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ORDER PROPERTIES OF COMPACT MAPS ON L^p-SPACES ASSOCIATED WITH VON NEUMANN ALGEBRAS

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1. Introduction.

Compact operators on Banach lattices were studied by several mathematicians (see [9], Chap. 18). Prominent examples for Banach lattices are the spaces $L^p(\mu)$, where μ is some measure. Therefore it is natural to investigate also compact maps on non-commutative L^p-spaces. In this paper we show that the completely positive compact maps from $L^p(\mathscr{M})$ into $L^q(\mathscr{N})$, $\mathscr{M}$ and $\mathscr{N}$ von Neumann algebras, form an order ideal whenever $p > 1$ and $q < \infty$. This means that every completely positive map which is dominated by a compact map is itself compact. Since a positive map from $L^p(\mathscr{M})$ into $L^q(\mathscr{N})$ is already completely positive if $\mathscr{M}$ or $\mathscr{N}$ is abelian, this includes the abelian result. The main idea to the non-commutative extension consists in replacing formulas for the infimum of two linear operators by representation theorems for algebras and linear functionals. The order ideal of completely positive compact maps from $\mathscr{M}$ into $L^q(\mathscr{N})$, $q < \infty$, is monotone closed. Hence every completely positive map from $\mathscr{M}$ into $L^q(\mathscr{N})$ is the unique sum of two completely positive maps of which one is compact and the other dominates no nonzero compact map. This can be considered as a non-commutative analogue of the band decomposition ([9], Thm. 123.5).

2. Convergence of Cauchy sequences in L^p-spaces.

There are different ways of constructing L^p-spaces associated with a von Neumann algebra. In relation with order properties, Haagerup's construction seems to be the most useful. Therefore we give a short description of this construction. The details can be found in [8].

For von Neumann algebras and linear operators on a Hilbert space we use the definitions and notations of [6]. Let $\mathscr{M} \subset \mathscr{B}(\mathscr{H})$ be a von Neumann algebra with a semifinite normal faithful weight φ and the corresponding modular group σ^{φ}. $\mathscr{M}_1$ denotes the crossed product $\mathscr{R}(\mathscr{M}, \sigma^{\varphi}) \subset \mathscr{B}(L^2(\mathsf{R}, \mathscr{H}))$. $\mathscr{M}_1$ has a semifinite normal faithful trace μ naturally associated with φ ([2], Lemma 5.2). The set

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$L(\mathcal{M}_1, \mu)$ of all μ-measurable operators has a topology which is generated by the 0-neighbourhoods

$$(1) \qquad N(\varepsilon, \delta) = \{a \in L(\mathcal{M}_1, \mu) \mid \exists\, p \in \mathcal{M}_1,\ \|ap\| \leqq \varepsilon,\ \mu(1-p) \leqq \delta\}.$$

For the calculus of these neighbourhoods we refer to [4]. In [8], Chap. I, Lemma 7, it is shown that for $a \in L(\mathcal{M}_1, \mu)$

$$(2) \qquad a \in N(\varepsilon, \delta) \Leftrightarrow \mu(\chi_{(\varepsilon,\infty)}(|a|)) \leqq \delta.$$

With this topology $L(\mathcal{M}_1, \mu)$ is a complete Hausdorff topological $*$-algebra. The spaces $L^p(\mathcal{M})$ are realized as subspaces of $L(\mathcal{M}_1, \mu)$. We denote the norm on $L^p(\mathcal{M})$ by $\|\cdot\|_p$ and the duality between $L^p(\mathcal{M})$ and $L^q(\mathcal{M})$ for $1/p + 1/q = 1$ by $\langle \cdot, \cdot \rangle$. We identify $\mathcal{M}$ with $L^\infty(\mathcal{M})$ and $\mathcal{M}_*$ with $L^1(\mathcal{M})$. For $p < \infty$ and $\alpha, \beta > 0$ we have

$$(3) \qquad L^p(\mathcal{M}) \cap N(\alpha, \beta) = \{a \in L^p(\mathcal{M}) \mid \|a\|_p \leqq \alpha\beta^{1/p}\},$$

thus the norm topology on $L^p(\mathcal{M})$ is equivalent to the relative topology of the μ-measurable operators.

For $n \in \mathrm{N}$ $M_n(\mathrm{C})$ denotes the complex $n \times n$ matrices. An element $f = [f_{ij}] \in L^p(\mathcal{M}) \otimes M_n(\mathrm{C})$ can be considered as a densely defined preclosed linear operator on $L^2(\mathrm{R}, \mathcal{H})^n$. In fact, its closure is uniquely determined, and identifying f with its closure we get an identification of $L^p(\mathcal{M}) \otimes M_n(\mathrm{C})$ with $L^p(\mathcal{M} \otimes M_n(\mathrm{C}))$. If $1/p + 1/q = 1$, this furnishes a duality between $L^p(\mathcal{M}) \otimes M_n(\mathrm{C})$ and $L^q(\mathcal{M}) \otimes M_n(\mathrm{C})$ given by

$$(4) \qquad \langle f, g \rangle = \sum_{i,j=1}^{n} \langle f_{ij}, g_{ji} \rangle, \qquad \begin{aligned} f &= [f_{ij}] \in L^p(\mathcal{M}) \otimes M_n(\mathrm{C}) \\ g &= [g_{ij}] \in L^q(\mathcal{M}) \otimes M_n(\mathrm{C}). \end{aligned}$$

$L^p(\mathcal{M}) \otimes M_n(\mathrm{C})$ is naturally ordered by the cone of the positive operators $(L^p(\mathcal{M}) \otimes M_n(\mathrm{C}))_+$, and $f \in L^p(\mathcal{M}) \otimes M_n(\mathrm{C})$ is positive if and only if $\langle f, g \rangle \geqq 0$ for all $g \in (L^q(\mathcal{M}) \otimes M_n(\mathrm{C}))_+$, where $1/p + 1/q = 1$.

Now we investigate the convergence of sequences in $L^p(\mathcal{M})$. We denote the lattice of projections in $\mathcal{M}_1$ by $\mathcal{P}_{\mathcal{M}_1}$. If a sequence $(e_n)_{n=1}^\infty$ in $\mathcal{M}_1$ converges monotone to 0, we write $e_n \downarrow 0$.

2.1. Lemma. *Let* $p < \infty$, $f \in L^p(\mathcal{M})$, *and* $(e_n)_{n=1}^\infty \subset \mathcal{P}_{\mathcal{M}_1}$, $e_n \downarrow 0$. *Then* $\lim_{n\to\infty} e_n f = 0$ *in* $L(\mathcal{M}_1, \mu)$.

Proof. We may assume that f is positive and $\|f\|_p = 1$. For $0 < \alpha < \beta$ we put

$$f_1 = f\chi_{[0,\alpha]}(f),\ f_2 = f\chi_{(\alpha,\beta]}(f),\ f_3 = f\chi_{(\beta,\infty)}(f),\ e = \chi_{(\alpha,\beta]}(f).$$

Then (3) implies $\mu(e) \leqq \alpha^{-p}$. For ε, $\delta > 0$ there exist α, β such that $e_n f_1$, $e_n f_3 \in N(\varepsilon, \delta)$ for all $n \in \mathrm{N}$. Hence we have to show that $\lim_{n\to\infty} e_n f_2 = 0$. Now

$$\mu(|e_n f_2|)^2 \leqq \mu(e)\,\mu(e_n f_2^2) \leqq \beta^2\,\mu(e)\,\mu(e_n e) \to 0 \text{ as } n \to \infty.$$

Since the convergence in the trace norm implies the convergence in $L(\mathcal{M}_1, \mu)$, the assertion is proved.

2.2. DEFINITION. A subset $\mathcal{A} \subset L^p(\mathcal{M})$ is called uniformly μ-continuous, if for every sequence $(e_n)_{n=1}^{\infty}$ in $\mathcal{P}_{\mathcal{M}_1}$, $e_n \downarrow 0$, the sequences $(e_n f e_n)_{n=1}^{\infty}$ converge uniformly to 0 in $L(\mathcal{M}_1, \mu)$, i.e. for every $\varepsilon, \delta > 0$ there exists $k \in \mathbf{N}$ such that $e_n f e_n \in N(\varepsilon, \delta)$, $n \geqq k$, $f \in \mathcal{A}$.

2.3. LEMMA. *Let $p < \infty$, $\mathcal{A} \subset L^p(\mathcal{M})_+$ bounded and $(e_n)_{n=1}^{\infty} \subset \mathcal{P}_{\mathcal{M}_1}$, $e_n \downarrow 0$. Then there are equivalent:*

i) *The sequences $(e_n f e_n)_{n=1}^{\infty}$, $f \in \mathcal{A}$, converge uniformly to* 0.

ii) *The sequences $(e_n f)_{n=1}^{\infty}$, $f \in \mathcal{A}$, converge uniformly to* 0.

PROOF. The implication ii) $\Rightarrow$ i) is obvious.

i) $\Rightarrow$ ii): Let $\|f\|_p \leqq 1$ for all $f \in \mathcal{A}$. By (3), $f \in N(k, k^{-p})$ and hence $f^{1/2} \in N(k^{1/2}, k^{-p})$ for all $k \in \mathbf{N}$. If $e_n f e_n \in N(\varepsilon, \delta)$, then $e_n f = e_n f^{1/2} f^{1/2} \in N((\varepsilon k)^{1/2}, \delta + k^{-p})$.

2.4. LEMMA. Let $1 < p < \infty$, $1/p + 1/q = 1$, *and $(f_n)_{n=1}^{\infty}$ be a bounded sequence in $L^p(\mathcal{M})_+$. Then there are equivalent:*

i) *$(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L^p(\mathcal{M})$.*

ii) *The set $\{f_n \mid n \in \mathbf{N}\}$ is uniformly μ-continuous and $(h f_n h)_{n=1}^{\infty}$ is a Cauchy sequence in $L^1(\mathcal{M})$ for all $h \in L^{2q}(\mathcal{M})_+$.*

iii) *The set $\{f_n \mid n \in \mathbf{N}\}$ is uniformly μ-continuous and $(h f_n h)_{n=1}^{\infty}$ is a Cauchy sequence in $L^1(\mathcal{M})$ for some $h \in L^{2q}(\mathcal{M})_+$ such that $s(h) \geqq s(f_n)$ for all $n \in \mathbf{N}$.*

PROOF. It suffices to prove the convergence of all sequences in the topology of $L(\mathcal{M}_1, \mu)$.

i) $\Rightarrow$ ii): For $h \in L^{2q}(\mathcal{M})_+$ the convergence of the sequence $(h f_n h)_{n=1}^{\infty}$ follows from the continuity of the multiplication in $L(\mathcal{M}_1, \mu)$. Since $(f_n)_{n=1}^{\infty}$ has a limit in $L^p(\mathcal{M})$, Lemma 2.1 implies that $\{f_n \mid n \in \mathbf{N}\}$ is uniformly μ-continuous.

ii) $\Rightarrow$ iii) is obvious.

iii) $\Rightarrow$ i): By considering the reduced algebra $s(h)\mathcal{M}s(h)$ we may assume that $s(h) = 1$. Then we put $e_k = \chi_{(1/k,\infty)}(h)$, $k \in \mathbf{N}$. For every $m, n \in \mathbf{N}$ we have

$$(5) \qquad f_n - f_m = (1 - e_k)(f_n - f_m) + e_k(f_n - f_m)(1 - e_k) + e_k(f_n - f_m)e_k.$$

By Lemma 2.3 the first two summands of (5) are small if k is large enough. Therefore we have only to show: If $k \in \mathbf{N}$ is fixed then $(e_k f_n e_k)_{n=1}^{\infty}$ is a Cauchy sequence. But this is easily seen by writing

$$(6) \qquad e_k(f_n - f_m)e_k = (h^{-1}e_k)h(f_n - f_m)h(h^{-1}e_k)$$

and observing that $h^{-1}e_k$ is a bounded operator in a natural way.

3. Compact maps.

Let $\mathcal{N}$ be another von Neumann algebra with a semifinite normal faithful weight ψ, $\mathcal{N}_1 = \mathcal{R}(\mathcal{N}, \sigma^\psi)$, and ν the canonical trace on $\mathcal{N}_1$. Let $1 \leqq p, q, q' \leqq \infty$, $1/q + 1/q' = 1$. For $h \in L(\mathcal{N}_1, \nu)$ we define

$$M(h): L(\mathcal{N}_1, \nu) \to L(\mathcal{N}_1, \nu),\ a \mapsto h^*ah.$$

3.1. Definition. Let $T: L^p(\mathcal{M}) \to L^q(\mathcal{N})$ be a linear map. We define

$$T_n: L^p(\mathcal{M}) \otimes M_n(\mathrm{C}) \to L^q(\mathcal{N}) \otimes M_n(\mathrm{C}),\ [f_{ij}] \mapsto [Tf_{ij}].$$

T is completely positive ($T \underset{\mathrm{cp}}{\geqq} 0$), if $T_n(L^p(\mathcal{M}) \otimes M_n(\mathrm{C}))_+ \subset (L^q(\mathcal{N}) \otimes M_n(\mathrm{C}))_+$ for all $n \in \mathrm{N}$. $\mathrm{CP}(L^p(\mathcal{M}), L^q(\mathcal{N}))$ denotes the cone of all completely positive maps from $L^p(\mathcal{M})$ into $L^q(\mathcal{N})$. $S \in \mathrm{CP}(L^p(\mathcal{M}), L^q(\mathcal{N}))$ is dominated by T if $T - S \underset{\mathrm{cp}}{\geqq} 0$. T is uniformly ν-continuous if $\{Tf \mid f \in L^p(\mathcal{M}),\ \|f\|_p \leqq 1\}$ is uniformly ν-continuous.

3.2. Proposition. *Let $1 < q < \infty$, and T: $L^p(\mathcal{M}) \to L^q(\mathcal{N})$ be a positive linear map. Then there are equivalent:*

i) T is compact.

ii) T is uniformly ν-continuous and $M(h)T$: $L^p(\mathcal{M}) \to L^1(\mathcal{N})$ is compact for all $h \in L^{2q'}(\mathcal{N})_+$.

Proof. i) $\Rightarrow$ ii): The compactness of $M(h)T$ follows from the compactness of T. Suppose T is not uniformly ν-continuous. Then there exist a sequence $(e_l)_{l=1}^\infty$ in $\mathcal{P}_{\mathcal{N}_1}$, $e_l \downarrow 0$, and $\alpha > 0$ such that for every $n \in \mathrm{N}$ there exist $k(n) \in \mathrm{N}$, $k(n) \geqq n$ and $f_n \in L^p(\mathcal{M})_+$, $\|f_n\|_p \leqq 1$, with

$$e_{k(n)}(Tf_n)e_{k(n)} \notin N(\alpha, \alpha). \tag{7}$$

By considering a subsequence of $(f_n)_{n=1}^\infty$ we may assume that $(Tf_n)_{n=1}^\infty$ converges to $g \in L^q(\mathcal{N})$. Then

$$e_{k(n)}(Tf_n)e_{k(n)} = e_{k(n)}(Tf_n - g)e_{k(n)} + e_{k(n)}ge_{k(n)}. \tag{8}$$

If n is large enough, the right side of (8) is an element of $N(\alpha, \alpha)$ which contradicts (7).

ii) $\Rightarrow$ i): Let $(f_n)_{n=1}^\infty$ be a bounded sequence in $L^p(\mathcal{M})_+$. We choose $h \in L^{2q'}(\mathcal{N})_+$ with $s(h) \geqq s(Tf_n)$ for all $n \in \mathrm{N}$. Then the set $\{Tf_n \mid n \in \mathrm{N}\}$ is uniformly ν-continuous and $(h(Tf_n)h)_{n=1}^\infty$ has a convergent subsequence. Hence Lemma 2.4 implies that $(Tf_n)_{n=1}^\infty$ has a convergent subsequence.

3.3. Proposition. *Let $1 < q < \infty$, T: $\mathcal{M} \to L^q(\mathcal{N})$ be a positive linear map, $h \in L^{2q'}(\mathcal{N})_+$, $s(h) \geqq s(T1)$. Then T is compact if and only if $M(h)T$: $\mathcal{M} \to L^1(\mathcal{N})$ is compact.*

PROOF. For $x \in \mathcal{M}$, $0 \leqq x \leqq 1$, and $e \in \mathscr{P}_{\mathcal{N}_1}$ we have $eT(x)e \leqq eT(1)e$. Hence T is uniformly ν-continuous. Clearly the compactness of T implies the compactness of $M(h)T$. Conversely if $(x_n)_{n=1}^{\infty}$ is a bounded sequence in $\mathcal{M}_+$, then $s(h) \geqq s(Tx_n)$. Hence by Lemma 2.4 $(Tx_n)_{n=1}^{\infty}$ has a convergent subsequence if and only if $(h(Tx_n)h)_{n=1}^{\infty}$ has a convergent subsequence.

$\mathcal{N}^{\mathrm{op}}$ denotes the opposite algebra which has the same elements as $\mathcal{N}$ and its vector space structure but reversed order of multiplication. We attach a$^{\circ}$ to the elements of $\mathcal{N}^{\mathrm{op}}$. To each $T \in \mathrm{CP}(\mathcal{M}, L^1(\mathcal{N}))$ we assign the positive linear functional

$$\varphi_T\colon \mathcal{M} \otimes \mathcal{N}^{\mathrm{op}} \to \mathbf{C}, \ \sum_{i=1}^{n} a_i \otimes b_i^{\circ} \mapsto \sum_{i=1}^{n} \langle Ta_i, b_i \rangle.$$

Let $\{\mathscr{H}_T, \pi_T, \xi_T\}$ denote the cyclic representation of $\mathcal{M} \otimes \mathcal{N}^{\mathrm{op}}$ induced by φ_T. By [7], Chap. IV, Prop. 3.10, the map

$$\Theta\colon \{S \in \mathrm{CP}(\mathcal{M}, L^1(\mathcal{N})) \mid S \underset{\mathrm{cp}}{\leqq} \lambda T, \lambda \geqq 0\} \to \pi_T(\mathcal{M} \otimes \mathcal{N}^{\mathrm{op}})'_+$$

given by

$$\langle Sx, y \rangle = (\pi_T(x \otimes y^{\circ})\, \Theta(S)\, \xi_T \mid \xi_T), \quad x \in \mathcal{M}, y \in \mathcal{N} \tag{9}$$

is bijective, additive, and monotone.

Now we can prove the main theorem of this paper.

3.4. THEOREM. *Let $p > 1, q < \infty$, and $S, T\colon L^p(\mathcal{M}) \to L^q(\mathcal{N})$ be linear maps such that $0 \underset{\mathrm{cp}}{\leqq} S \underset{\mathrm{cp}}{\leqq} T$. If T is compact, then S is compact.*

PROOF. First we reduce to the case $p = \infty$ and $q = 1$. Suppose that $q > 1$. T is uniformly ν-continuous because it is compact. Since $0 \leqq S \leqq T$, S is also uniformly ν-continuous. Hence by Prop. 3.2, it suffices to show that $M(h)S\colon L^p(\mathcal{M}) \to L^1(\mathcal{N})$ is compact for all $h \in L^{2q'}(\mathcal{N})_+$. Since $0 \underset{\mathrm{cp}}{\leqq} M(h)S \underset{\mathrm{cp}}{\leqq} M(h)T$, we may assume that $q = 1$. Considering the adjoint maps and repeating the above arguments we may also assume $p = \infty$.

Now we put $y = \Theta(S)^{1/2} \in \pi_T(\mathcal{M} \otimes \mathcal{N}^{\mathrm{op}})'_+$. Since ξ_T is a cyclic vector, we can find a sequence $(y_n)_{n=1}^{\infty}$ in $\mathcal{M} \otimes \mathcal{N}^{\mathrm{op}}$ such that

$$y\xi_T = \lim_{n \to \infty} \pi_T(y_n)\xi_T. \tag{10}$$

For $y_n = \sum_{i=1}^{k(n)} c_{in} \otimes d_{in}^{\circ}$ we define

$$S_n\colon \mathcal{M} \to L^1(\mathcal{N}), x \mapsto \sum_{i,j=1}^{k(n)} d_{in}^* T(c_{in}^* x c_{jn}) d_{jn}. \tag{11}$$

Then all S_n are completely positive compact maps. For $a \in \mathcal{M}$, $b \in \mathcal{N}$ we have

(12) $$\langle S_n a, b\rangle = (\pi_T(a \otimes b^\circ)\pi_T(y_n)\xi_T \mid \pi_T(y_n)\xi_T)$$

and

(13) $$|\langle Sa - S_n a, b\rangle| \leqq \|y\xi_T - \pi_T(y_n)\xi_T\| \, \|a\| \, \|b\| \, (\|y\xi_T\| + \|\pi_T(y_n)\xi_T\|).$$

Hence $\lim_{n\to\infty} \|S - S_n\| = 0$ and S is compact.

For $p = 1$ or $q = \infty$ this theorem does not hold even in the abelian case. Counterexamples can be found in [9], p. 522f.

We close this section with the decomposition of a completely positive map from $\mathcal{M}$ into $L^q(\mathcal{N})$, $q < \infty$, into a compact map and a map which dominates no nonzero compact map.

An order ideal $\mathfrak{J}$ of $\mathcal{M}$ is a subset of $\mathcal{M}_+$ which has the property:

$$a, b \in \mathfrak{J},\ c \in \mathcal{M}_+,\ c \leqq a + b \ \Rightarrow\ c \in \mathfrak{J}.$$

$\mathfrak{J}$ is monotone closed if $0 \leqq y_i \uparrow y \in \mathcal{M}$, $y_i \in \mathfrak{J}$ for all i implies $y \in \mathfrak{J}$.

3.5. Lemma. *Let* $1 < q < \infty$, $T\colon \mathcal{M} \to L^q(\mathcal{N})$ *be completely positive,* $h = T(1)^{q/2}$, *and* $\tilde{S}\colon \mathcal{M} \to L^1(\mathcal{N})$ *such that* $0 \underset{\text{cp}}{\leqq} \tilde{S} \underset{\text{cp}}{\leqq} M(h^{1/q'})T$. *Then there exists a unique map* $S\colon \mathcal{M} \to L^q(\mathcal{N})$ *such that* $0 \underset{\text{cp}}{\leqq} S \underset{\text{cp}}{\leqq} T$ *and* $M(h^{1/q'})S = \tilde{S}$.

Proof. By [5], Lemma 2.2, there exists for each $x \in \mathcal{M}$, $0 \leqq x \leqq 1$, a unique $y \in \mathcal{N}, 0 \leqq y \leqq s(h)$, such that $\tilde{S}(x) = hyh$. We define $S(x) = h^{1/q}yh^{1/q}$. Since y is unique, S is additive and positive homogeneous. Hence it can be extended to a linear map $S\colon \mathcal{M} \to L^q(\mathcal{N})$. Clearly $M(h^{1/q'})S = \tilde{S}$. For $x \in \mathcal{M} \otimes M_n(\mathbf{C})$ we have $s(S_n(x)) \leqq s(h) \otimes 1_n$ and $(M(h^{1/q'})S)_n(x) = \tilde{S}_n(x)$. Again [5], Lemma 2.2, implies $0 \leqq S_n(x) \leqq T_n(x)$. The uniqueness of S follows from the same lemma.

3.6. Theorem. *Let* $q < \infty$ *and* $T\colon \mathcal{M} \to L^q(\mathcal{N})$ *be completely positive. Then* T *has a unique decomposition* $T = K + L$ *where* $K, L\colon \mathcal{M} \to L^q(\mathcal{N})$ *are completely positive,* K *is compact, and* L *dominates no nonzero compact map.*

Proof. We first prove the case $q = 1$. By Theorem 3.4, the set $\mathfrak{J} = \{K\colon \mathcal{M} \to L^1(\mathcal{N}) \mid 0 \underset{\text{cp}}{\leqq} K \underset{\text{cp}}{\leqq} \lambda T,\ \lambda \geqq 0,\ K \text{ compact}\}$ is an order ideal. Hence $\Theta(\mathfrak{J}) \subset \pi_T(\mathcal{M} \otimes \mathcal{N}^{\text{op}})'_+$ is an order ideal. If $\Theta(K_i) \uparrow \Theta(K) \in \pi_T(\mathcal{M} \otimes \mathcal{N}^{\text{op}})'_+$, $K_i \in \mathfrak{J}$ for all i, then

(14) $$\lim_i \|K - K_i\| \leqq 4 \lim_i ((\Theta(K) - \Theta(K_i))\xi_T \mid \xi_T) = 0.$$

Hence $\mathfrak{J}$ is monotone closed and there exists a projection $e \in \pi_T(\mathcal{M} \otimes \mathcal{N}^{\text{op}})'$ such that $\Theta(\mathfrak{J}) = e\pi_T(\mathcal{M} \otimes \mathcal{N}^{\text{op}})'_+ e$. Let $K\colon \mathcal{M} \to L^1(\mathcal{N})$ be the map with $\Theta(K) = e$ and $L = T - K$. Then $T = K + L$ is the desired decomposition.

If $q > 1$, we put $h = T(1)^{q/2}$. Then $M(h^{1/q'})T = \tilde{K} + \tilde{L}$ by the first part of the proof. Let K and L be the maps according to Lemma 3.5. Then $T = K + L$. By Prop. 3.2, K is compact. If L would dominate a nonzero compact map, then $\tilde{L}$ would dominate a nonzero compact map which is impossible.

3.7. COROLLARY. *Let $p > 1$ and T: $L^p(\mathcal{M}) \to L^1(\mathcal{N})$ be completely positive. Then T has a unique decomposition $T = K + L$, where K, L: $L^p(\mathcal{M}) \to L^1(\mathcal{N})$ are completely positive, K is compact, and S dominates no nonzero compact map.*

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