\{k\}$ and elements $\gamma_k \in \Gamma$, $g_k = \varphi_k(\gamma_k) \in G_k$ is such that $h = \lim g_k$ exists with $h \notin G$. Fix a compact set $K \subset \Omega(G)$ and choose another compact set $K'$ such that $K \subset \operatorname{Int} K' \subset K' \subset \Omega(G)$. The sequence $g_k(K)$ converges to $h(K)$. We claim that $h(K) \subset \Omega(G)$.

If not, the interior $\operatorname{Int} h(K') = \lim g_k(\operatorname{Int} K')$ containts limit points of $G$ and hence loxodromic fixed points of elements of $G$. If $\operatorname{Int} h(K')$ contains a loxodromic fixed point of $\varphi(A)$, for $A \in \Gamma$, then $\operatorname{Int} g_k(K')$ also does, for $n$ large, say $n \geqq N_1$. There exists $N_2 = N_2(N_1)$ such that $\operatorname{Int} g_k(K')$ contains a fixed point of $\varphi_m(A)$ for $m \geqq N_2$. That is, $g_k(K')$ contains a fixed point of $\varphi_k(A)$ for $k \geqq \max(N_1, N_2)$. This contradicts the assumption that $K' \subset \Omega(G_k)$ for all sufficiently large $k$.

It follows that $h(K) \subset \Omega(G)$ for every compact subset $K$ of $\Omega(G)$. Consequently $h\Omega(G) \subset \Omega(G)$. The same argument can be applied to $h^{-1} = \lim \varphi_k(\gamma_\kappa^{-1})$. We conclude that $h\Omega(G) = \Omega(G)$. In particular, if $h$ is loxodromic or parabolic its fixed points lie in $\Lambda(G)$.

We may assume that the geometric limit $H \supset G$ of $\{G_k\}$ exists (Proposition 3.8). At this point we apply an important theorem of Leon Greenberg [4] that states that if $\Lambda(H) = \Lambda(G)$ is not a circle on $\partial \mathsf{H}^3$, then $G$ has finite index in $H$. But the condition of finite index holds when $\Lambda(G)$ is a circle too. For then $G$ is a fuchsian group of finite area or a $\mathsf{Z}_2$-extension of one so that the larger discrete group $H$ must contain $G$ as a subgroup of finite index.

In fact the only possibility is that $H = G$. For assume that $h = \lim \varphi_k(\gamma_k) \in H$, $h \notin G$. If $h$ has infinite order, then for some $m$, $h^m \neq \mathrm{id} \in G$ and so $h^m = \varphi(A)$ for some $A \in \Gamma$. Therefore, $\lim \varphi_k(\gamma_\kappa^m A^{-1}) = \mathrm{id}$. By Lemma 3.6, $A = \gamma_\kappa^m$ for all large $k$. An element of infinite order in a kleinian group can have at most one $m$th root in the group. Thus all the elements $\gamma_k$ are the same and $h \in G$. This is a contradiction.

The other possibility is that $h$ is elliptic. Choose a loxodromic element $g \in G$ whose fixed points are not interchanged by $h$ For some integer $p$, $g^p h$ is not elliptic (cf. proof of Proposition 3.12). From above, $g^p h \in G$, again a contradiction.

REMARK. Suppose instead that $\varphi_n : \Gamma \to G_n$ is a sequence of representations of $\Gamma$ onto kleinian groups $G_n$ converging algebraically to $G$ and suppose $\{G_n\}$ converges geometrically to $H$. If $\Omega(G_n) \to \Omega(G)$ in the sense of Carathéodory, then $G$ has finite index in $H$. For the first part of the proof above applies to this case.

4.3. For the remainder of this Chapter we will consider only geometrically finite groups without elliptic elements. The reason for the requirement of geometric finiteness is that we will base our study of deformations on fundamental polyhedra. When there are no elliptic elements, we can use the nice generic polyhedra constructed in [10]. In view of Proposition 3.12, from the point of view of the ordinary sets, restricting our attention to torsion free groups is not a serious limitation.

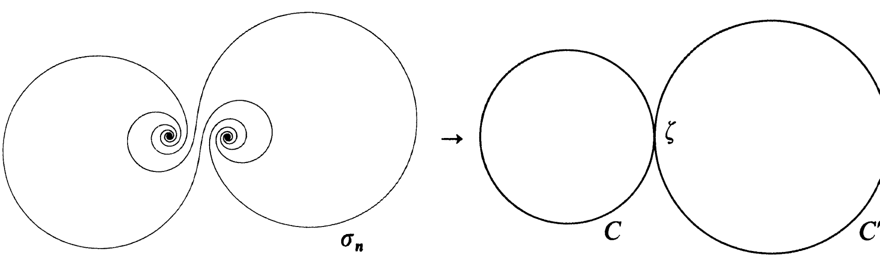
We will start out, in this and the next section, with a discussion of geometric convergence of cyclic loxodromic groups. The basic analysis was carried out in [5] in reference to the isometric (Ford) fundamental polyhedron. However, it carries over the polyhedra centered at any point $O \in \mathrm{H}^3$. Although the number of faces depends on the choice of center $O$, the combinatorial possibilities for each of the polyhedra are the same, for each cyclic group. In particular, all Dirichlet fundamental polyhdra meet the sphere at infinity in a connected set, and the cycles about all edges in $\mathrm{H}^3$ have order three (see §4.5).

4.4. Suppose then that $S_n$ is loxodromic and $\{\langle S_n \rangle\}$ converges polyhedrally to a discrete parabolic group $P$. Given $O \in \mathrm{H}^3$ let $\mathscr{P}_n$ denote the fundamental polyhedron for $\langle S_n \rangle$ centered at $O$ and $\mathscr{P}$ denote that for $P$. Set $F_n = \mathscr{P}_n^- \cap \Omega(\langle S_n \rangle)$ and $F = \mathscr{P}^- \cap \Omega(P)$, where $\mathscr{P}_n^-, \mathscr{P}^-$ denote euclidean closure in $\mathrm{H}^3 \cup \partial \mathrm{H}^3$.

A modification of the arguments of [5] shows that $F_n$ is connected; in fact this holds for any cyclic loxodromic group. When $P$ has rank two we can verify this directly for large $n$ since $F$ has four or six sides so that $F_n$ must as well. When $P$ has rank one, it is true that $F_n$ has two, four, or six sides (and is connected) but we do not need the full force of this statement. What we require can be deduced from the fact of polyhedral convergence alone. In any case, $\mathscr{T}_n = \Omega(\langle S_n \rangle)/\langle S_n \rangle$ is a torus.

*Case* 1. $\{\langle S_n \rangle\}$ converges to the cyclic parabolic group $\langle S \rangle$. Then necessarily, $S$ is the limit of generators of the $\{\langle S_n \rangle\}$ so we may assume $\lim S_n = S$. The polyhedra $\{\mathscr{P}_n\}$ converge to $\mathscr{P}$, the fundamental polyhedron for $\{S\}$. The two faces of $\mathscr{P}$ determines disks $D$, $D'$ in $\partial \mathsf{H}^3$ which are externally tangent at the fixed point $\zeta$ of $S$. Two of the sides of $F_n$ converge to $\partial D$, $\partial D'$ and any other shrink in $\partial \mathsf{H}^3 \backslash (D \cup D')$ towards $\zeta$. Likewise in $\mathsf{H}^3$, two of the faces of $\mathscr{P}_n$ converge to those of $\mathscr{P}$ while all the other faces collapse towards $\zeta$.

The quotient torus $\mathscr{T}_n$ conveges to the doubly punctured sphere $\mathscr{T}_\infty = \Omega(\langle S \rangle)/\langle S \rangle$. Let $C$, $C'$ be a pair of horocircles for $S$, externally tangent at $\zeta$. On $\mathscr{T}_\infty$, $C$ and $C'$ becomes two disjoint simple loops separating the punctures. On $\mathscr{T}_n$ we can find a pair of simple loops which converges to these. Back up in $\partial \mathsf{H}^3$, this means that we can find a sequence of Jordan curves $\{\sigma_n\}$, where $\sigma_n$ passes through the fixed points of $S_n$ and is invariant under $S_n$, and $\lim \sigma_n = C \cup C'$, the convergence being uniform away from $\zeta$.




*In* $\mathsf{H}^3$ it will be convenient to have some canonical neighborhoods of $\zeta$. With respect to $\mathsf{H}^3$, construct the convex hull of the closed set consisting of $C \cup C'$ and their disjoint interiors. In boundary $\mathscr{S}$ in $\mathsf{H}$ is a smooth simply connected surface meeting $\partial \mathsf{H}^3$ orthogonally in $C \cup C' \backslash \{\zeta\}$. (This is easiest to visualize when $\zeta = \infty$.) Let $\mathscr{H}$ denote the interior of the convex hull and set $\mathscr{H}' = \mathsf{H}^3 \backslash \mathscr{H}$; $\mathscr{H}'$ is closed and contains a horoball at $\xi$. For the quotients, $\mathscr{S}/\langle S \rangle$ is a cylinder and $(\mathscr{H}'^{-} \backslash \{\zeta\})/\langle S \rangle$ is a pairing tube homemomorphic to $\{z : 0 < |z| \leqq 1\} \times [0, 1]$.

Correspondingly, let $\mathscr{H}_n$ denote the interior of the convex hull of the union of $\sigma_n$ and its "interior"; set $\mathscr{H}_n' = \mathsf{H}^3 \backslash \mathscr{H}_n$. The relative boundary $\mathscr{S}_n$ of $\mathscr{H}_n'$ is also smooth (if we so choose $\sigma_n$, except at the fixed points $p_n$, $q_n$ of $S_n$). The quotient $\mathscr{S}_n/\langle S_n \rangle$ is again a cylinder but now $\mathscr{H}_n'^{-} \backslash \{p_n, q_n\}/\langle S_n \rangle$ is a solid torus.

The point of the construction is that the truncated $\mathscr{P}_n \cap \mathscr{H}_n$ converges uniformly to $\mathscr{P} \cap \mathscr{H}$. Note too that the orbit of $\mathscr{P}_n \cap \mathscr{H}_n$ under $\langle S_n \rangle$ is just $\mathscr{H}_n$, which is simply connected in $\mathsf{H}^3$.

*Case* 2. $\{\langle S_n \rangle\}$ converges geometrically to a rank two parabolic group $\langle S_1, S_2 \rangle$ where $S_1 = \lim S_n^k$ and $S_2 = \lim S_n^l$ with $k = k(n)$, $l = l(n)$. The fundamental polyhedron $\mathscr{P}$ for $\langle S_1, S_2 \rangle$ is a 4 or 6-faced chimney rising toward the common fixed point $\zeta$. $F = \mathscr{P}^- \cap \Omega(\langle S_1, S_2 \rangle)$ has 4 or 6 sides, and is a fundamental polygon for $\langle S_1, S_2 \rangle$.

Let $B_r$ designate the open ball centered at $O$ with radius $r$ so large that it crosses all the edges of $\mathscr{P}$. Let $(\mathscr{H}_r')$ denote the component of $\mathscr{P}\backslash B_r \cap \mathscr{P}$ that is adjacent to $\zeta$ and write $\mathscr{H}_r'$ for its $\langle S_1, S_2\rangle$-orbit. The opposite faces of $(\mathscr{H}_r')$ match up under $\langle S_1, S_2\rangle$ and the quotient under this identification is a solid cusp torus homeomorphic to $\{z:0 < |z| \leqq 1\} \times S^1$. The full orbit $\mathscr{H}_r'$ "looks" like a horoball at $\zeta$.

Away from $\zeta$, $\mathscr{P}_n$ converges to $\mathscr{P}$. The polyhedron $\mathscr{P}_n$ will have an increasing number of faces but all except 4 or 6 appear closer and closer to $\zeta$ and disappear in the limit. Thus, $F_n = \mathscr{P}_n^- \cap \Omega(\langle S_n\rangle)$ converges uniformly to $F$.

For large $n$, let $(\mathscr{H}_{rn}')$ denote the component of $\mathscr{P}_n\backslash B_r \cap \mathscr{P}_n$ which converges to $(\mathscr{H}_r')$. The faces of $(\mathscr{H}_{rn}')$ match up under $\langle S_n\rangle$ and the result of the identification is a solid torus. In this case, however, the full orbit $\mathscr{H}_{rn}'$ of $(\mathscr{H}_{rn}')$ under $\langle S_n\rangle$ is a simply connected, banana-shaped neighborhood of the axis of $S_n$. It will be important to remember that the complement $\mathsf{H}^3\backslash\mathscr{H}_m'$ is not simply connnected.

There is a natural homomorphism $\psi_n$ of $\langle S_1, S_2\rangle$ onto $\langle S_n\rangle$ which sends the face pairing transformations of $\mathscr{P}$ to the 2 or 3 elements of $\langle S_n\rangle$ which pair the corresponding faces of $\mathscr{P}_n$. The kernel of $\psi_n$ is generated by a transformation $T$ that gives rise to a simple loop on the boundary of the solid tours $\mathscr{H}_{rn}'/\langle S_n\rangle$. This loop bounds a disk in its interior. In $\mathsf{H}^3$, this is a simple loop on the banana skin $\partial\mathscr{H}_{rn}'$ which separates the tips, $p_n$ and $q_n$.

4.5. Generic fundamental polyhedra. Suppose that $G$ is a geometrically finite, torsion free kleinian group and $\mathscr{P}_0$ is a fundamental polyhedron with center $O \in \mathsf{H}^3$. Set $\mathscr{P} = \mathscr{P}_0^- \cap (\mathsf{H}^3 \cup \Omega(G))$. The order $k$ of a vertex $v$ of $\mathscr{P}_0$ is the number of distinct vertices of $\mathscr{P}_0$ in the equivalence class of $v$: it is the number of polyhedra $\mathscr{P}_0, T_2\mathscr{P}_0, \dots, T_k\mathscr{P}_0$ in the $G$-orbit of $\mathscr{P}_0$ that share the vertex $v$; the transformations $T_i \in G$ are said to be associated with $v$. The order $k$ of an edge $e$ of $\mathscr{P}_0$ is the number of distinct edges of$_0$ in the equivalence class of $e$: it is the number of polyhedra $\mathscr{P}_0$, $T_2\mathscr{P}_0, \dots, T_k\mathscr{P}_0$ in the $G$-orbit of $\mathscr{P}_0$ that share the edge $e$; the transformations $T_i \in G$ are said to be associated with $e$.

A cusp of $\mathscr{P}$ is a parabolic fixed point that lies in the euclidean closure $\mathscr{P}^-$. It is of rank one or rank two. Boundary vertices and edges are those that lie in $\Omega(G)$. Associated with each boundary vertex is a vertex cycle.

The main result of [10] is that there is a dense set of points in $\mathsf{H}^3$ so that if $O$ is chosen from among these, $\mathscr{P}$ will be generic in the following sense.

(i) Each edge $e$ of $\mathscr{P}_0$ for which the line $l(e) \subset \mathsf{H}^3$ containing $e$ does not end at a parabolic fixed point has an edge cycle of order three. If $l(e)$ ends at a parabolic fixed point $\zeta$, then $e$ has an edge cycle of length three or four, and every transformation associated with $e$ fixes $\zeta$.

(ii) Three edges emanate from each vertex $v$ of $\mathscr{P}_0$. For at most one of them $e$, $l(e)$ ends at a parabolic fixed point $\zeta$. The order of $v$ is either four or five. In the latter case three of the four transformations $\neq \mathrm{id}$ associated with $v$ are parabolic and fix the end point $\zeta$ of $l(e)$ for some edge $e$ from $v$.

(iii) Every boundary vertex $v^*$ is the end point of exactly one edge of $\mathscr{P}_0$. The vertex cycle at $v^*$ has length three.

(iv) No edges of $\mathscr{P}_0$ end at a rank one cusp $\zeta$ of $\mathscr{P}$ but two faces of $\mathscr{P}$ are tangent to $\zeta$ with a face pairing transformation that fixes $\zeta$. Each rank two cusp $\zeta$ is the end point of four or six edges of $\mathscr{P}_0$. The corresponding four or six faces are paired by elements of Stab($\zeta$).

REMARK. In (i) and (ii). If $e$ has an edge cycle of order four, then Stab($\zeta$) is a rank two parabolic group that represents a rectangular torus.

4.6. LEMMA. *Suppose $H$ is a geometrically finite group and $\psi_n: H \to G_n$ is a sequence of homomorphisms onto torsion free kleinian groups such that $\lim \psi_n(h) = h$ for all $h \in H$. Assume that $\{\psi_n(H_0)\}$ converges geometrically to $H_0$ for every maximal parabolic subgroup $H_0$ of $H$. Let $\mathscr{P}$ be a generic fundamental polyhedron for $H$ with center at $O \in \mathbf{H}^3$ and*

$$F(\mathscr{P}) = \{S \in H: S\mathscr{P}^- \cap \mathscr{P}^- \neq \varnothing\}$$

*the set of face pairing, edge pairing, vertex pairing, and cusp fixing transformations of the euclidean closure $\mathscr{P}^-$ of $\mathscr{P}$. Then for all large n,*

$$\mathscr{P}_n^* = \{x \in \mathbf{H}^3: d(x, O) \leqq d(x, TO),\ T \in \psi_n(F(\mathscr{P}))\}$$

*is the (generic) fundamental polyhedron for $G_n$ centered at $O$. If $\psi_n(h)$ is parabolic whenever $h \in H$ is parabolic, then for all large n, $\psi_n$ is an isomorphism and converges algebraically to* id.

PROOF. To verify that $\mathscr{P}_n^*$ is a fundamental polyhedron we need only to verify the hypothesis of Poincaré's theorem (see [13]): that the dihedral angles for each edge cycle of $\mathscr{P}_n^*$ add up to $2\pi$.

Consider first the situation about an edge $e$ of $\mathscr{P}_0$ which has order three. In the orbit of $\mathscr{P}$ under $H$ there is a cycle of polyhedra that share $e$:

$$\mathscr{P},\ S_1\mathscr{P},\ S_1S_2\mathscr{P},\ S_1S_2S_3\mathscr{P} = \mathscr{P}$$

where $S_1$, $S_2$ and $S_3$ are face pairing transformations of $\mathscr{P}$ and $e$ is contained in the intersection of bisecting planes of the line segments $[O, S_1O]$, $[O, S_1S_2O]$.

There is a corresponding edge $e_n$ of $\mathscr{P}_n^*$ which is contained in the intersection of the bisecting planes of $[O, \psi_n(S_1)O]$, $[O, \psi_n(S_1S_2)O]$. Automatically, the cycle of polyhedra,

$$\mathscr{P}_n^*,\ \psi_n(S_1)\mathscr{P}_n^*,\ \psi_n(S_1S_2)\mathscr{P}_n^*,\ \psi_n(S_1S_2S_3]P_n^* = \mathscr{P}_n^*$$

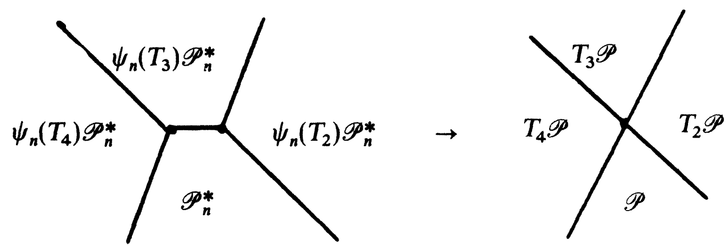
are arranged about $e_n$ without overlap.

If instead $e$ has order four then the cycle about $e$ is

$$\mathscr{P},\ S_1\mathscr{P},\ S_1S_2\mathscr{P},\ S_1S_2S_3\mathscr{P},\ S_1S_2S_3S_4\mathscr{P} = \mathscr{P}.$$

The three transformations $T_2 = s_1$, $T_3 = s_1 s_2$, $T_4 = s_1 s_2 s_3$ associated with $e$ are parabolic with a common fixed point $\zeta$ which is also an end point of the line $l(e)$ containing $e$. Indeed, $l(e)$ is an edge of the fundamental polyhedron $\mathscr{P}(\text{Stab}\,\zeta)$ for $\text{Stab}\,\zeta = \{h \in H: h\zeta = \zeta\}$ centered at $O$, where Stab $\zeta$ has rank two.

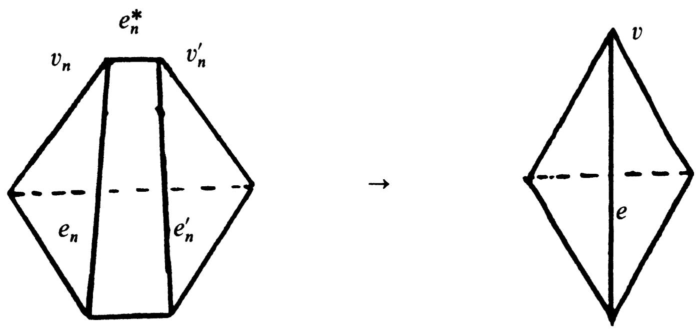
Suppose $\psi_n(\text{Stab}\,\zeta)$ remains parabolic of rank two. Depending on whether $\mathscr{P}(\psi_n(\text{Stab}\,\zeta))$ has four or six edges, one or two of these edges converge to $l(e)$ as $n \to \infty$. Assume for definiteness the latter case, which is the general case, holds for all large $n$. Then $\mathscr{P}_n^*$ will have two edges $e_n$, $e_n'$ both of which converge to $e$, and also one face of $\mathscr{P}_n^*$ converges to $e$. The situation as $n \to \infty$ is described by the following diagram illustrating the situation in a small disk $D$ transverse to $e$.




If $\psi_n(\text{Stab}\,\zeta)$ is a cyclic loxodromic group, the diagram remains valid for all large $n$ (assuming $D$ is transverse at an interior point of $e$). There will be two edges $e_n$, $e_n'$ of $\mathscr{P}_n^*$ which converge to $e$. The $e_n$ and $e_n'$ will be contained in edges of the fundamental polyhedron $\mathscr{P}(\psi_n(\text{Stab}\,\zeta))$ centered at $O$, all of whose edges have order three. The transformations associated with the edges $e_n$, $e_n'$, namely $\psi_n(T_i)$, $i = 1, 2, 3, 4$, lie in $\psi_n(\text{Stab}\,\zeta)$.

Now let $v$ be a vertex of $\mathscr{P}_0$. Exactly three faces of $\mathscr{P}_0$ intersect at $v$. If the order of $v$ is four, and $T_1 = \text{id}$, $T_2, T_3, T_4$ denote the associated transformations, then $v$ is the point of intersection of the three bisecting planes of the three segments $[O, T_i O]$, $i = 2, 3, 4$. Correspondingly, $\mathscr{P}_n^*$ has a vertex $v_n$, also of order four, which is the intersection of the planes bisecting $[O, \psi(T_i)O]$, $i = 2, 3, 4$. The vertex $v_n$ converges to $v$, and $\mathscr{P}_n^*$ has no edges which collapse to $v$.

On the other hand suppose the order of $v$ is five. Denote the associated transformations by $T_1 = \text{id}$, $T_2, T_3, T_4, T_5$. Three of these, say $T_2, T_3, T_4$ are parabolic with a common fixed point $\zeta$. Furthermore, one of the three edges $e$ emanating from $v$ is contained in an edge $l(e)$ of $\mathscr{P}(\text{Stab}\,\zeta)$ (Stab $\zeta$ is necessarily of rank two; in fact, it represents a rectangular torus). The edge $e$ has order four and its associated transformations $\neq$ id are $T_2, T_3, T_4$. The vertex $v$ is the intersection of $l(e)$ with the plane bisecting $[O, T_5 O]$.




Assume first that $\psi_n(\mathrm{Stab}\,\zeta)$ is also a parabolic group and for definiteness assume the general case that $\mathscr{P}(\psi_n(\mathrm{Stab}\,\zeta))$ has six edges and faces rather than the four of $\mathscr{P}(\mathrm{Stab}\,\zeta)$. Two of the edges $l(e_n)$, $l(e_n')$ of $\mathscr{P}(\psi_n(\mathrm{Stab}\,\zeta))$ converge to $l(e)$ and two edges $e_n \subset l(e_n)$, $e_n' \subset l(e_n')$ of $\mathscr{P}_n^*$ converge to $e$. Instead of one vertex near $v$, $\mathscr{P}_n^*$ has two vertices $v_n$, $v_n'$ and an edge $e_n^*$ between them all of which converge to $v$. The edge $e_n^*$ is contained in the intersection with $\mathscr{P}(\psi_n(\mathrm{Stab}\,\zeta))$ of the plane bisecting $[O, \psi_n(T_5)O]$; $e_n^*$ lies in the line of intersection of the bisecting planes of $[O, \psi_n(T_5)O]$ and $[O, \psi_n(T_4)O]$, if $T_4$ is that transformation associated with $e$ for which $T_4\mathscr{P} \cap \mathscr{P} = e$.

The three polyhedra $\mathscr{P}_n^*$, $\psi_n(T_5)P_n^*$, share the edge $e_n^*$. We see that these form the edge cycle about $e_n^*$.

The analysis about $v$ when $\psi_n(\mathrm{Stab}\,\zeta)$ is a cyclic loxodromic is the same as that given above. In particular, $\mathscr{P}_n^*$ has two vertices $v_n$, $v_n'$ and one edge $e_n^*$ all of which converge to $v$.

It remains to analyze the situation about each cusp $\zeta$ of $\mathscr{P}$ for which $\psi_n(\mathrm{Stab}\,\zeta)$ is not a parabolic group for large $n$. Necessarily, it is then a cyclic loxodromic group which is converging geometrically to $\mathrm{Stab}(\zeta)$: $\psi_n(\mathrm{Stab}(\zeta))$ is generated by the $\psi_n$-image of the generator or generators of $\mathrm{Stab}(\zeta)$.

We deal with this by isolating the end of $\mathscr{P}$ near $\zeta$ using one of the objects $\mathscr{H}'$ of §4.3 to truncate $\mathscr{P}$ and correspondingly $\mathscr{H}_n'$ to truncate $\mathscr{P}_n^*$. We can choose $\mathscr{H}'$, $\mathscr{H}_n'$ to be mutually disjoint for distinct parabolic vertices on $\mathscr{P}^-$. The situation in the ends $\mathscr{P} \cap \mathscr{H}'$ and $\mathscr{P}_n^* \cap \mathscr{H}_n'$ depends only on the groups $\mathrm{Stab}(\zeta)$ and $\psi_n(\mathrm{Stab}(\zeta))$ as described in §4.3. The remainder $\mathscr{P} \setminus \mathscr{P} \cap \mathscr{H}'$ and $\mathscr{P}_n^* \setminus \mathscr{P}_n^* \cap \mathscr{H}_n'$ has been analyzed above. In short, $\mathscr{P}_n^*$ is a fundamental polyhedron for $G_n$.

The argument shows that if $\psi_n$ preserves parabolic transformations then all the edge relations of $\mathscr{P}_n^*$ already appears as edge relations of $\mathscr{P}$. That is, $\psi_n$ is an isomorphism.

4.7. The analysis of §§4.4–4.6 gives detailed information about the relation of $\mathscr{P}_n^*$ to $\mathscr{P}$ for large $n$. We put this together in the following statement.

PROPOSITION. *Suppose H is a geometrically finite kleinian group and* $\psi_n: H \to G_n$ *is a sequence of homomorphisms onto torsion free kleinian groups* $G_n$ *such that* $\lim \psi_n(h) = h$ *for all* $h \in H$ *and for any maximal parabolic subgroup* $H_0$ *of* $H$, $\{\psi_n(H_0)\}$ *converges geometrically to* $H_0$. *Then,*

(i) $\{G_n\}$ *converges geometrically to* $H$,

(ii) *The ordinary sets converge,* $\Omega(G_n) \to \Omega(H)$, *in the sense of Carathéodory,*

*Denote by* $\mathrm{Stab}(\zeta_i)$, $1 \leqq i \leqq p$, *and* $\mathrm{Stab}(\eta_j)$, *the rank one and rank two, respectively, parabolic subgroups of* $H$ *for which* $\psi_n(\mathrm{Stab}(\zeta_i))$, $\psi_n(\mathrm{Stab}(\eta_j))$ *are cyclic loxodromic, in each case one representative from each conjugacy class; possibly one or both classes will be empty.*

(iii) $\mathrm{Ker}\, \psi_n$ *is the normal closure of the subgroup generated by* $\{T^*_{jn}\}$, $1 \leqq j \leqq q$, *for all large n, where* $T^*_{jn}$ *is a generator of*

$$\ker(\psi_n\colon \mathrm{Stab}(\eta_j) \to \psi_n(\mathrm{Stab}(\eta_j))),$$

(iv) *Associated to each* $\mathrm{Stab}(\zeta_i)$ *is a pairing tube* $\mathscr{T}_i \cong \{0 < |z| < 1\} \times [0,1]$ *in* $\mathscr{M}(H)$ *and to each* $\mathrm{Stab}(\eta_j)$ *a solid cusp torus* $\mathscr{T}_j \sim \{0 < |z| < 1) \times S^1$. *Corresponding to each* $\mathscr{T}_j$ *there is in* $\mathrm{Int}\, \mathscr{M}(G_n)$ *a solid torus* $\mathscr{T}'_{jn} \sim \{|z| < 1\} \times S^1$ *such that, for all large n,*

$$\psi_{n*}\colon \pi_1(\mathscr{M}(H)) \to \pi_1(\mathscr{M}(G_n))$$

*is induced by a quasiconformal homeomorphism,*

$$\psi_n\colon \mathscr{M}(H) \setminus (\cup \mathscr{T}_i \cup \mathscr{T}_j) \to \mathscr{M}(G_n) \setminus \cup \mathscr{T}'_{jn}.$$

In short, the rank one groups $\mathrm{Stab}(\zeta_i)$ have been "opened up" in $\mathscr{M}(G_n)$, and the rank two groups $\mathrm{Stab}(\eta_j)$ arise as the end result of a sequence of Dehn surgeries on the solid tori $\mathscr{T}'_{jn}$.

PROOF. Let $\mathscr{P}$ be a generic polyhedron for $H$. The point of Lemma 4.6 was to show that the polyhedron $\mathscr{P}^*_n$ for $G_n$ has the same combinatorial structure as $\mathscr{P}$, except near the parabolic vertices of the type $\{\zeta_i\}$ or $\{\eta_j\}$. This implies that when $\mathscr{P}$ and correspondingly $\mathscr{P}^*_n$ are truncated near the $\{\zeta_i\}$ and $\{\eta_j\}$, using the constructions of §4.4, to get $\mathscr{P}'$ and $\mathscr{P}^{*\prime}_n$, the results of identifying the faces of $\mathscr{P}'$ and of $\mathscr{P}^{*\prime}_n$ are homomorphic manifolds. This is also true of the surface formed from $\mathscr{P}'^{-} \cap \Omega(H)$ and $\mathscr{P}^{*\prime -}_n \cap \Omega(G_n)$.

The manifold resulting from the face identification of $\mathscr{P}'$ is just $\mathscr{M}(H)$ less the pairing tubes $\mathscr{T}_i$ and cusp tori $\mathscr{T}_j$. The complement in $\mathscr{M}(G_n)$ of the $\mathscr{P}^{*\prime}_n$-manifold is a union of solid tori. Those $\mathscr{T}'_{jn}$ corresponding to $\mathscr{T}_j$ are relatively compact in $\mathrm{Int}\, \mathscr{M}(G_n)$ but those $\mathscr{T}'_{in}$ corresponding to $\mathscr{T}_j$ are retractable onto an annular region in $\partial \mathscr{M}(G_n)$, as indicated in §4.4. Thus the structure is as stated in (iv).

The convergence of $\mathscr{P}^*_n$ to $\mathscr{P}$ proves (i), and it also proves (ii). To prove (iii), we have to understand the situation in more detail.

The orbit $\mathcal{H}$ of $\mathcal{P}'$ under $H$ is an open, simply connected region. The orbit $\tilde{e}$ of the edges of $\mathcal{P}'$ form a network in $\mathcal{H}$. A word $W$ in the face pairing transformations of $\mathcal{P}'$ can be interpreted as a path in $\mathcal{H}\setminus\tilde{e}$, starting from $O\in\mathcal{P}'$, running through a face of $\mathcal{P}'$ to the neighboring polyhedron, etc., until ending up at $W(O)$. If $W=\text{id}$, that is, if $W$ is a relation in $H$, then $\gamma$ is a closed loop in $\mathcal{H}\setminus\tilde{e}$. As such it is homotopic to a product of simple loops $\alpha$, where each $\alpha$ is freely homotopic in $\mathcal{H}\setminus\tilde{e}$ to a small circle about an edge of $\tilde{e}$. This is the proof that every relation $W=\text{id}$ is the consequence of the relations about the edges of $\mathcal{P}'$. For example $\alpha$ itself determines the relation conjugate to the one about the corresponding edge of $\mathcal{P}'$.

In contrast, the orbit $\mathcal{H}_n$ of $\mathcal{P}_n^{*\prime}$ is not simply connected. Each component $N(S_{jn})$ of $\mathsf{H}^3\setminus\mathcal{H}_n$ is a banana shaped neighborhood of the axis of some $S_{jn}$, or conjugate of $S_{jn}$, in $G_n$. Here $S_{jn}$ is a generator of $\psi_n(\text{Stab})(\eta_j))$. Its boundary $\partial N(S_{jn})$ is itself the union of fundamental polygons. These come from the $G_n$-orbit of a face $f_n$ of $\mathcal{P}_n^{*\prime}$ that lies on $\partial\mathcal{H}_n$. Such a face $f_n$, in turn, corresponds to a face $f$ of $\mathcal{P}'$ that lies on $\partial\mathcal{H}$.

Consequently a non-trivial simple loop $\beta\subset\partial N(S_{jn})$ that passes through a succession of fundamental polygons, crossing from one to the next over a common side, can be interpreted as follows. The loop $\beta$ is conjugate in $G_n$ to a word in the side pairing transformations of $f_n$. This word is the identity in $G_n$ but it corresponds to a word in the side pairing transformations of $f$ which is not the identity in $H$. Instead, it is an element of $\text{Stab}(\eta_j)$.

Now examine $\mathcal{H}_n$ and the $G_n$-orbit $\tilde{e}_n$ of the edges of $\mathcal{P}_n^{*\prime}$. Some of these edges end on the $\partial N(S_{jn})$. Given a word $W$ in the face pairing transformations of $\mathcal{P}'$, there is a corresponding word $W_n$ in those of $\mathcal{P}_n^{*\prime}$. The word $W_n$ may be interpreted as a path $\gamma_n$ from $O$ in $\mathcal{H}_n\setminus\tilde{e}_n$. We are interested in the case that $\gamma_n$ is closed, that is, that $W_n=\text{id}$.

Suppose first that $\gamma_n\sim\text{id}$ in $\mathcal{H}_n$. Then $\gamma_n$ is homotopic in $\mathcal{H}_n\setminus\tilde{e}_n$ to a product of simple loops, each surrounding an edge of $\tilde{e}_n$. In this case we conclude that the relation $W_n=\text{id}$ is a consequence of the edge relations of $\mathcal{P}_n^{*\prime}$. But these come from the edge relations of $\mathcal{P}'$. That is, $W=\text{id}$ in $H$.

On the other hand, if $\gamma_n$ is not homotopic to 1 in $\mathcal{H}_n$, then in $\mathcal{H}_n\setminus\tilde{e}_n$, $\gamma_n$ is homotopic to a product of simple loops about the edges in $\tilde{e}_n$ as before, and also simple loops $\{\beta\}$ where each $\beta$ is freely homotopic in $\mathcal{H}_n\setminus\tilde{e}_n$ to a non-trivial simple loop $\beta'$ in some $\partial N(S_{jn})$, passing through a succession of fundamental polygons there. Returning to $W$ from this decomposition of $W_n$, we can conclude as follows. That $W$ is a product of parabolic transformations each one of which is conjugate to an element of some $\text{Stab}(\eta_j)$. This completes the argument for (iii).

4.8. On combining Propositions 4.2, 4.7 and 3.12, we are led to the following statement.

THEOREM. *Suppose $\varphi_n: \Gamma \to G_n$ is a sequence of isomorphisms of a group $\Gamma$ onto kleinian groups $G_n$ that converges algebraically to $\varphi: \Gamma \to G$. Suppose $G$ is geometrically finite with $\Omega(G) \neq \emptyset$. Then $\{G_n\}$ converges geometrically to $G$ if and only if the ordinary sets converge, $\Omega(G_n) \to \Omega(G)$, in the sense of Carathéodory.*

4.9. The next result is also based on Proposition 4.7 but focuses on algebraic vis-à-vis geometric convergence.

THEOREM. *Suppose that $\Gamma$ is a finitely generated, torsion free group and $\theta_n$: $\Gamma \to G_n$ a sequence of isomorphisms onto kleinian groups that converges algebraically to $\theta: \Gamma \to G$. Assume that $\{G_n\}$ converges geometrically to a geometrically finite kleinian group $H$. Then*

(i) *The ordinary sets converge, $\Omega(G_n) \to \Omega(H)$, in the sense of Carathéodory.*

(ii) *For all large $n$, there is a homomorphism $\psi_n: H \to G_n$ such that $\lim \psi_n(h) = h$ for all $h \in H$ and for $g \in G$, $\psi_n(g) = \theta_n\theta^{-1}(g)$.*

(iii) Denote by $\{\mathrm{Stab}(\eta_j)\}$, $1 \leqq j \leqq q$, *the rank two parabolic subgroups of $H$ for which $\psi_n(\mathrm{Stab}(\eta_j))$ is cyclic loxodromic, one representative from each conjugacy class in $H$. Denote by $T_{jn}^*$ a generator of the kernel of the homomorphism $\psi_n$: $\mathrm{Stab}(\eta_j) \to \psi_n(\mathrm{Stab}(\eta_j))$. Then $\ker \psi_n$ is the normal closure of the subgroup of $H$ generated by $\{T_{jn}^*\}$, $1 \leqq j \leqq q$.*

(iv) *Assume in addition that each parabolic subgroup $\mathrm{Stab}(\eta_j)$ contains an element of $G$. Then for some $T_j^* \in \mathrm{Stab}(\eta_j)$, $T_j^* \notin \mathrm{Stab}(\eta_j)$, $T_j^* \notin G$,*

$$H = \langle G, T_1^*, \ldots, T_q^* \rangle.$$

REMARK. In (iii) and (iv) we allow the possibility that there are no such subgroups $\{\mathrm{Stab}(\eta_j)\}$ of $H$. Theorem 4.9 asserts that this is the case if and only if $H = G$.

PROOF. Statement (i) and the existence of $\psi_n$ are contained in Proposition 3.8 and 4.6. The restriction of $\psi_n$ to $G \subset H$ satisfies $\lim \theta_n\theta^{-1}(g^{-1})\psi_n(g) = \mathrm{id}$, $g \in G$. By Lemma 3.6, there exists $N$ such that

$$\theta_n\theta^{-1}(g) = \psi_n(g), \quad n \geqq N,$$

first for a set of generators $g$ of $G$, and then for all $g \in G$.

Statement (iii) is also contained in Proposition 4.7. For (iv), let $T_j$ be a generator of the subgroup $\mathrm{Stab}_0(\eta_j)$ of $G$ that fixes $\eta_j$. Then $\psi_n(T_j)$ is a generator of $\psi_n(\mathrm{Stab}(\eta_j)) = \theta_n\theta^{-1}(\mathrm{Stab}_0(\eta_j))$. Consequently, $\mathrm{Stab}(\eta_j) = \langle T_j, T_{jn}^* \rangle$. Fix $T_j^*$ in $\mathrm{Stab}(\eta_j)$ so that $\mathrm{Stab}(\eta_j) = \langle T_j, T_j^* \rangle$.

REMARK. As pointed out by Thurston [17], it is a consequence of the Ahlfors finiteness theorem that if $H$ is a geometrically finite group whose limit is not the whole sphere and if $G$ is a finitely generated subgroup, then $G$ is geometrically finite as well. Thus for the situation above, the algebraic limit $G$ is geometrically finite.

## 5. An example

5.1. In conjunction with §§2–4, it is very illuminating to study the following explicit example of conformal Dehn surgery.

We start with the parabolic group.

$$\Gamma = \langle T_1 z = z + \omega_1, T_2 z = z + \omega_2 \rangle, \quad \tau = \omega_2/\omega_1, \operatorname{Im} \tau > 0,$$

which represents an abstract torus $\mathscr{T} = \mathbf{C}/\Gamma$. The generating pair $(\omega_1, \omega_2)$ corresponds to a pair of simple loops $\alpha, \beta$ on $\mathscr{T}$.

Change the basis by the rule,

$$\omega_{1n} = \omega_1 + n\omega_2, \quad \omega_{2n} = \omega_2; \quad \tau_n = \omega_{2n}/\omega_{1n} = \tau/(1 + n\tau),$$

so that $T_{1n}z = z + \omega_{1n}$, $T_{2n} = z + \omega_{2n}$ also generate $G$. The pair $(\omega_{1n}, \omega_{2n})$ represents the simple loops $\alpha + n\beta$, $\beta$ on $\mathscr{T}$

Map $\mathbf{C}$ onto $\mathbf{C}\backslash\{0\}$ by

$$w_n(z) = \exp(-2\pi i z/\omega_{1n}).$$

We find that,

$$(w_n \circ T_1)(z) = (\exp 2\pi i n \tau_n) w_n(z) = (U_n^{-n} \circ w_n)(z)$$

$$(w_n \circ T_2)(z) = (\exp -2\pi i \tau_n) w_n(z) = (U_n \circ w_n)(z)$$

$$(w_n \circ T_{1n})(z) = w_n(z)$$

$$(w_n \circ T_{2n})(z) = (\exp -2\pi i \tau_n) w_n(z) = (U_n \circ w_n)(z),$$

where $U_n(w)$ is the loxodromic transformation,

$$U_n(w) = (\exp -2\pi i \tau_n) w = a_n w.$$

In short, $w_n$ determines a conformal mapping

$$\mathscr{T} \to (\mathbf{C}\backslash\{0\})/\langle U_n \rangle$$

in which the image of $\alpha + n\beta$ bounds a disk in the solid torus

$$\mathbf{H}^3 \cup (\mathbf{C}\backslash\{0\})/\langle U_n \rangle.$$

5.2. As $n \to \infty$, $\lim \tau_n = 0$ and $\lim U_n = \mathrm{id}$. We can replace $U_n$ by a conjugate $V_n$ so that $\lim V_n(w) = w + \omega_2$. To carry out this renormalization, set

$$A_n(w) = w + \frac{\omega_2}{1 - a_n}.$$

We find that

$$V_n(w) = A_n U_n A_n^{-1}(w) = a_n w + \omega_2.$$

At the same time we compute for any integer $k$,

$$V_n^k(w) = A_n U_n^k A_n^{-1} = a_n^k w + \frac{a_n^k - 1}{a_n - 1}\omega_2.$$

Consequently, set

$$\tilde{w}_n(z) = \left(\frac{\omega_2}{a_n - 1}\right) w_n(z),$$

$$f_n(z) = A_n \circ \tilde{w}_n(z) = \frac{\omega_2}{a_n - 1}(w_n(z) - 1).$$

Since any transformation $w \to bw$ commutes with $U_n$, we find that $f_n$ satisfies the formulas

$$f_n \circ T_1(z) = V_n^{-n} \circ f_n(z)$$

$$f_n \circ T_2(z) = V_n \circ f_n(z)$$

$$f_n \circ T_{1n}(z) = f_n(z)$$

$$f_n \circ T_{2n}(z) = V_n \circ f_n(z).$$

In short, $f_n$ is a conformal mapping

$$\mathcal{T} \to \mathbf{C} \backslash \{\omega_2/(1 - a_n)\}/\langle V_n \rangle$$

such that the image of $\alpha + n\beta$ bounds a disk in the associated solid torus. The image solid tori have been renormalized.

5.3. PROPOSITION. (i) $\lim f_n(z) = z$, *uniformly on compact subsets of* $\mathbf{C}$.

(ii) $\lim V_n(z) = z + \omega_2$,

(iii) $\lim V_n^{-n}(z) = z + \omega_1$,

(iv) $\{\langle V_n \rangle\}$ *converges geometrically and polyhedrally to* $\langle z \to z + \omega_1, z \to z + \omega_2 \rangle = \Gamma$.

PROOF. By Taylor's formula, for $z$ on a compact subset of $\mathbf{C}$,

$$w_n(z) - 1 = \frac{-2\pi i}{\omega_2}\tau_n z + o(n^{-1}),$$

$$a_n - 1 = -2\pi i \tau_n + o(n^{-1}).$$

Therefore $f_n(z) = z + o(1)$, uniformly on compact subsets of $\mathbf{C}$. The transformation $V_n$ has fixed point $\infty$ and $\omega_2/(1 - a_n) \to \infty$. Therefore (ii) and (iii) follow from (i) by the formulas above.

Because of (ii), the algebraic limit of $\{\langle G_n \rangle\}$ is the cyclic parabolic group $\langle T_2 \rangle$. To prove the polyhedral limit is $\Gamma$ we must prove the fundamental polyhedra converge to that of $\Gamma$ (cf. proof of Proposition 3.8). In view of (ii) and (iii) it suffices to prove that if for a sequence $k = k(m) \to \infty$, the sequence $\{V_m^k\}$ converges to Möbius transformation, then that limit is an element of $\Gamma$, and is not the identity.

Fist we look at

$$a_m^k = \exp(-2\pi i k \tau_m).$$

We claim that $k = k(m) = o(m^2)$ as $m \to \infty$. For $\lim a_m^k = 1$ since the ratio

$$\text{(1)} \qquad \frac{a_m^k - 1}{a_m - 1} = \frac{\exp(-2\pi i k \tau_m) - 1}{\exp(-2\pi i \tau_m) - 1}$$

must remain bounded for $\{V_m^k\}$ to converge to a Möbius transformation. And in addition,

$$\operatorname{Im} \tau_m = \frac{\operatorname{Im} \tau}{|1 + m\tau|^2}.$$

Next, set

$$k(m) = p(m)m + O(m), \quad 0 \leqq O(m) < m,$$

where $p(m) \geqq 0$ and $O(m)$ are integers. Because $k(m) = o(m^2)$, it follows that $p(m) = o(m)$ as $m \to \infty$.

Take a subsequence $\{r\}$ so that $\lim O(r)/r = c$ exists, $0 \leqq c \leqq 1$. We claim that either $c = 0$ or $c = 1$. For consider the relation,

$$\exp[-2\pi i k \tau_r] = \exp[-2\pi i (k\tau_r - p(r))] = \exp[-2\pi i (p(r) r \tau_r - p(r) + O(r)\tau_r)].$$

Since $\tau_r = \tau/(1 + r\tau)$, so $r\tau_r \to 1$, and

$$\lim p(r)(r\tau_r - 1) = -\lim p(r)/(1 + r\tau) = 0,$$

$$\lim O(r)\tau_r = c.$$

Consequently, since $a_r^k = \exp[-2\pi i k \tau_r] \to 1$, $c$ is an integer.

Finally we have to examine the ratio (1) in more detail. Write,

$$\exp[-2\pi i k \tau_r] = \exp[-2\pi i (k\tau_r - p(r) - c)],$$

so that the exponent on the right approaches zero. By Taylor's formula, the limit of the ratio (1) is the limit of

$$\frac{\kappa\tau_r - p(r) - c}{\tau_r} = \frac{p(r)(r\tau_r - 1) + O(r)\tau_r - c}{\tau_r} = [-p(r) + (O(r) - cr)\tau - c]\,\tau^{-1}.$$

Since $\operatorname{Im} \tau > 0$, if this is to have a finite limit, then $\lim(O(r) - cr)$ must exist, necessarily as an integer. Then $\lim p(r)$ must exist as an integer too. We conclude that

$$(\lim V_r^k)(w) = w - \omega_1(c + \lim p(r)) + \omega_2 \lim(O(r) - cr).$$

Moreover, lim $V_r^k \neq$ id. Suppose otherwise. There are two cases. If $c = 0$, then for large $r$, $p(r) = O(r) = 0$. This implies that $k(r) = 0$ which is impossible. If $c = 1$, then for large $r$, $p(r) = -1$. This too is impossible.

## REFERENCES


1. L. V. Ahlfors, *Finitely generated Kleinian groups*, Amer. J. Math. 86 (1964), 413–429.
2. A. F. Beardon and T. Jørgensen, *Fundamental domains for finitely generated Kleinian groups*, Math. Scand. 36 (1975), 21–26.
3. C. Earle and A. Marden, *Geometric complex coordinates for Teichmüller space*, to appear.
4. L. Greenberg, *Commensurable groups of Möbius transformations*, in "*Discrete Groups and Riemann Surfaces*," L. Greenberg, ed., Ann. of Math. Studies 79, Princeton University Press, 1974.
5. T. Jørgensen, *On cyclic groups of Möbius transformations*, Math. Scand. 33 (1973), 250–260.
6. T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. 98 (1986), 739–749.
7. T. Jørgensen, *On reopening of cusps*, to appear.
8. T. Jørgensen, *On pairs of once punctured tori*, to appear.
9. T. Jørgensen and P. Klein, *Algebraic convergence of finitely generated kleinian groups*, Quart. J. Math. Oxford (2) 33 (1982) 325–332.
10. T. Jørgensen and A. Marden, *Generic fundamental polyhedra for kleinian groups*, in "*Holomorphic Functions and Moduli I*," D. Drasin et al, eds., Springer-Verlag, New York, 1988.
11. T. Jørgensen and A. Marden, *Some universal properties of kleinian groups*, to appear.
12. A. Marden, *The geometry of finitely generated kleinian groups*, Ann. of Math. 99 (1974), 383–462.
13. B. Maskit, *On Poincare's theorem for fundamental polygons*, Adv. in Math. 7 (1971), 219–230.
14. G. P. Scott, *Finitely generated 3-manifold groups are finitely presented*, London Mathh. Soc. (2) 6 (1973), 437–440.
15. A. Selberg, *On discontinuous groups in higher-dimensional symmetric space*, in "Contribution to Function Theory," Bombay (1960.
16. W. P. Thurston, *The Geometry and Topology of 3-Manifolds*, Princeton University Press, to appear.
17. W. P. Thurston, *Hyperbolic structures on 3-manifolds: Overall logic*, Bowdoin notes, 1980.



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