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ANTIAUTOMORPHISMS OF $B(H)$

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1. Introduction.

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. The main results of the present paper provide a description of the conjugacy classes of $*$-antiautomorphisms of $B(H)$ and relate the conjugacy of two $*$-antiautomorphisms Φ and Ψ to that of Φ^2 and Ψ^2. More precisely, it is shown in Theorems 3.4 and 3.5 that the conjugacy classes of $*$-antiautomorphisms of $B(H)$ for some H are classified by pairs (X, m) where X is a closed subset of the unit circle S^1 with $\bar{X} = X$ and m is a multiplicity function on the finite measures on X, with $m(\mu) = m(\mu \circ \overline{\mathrm{id}})$ for each μ, such that $m(\delta_{-1})$ is even (possibly zero) or infinite. If Φ is a periodic $*$-antiautomorphism it is shown in Theorem 4.3 that there is at most one non-conjugate $*$-antiautomorphism Ψ for which Φ^2 is conjugate to Ψ^2; Example 4.4 shows however that for aperiodic $*$-antiautomorphisms there may be infinitely many non-conjugate $*$-antiautomorphisms with conjugate squares.

The proofs are based on the fact, proved in Lemma 3.6 of [4], that any $*$-antiautomorphism Φ of $B(H)$ is of the form $\Phi(x) = \Phi_J(x) = Jx^*J^*$ for some antilinear isometry J on H. It is easy to show that two $*$-antiautomorphisms Φ_J and Φ_K are conjugate if and only if J is unitarily equivalent to K: hence it is required to find a unitary classification of such antilinear isometries. This will be obtained from a generalization to the real case of the classification, up to unitary equivalence, of bounded normal operators on a complex Hilbert space, as described in [1].

To complete this introduction we prove the simple connections between the unitary equivalence of J and K and the conjugacy firstly of Φ_J and Φ_K and secondly of Φ_J^2 and Φ_K^2.

PROPOSITION 1.1. *Φ_J is conjugate to Φ_K if and only if J is unitarily equivalent to K.*

PROOF. Φ_J is conjugate to Φ_K if and only if there exists a unitary U such that

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$$Jx^*J^* = UKU^*x^*UK^*U^*$$

for all $x \in B(H)$. This holds if and only if $J = \lambda UKU^*$ for some $\lambda \in S^1$ and, replacing U by μU where $\mu^2 = \lambda$, we can assume that $J = UKU^*$ for some unitary U.

PROPOSITION 1.2. *Φ_J^2 is conjugate to Φ_K^2 if and only if J^2 is unitarily equivalent to λK^2 for some $\lambda \in S^1$.*

PROOF. Φ_J^2 is conjugate to Φ_K^2 if and only if there exists a unitary U such that $J^2xJ^{*2} = UK^2U^*xUK^{*2}U^*$ for each $x \in B(H)$, from which the result follows immediately.

2. Orthogonal equivalence of real-linear normal operators.

In this section we will parallel the spectral multiplicity theory and classification up to unitary equivalence of bounded normal operators on a complex Hilbert space, to obtain a classification up to orthogonal equivalence of bounded normal operators on a real Hilbert space. The proofs are fairly routine modifications of those for the complex case.

Let T be a real-linear bounded normal operator on a real Hilbert space H and let $\tilde{T}$ be the complexification of T on the complexification $\tilde{H} = H \otimes_{\mathbb{R}} \mathbb{C}$ of H. Recall that $\tilde{H}$ possesses an antilinear involution C, defined by $C(h_1 \otimes 1 + h_2 \otimes i) = h_1 \otimes 1 - h_2 \otimes i$, such that H is isomorphic to $\{h \in \tilde{H}: Ch = h\}$ and such that $\tilde{T}C = C\tilde{T}$. The first simple consequence of this fact is that the spectrum σ_T of $\tilde{T}$ is invariant under complex conjugation.

LEMMA 2.1. *$\lambda \in \sigma_T$ if and only if $\bar{\lambda} \in \sigma_T$.*

PROOF. An operator S in $B(\tilde{H})$ satisfies $(\tilde{T} - \lambda 1)S = 1$ if and only if $C(\tilde{T} - \lambda 1)SC = 1$ and hence if and only if $(\tilde{T} - \bar{\lambda} 1)CSC = 1$.

The remaining results in this section will all be concerned with the spectral measure P_T of $\tilde{T}$. An algebraic way to define this is by extending the functional calculus $f \mapsto f(\tilde{T})$ for continuous functions f on σ_T to a sequentially normal $*$-homomorphism from the bounded Borel functions on σ_T into $B(H)$ (as described in 4.5.9 and 4.5.14 of [3]). The value $P_T(E)$ of the spectral measure P_T on the Borel set E can then be defined to be the image $\chi_E(\tilde{T})$ of the characteristic function χ_E of E.

LEMMA 2.2. *For each Borel set E in σ_T, $CP_T(E)C = P_T(\bar{E})$, where $\bar{E} = \{\bar{\lambda}: \lambda \in E\}$.*

PROOF. The map $\Phi: f \mapsto f \circ \overline{\mathrm{id}}$ defines a sequentially normal $*$-automorphism of the Borel functions on σ_T and, by considering its effect on polynomials, it can be seen that $(\Phi f)(\tilde{T}) = Cf(\tilde{T})^*C$ for each Borel function f. Hence $P_T(\bar{E}) = (\Phi\chi_E)(\tilde{T}) = C\chi_E(\tilde{T})C = CP_T(E)C$, as required.

The next collection of results will use Lemma 2.2 to obtain information about the canonical description of $\tilde{T}$ as a direct sum of multiplication operators on L^2 spaces. The treatment and notation will be closely modelled on that of [1]. In particular, for each $x \in \tilde{H}$, Z_x will denote the projection onto the closed subspace generated by the vectors $P_T(E)x$, where E is a Borel subset of σ_T, and ρ_x^T will denote the finite measure on σ_T defined by $\rho_x^T(E) = \langle P_T(E)x, x\rangle$ for each Borel subset E of σ_T.

PROPOSITION 2.3. (i) *For each* $x \in \tilde{H}$, $CZ_xC = Z_{Cx}$.

(ii) *For each* $x \in \tilde{H}$ *and each Borel subset* E *of* σ_T, $\rho_x^T(E) = \rho_{Cx}^T(\bar{E})$.

(iii) *For each* $x \in \tilde{H}$ *there is a unitary map* U_x *from* $L^2(\sigma_T, \rho_x^T)$ *onto* $Z_x\tilde{H}$ *such that, for each bounded Borel function* f *on* σ_T, $U_x([f]) = f(\tilde{T})x$ *and* $U_{Cx}^* C U_x([f]) = [\overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}]$.

PROOF.(i) From Lemma 2.2 $CZ_xCP_T(E)Cx = CZ_xP_T(\bar{E})x = CP_T(\bar{E})x = P_T(E)Cx$, so that $CZ_xC \geqq Z_{Cx}$. The result follows by symmetry.

(ii) This is immediate from Lemma 2.2.

(iii) Apart from the formula for $U_{Cx}^* C U_x([f])$ this is just 60.1 of [1]. However, using the map Φ introduced in the proof of Lemma 2.2, $U_{Cx}^* C U_x([f]) = U_{Cx}^* C f(\tilde{T})x = U_{Cx}^* C(\overline{\mathrm{id}} \circ f)(\tilde{T})^* x = U_{Cx}^*(\Phi(\overline{\mathrm{id}} \circ f))(\tilde{T})Cx = [\Phi(\overline{\mathrm{id}} \circ f)] = [(\overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}]$.

Still following the terminology of [1], we will let the *column* $C_T(P)$ generated by a projection P in the commutant $\{\tilde{T}\}'$ of $\tilde{T}$ be defined to be the least projection in $\{\tilde{T}\}''$ which majorizes P. The column $C_T(Z_x)$, for $x \in \tilde{H}$, will be denoted for short by $C_T(x)$.

LEMMA 2.4. (i) *For each projection* $P \in \{\tilde{T}\}'$, $CC_T(P)C = C_T(CPC)$.

(ii) *For each* $x \in \tilde{H}$, $CC_T(x)C = C_T(Cx)$.

PROOF. (i) Let P be a projection in $\{\tilde{T}\}'$. Then $CC_T(P)C$ belongs to $\{\tilde{T}\}''$ and $CPC \leqq CC_T(P)C$, so that $C_T(CPC) \leqq CC_T(P)C$. The result then follows by symmetry.

(ii) This follows immediately from part (i) and Proposition 2.3 (i).

Persisting with the notation of [1], we define a *row* R to be a projection in $\{\tilde{T}\}'$ such that, for each projection P in $\{\tilde{T}\}'$,

$$P \leqq R \Rightarrow P = C_T(P)R.$$

It is proved in 63.1 of [1] that, for any projection F in $\{\tilde{T}\}''$, the cardinality of a maximal orthogonal family $\{R_j\}$ of rows with $C_T(R_j) = F$ for all j is an invariant of F, known as its *multiplicity* $m_T(F)$. A projection F is said to have *uniform multiplicity* if $m_T(F_0) = m_T(F)$ for each non-zero projection $F_0 \leqq F$. The following proposition shows how $\tilde{H}$ can be decomposed into pieces of uniform multiplicity, in a manner compatible with C.

PROPOSITION 2.5. *For each cardinal number m not exceeding the Hilbert space dimension of $\tilde{H}$ let F_m^T be the supremum of the projections F in $\{\tilde{T}\}''$ which have uniform multiplicity $m_T(F) = m$. Then $\{F_m^T\}$ is an orthogonal family of sum* 1 *and, for each m, $F_m^T C = CF_m^T$ and either $F_m^T = 0$ or F_m^T has uniform multiplicity m.*

PROOF. Apart from the claim that $F_m^T C = CF_m^T$, this is exactly 64.5 of [1]. To establish this claim, note that if $\{R_j\}$ is a maximal orthogonal family of rows with $C_T(R_j) = F$ then, by Lemma 2.4, $\{CR_jC\}$ is a maximal orthogonal family of rows with $C_T(CR_jC) = CFC$: hence $m_T(F) = m_T(CFC)$ for each projection F in $\{\tilde{T}\}''$. For each non-zero projection F_0, $F_0 \leqq F$ if and only if $CF_0C \leqq CFC$ so that F is of uniform multiplicity m if and only if CFC is, from which the result follows.

COROLLARY. *There exists an orthogonal family $\{F_j^T\}$ of separable projections of uniform multiplicity in $\{\tilde{T}\}''$ such that $\sum F_j^T = 1$ and $CF_j^T = F_j^T C$ for each j.*

PROOF. For each m let $\{F_{jm}^T\}$ be a maximal orthogonal family of non-zero separable projections in $\{\tilde{T}\}''$ with $CF_{jm}^T C = F_{jm}^T$ and $F_{jm}^T \leqq F_m^T$ for each j. Unless $\sum F_{jm}^T = F_m^T$ there exists a non-zero separable projection $F \leqq F_m^T$ in $\{\tilde{T}\}''$ which is orthogonal to each F_{mj}^T and a contradiction to maximality is obtained by considering the least upper bound of F and CFC.

PROPOSITION 2.6. *Let $F \in \{\tilde{T}\}''$ be a projection of uniform multiplicity such that $CF = FC$, Then there exists an orthogonal family $\{R_k\}$ of $m_T(F)$ rows such that $R_kC = CR_k$ for each k, $C_T(R_k) = F$ for each k and $F = \sum_k R_k$.*

PROOF. Apart from the claim that $R_kC = CR_k$ for each k this is contained in 64.4 of [1]. As in the proof of that result, a simple maximality argument shows that it is sufficient to demonstrate the existence of a non-zero projection $F_0 \leqq F$ in $\{\tilde{T}\}''$ with $CF_0 = F_0C$ and an orthogonal family of rows obeying the conditions of the Proposition with F replaced by F_0. To that end, following 62.4 of [1], let $\{Q_k\}$ be a maximal orthogonal family of rows such that $C_T(Q_k) = F$ and $Q_kC = CQ_k$ for each k, let $P = F - \sum_k Q_k$ and let $F_0 = F - C_T(P)$, for which it is easy to see that $F_0C = CF_0$.

If $F_0 = 0$ then, following 61.3 of [1], let $\{S_j\}$ be a maximal orthogonal family of rows such that $S_j \leqq P$ and $S_jC = CS_j$ for each j and $C_T(S_j)C_T(S_k) = 0$ for each $j \neq k$. Unless $P \leqq \sum_j C_T(S_j)$ there exists a vector x in $P\tilde{H}$ with $Cx = x$ and $C_T(S_j)x = 0$ for all j. By 55.2 and 60.2 of [1], Z_x is then a row with $C_T(S_j)C_T(Z_x) = C_T(C_T(S_j)Z_x) = 0$ for each j and, by Proposition 2.3 (i), $Z_xC = CZ_x$ so that a contradiction to the maximality of $\{S_j\}$ is obtained. Hence $\sum S_j \leqq P \leqq \sum C_T(S_j) \leqq C_T(\sum S_j)$ and so $C_T(P) = C_T(\sum S_j)$. However, by 61.2 of [1], $\sum S_j$ is a row with $\sum S_j \leqq P$ and, since $C\sum S_j = \sum S_jC$, this contradicts the maximality of the family $\{Q_k\}$. Hence $F_0 \neq 0$.

Now consider the orthogonal family $\{R_k\}$ of rows defined by $R_k = F_0 Q_k$. Then $R_k C = C R_k$ for each k, $C_T(R_k) = F_0 C_T(Q_k) = F_0$ for each k and $\sum_k R_k = F_0 \sum_k Q_k = F_0(F - P) = F_0$. The last property implies that $\{R_k\}$ is a maximal orthogonal family of rows with $C_T(R_k) = F_0$ for each k and so, by definition, the cardinality of the index set for k is $m_T(F_0) = m_T(F)$, as required.

Proposition 2.7. *Let P be a projection in $\{\tilde{T}\}'$ such that $C_T(P)$ is separable and $PC = CP$. Then there exists $x \in P\tilde{H}$ with $Cx = x$ and $C_T(P) = C_T(x)$.*

Proof. Apart from the claim that x can be chosen to satisfy $Cx = x$ this results from 58.3 of [1]. The extra claim can be obtained by modifying the proof of 58.3 in [1] in the following way. By imposing the condition $Cx_j = x_j C$, obtain a vector $x = \sum x_j$ such that $Cx = x$. Note, using Lemma 2.4, that

$$C(C_T(P) - C_T(x))P = (C_T(P) - C_T(x))PC$$

so that, unless $C_T(P) = C_T(x)$, there exists a non-zero vector y with $Cy = y$ in the range of $(C_T(P) - C_T(x))P$. Then, as in the proof of 58.3 of [1], a contradiction to the maximality of the family $\{x_j\}$ is obtained unless $C_T(P) = C_T(x)$.

Corollary. *Let R be a row in $\{\tilde{T}\}'$ with $RC = CR$ such that $C_T(R)$ is separable. Then there exists $x \in R\tilde{H}$ with $Cx = x$, $Z_x = R$ and $C_T(R) = C_T(x)$.*

Proof. Apply Proposition 2.7 to R and note that, since R is a row, $Z_x = C_T(x)R = C_T(R)R = R$.

Lemma 2.8. *Let $\nu = \nu \circ \overline{\mathrm{id}}$ be a finite measure on σ_T which is equivalent to ρ_y^T for some $y \in \tilde{H}$ with $Cy = y$. Then there exists $x \in Z_y\tilde{H}$ with $Cx = x$ and $\nu = \rho_x^T$.*

Proof. Apart from the claim that $Cx = x$, this follows from 65.3 of [1], with $x = U_y f$ where $f \geqq 0$, f^2 is the Radon-Nikodym derivative of ρ_y^T with respect to ν and U_y is the unitary of Proposition 2.3 (iii). From $Cy = y$ it follows using Proposition 2.3 (iii) that $U_y^* C U_y f = \overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}$, which is equal to $f \circ \overline{\mathrm{id}}$ since $f \geqq 0$. However, by Proposition 2.3 (ii) and the assumption $\nu = \nu \circ \overline{\mathrm{id}}$, $f = f \circ \overline{\mathrm{id}}$ so $Cx = CU_y f = x$, as required.

Proposition 2.9. *Let $F \in \{\tilde{T}\}''$ be a separable projection of uniform multiplicity such that $FC = CF$ and let $x = Cx \in F\tilde{H}$ be a vector satisfying $F = C_T(x)$ (whose existence is guaranteed by Proposition 2.7). Then there exists a family $\{x_k\}$ of $m_T(F)$ vectors with $Cx_k = x_k$ and $\rho_{x_k}^T = \rho_x^T$ for each k such that the corresponding cyclic projections form an orthogonal family of sum F.*

Proof. By Proposition 2.6 and the corollary to Proposition 2.7 there exists a family $\{y_k\}$ of $m_T(F)$ vectors with $C_T(y_k) = F$ and $Cy_k = y_k$ for each k, such that the corresponding cyclic projections Z_{y_k} form an orthogonal family of rows of sum F. By 65.2 of [1], $\rho_{y_k}^T$ is equivalent to ρ_x^T so, by Lemma 2.8, there exists $x_k \in Z_{y_k}\tilde{H}$ with $Cx_k = x_k$ and $\rho_{x_k}^T = \rho_x^T$. By 65.2 of [1] $C_T(x_k) = C_T(y_k)$ so $Z_{x_k} = C_T(x_k)Z_{y_k} = C_T(y_k)Z_{y_k} = Z_{y_k}$, as required.

THEOREM 2.10. *Let T be a normal real-linear operator on a real Hilbert space H, let $\tilde{T}$ be the complexification of T on $\tilde{H}$, the complexification of H, and let σ_T denote the spectrum of $\tilde{T}$. Then there exists a family $\{x_j\}$ of vectors in $\tilde{H}$ such that $\{C_T(x_j)\}$ is an orthogonal family of sum 1 and, for each j,*

(i) $CC_T(x_j) = C_T(x_j)C$,

(ii) *$C_T(x_j)$ is of uniform multiplicity m_j and*

(iii) *there exists an isomorphism U_j from $C_T(x_j)\tilde{H}$ onto $\oplus_k L^2(\sigma_T, \rho_{x_j}^T)$ such that $U_j\tilde{T}U_j^* = \oplus_k M_{\mathrm{id}}$ and $U_jCU_j^*([f_k]) = ([\overline{\mathrm{id}} \circ f_k \circ \overline{\mathrm{id}}])$, where k ranges over an index set of cardinality m_j and where $M_{\mathrm{id}}[f] = [\mathrm{id} \cdot f]$ for each $[f] \in L^2(\sigma_T, \rho_{x_j}^T)$.*

PROOF. By the corollary to Proposition 2.5 there exists an orthogonal family $\{F_j^T\}$ of separable projections of uniform multiplicity in $\{\tilde{T}\}''$ such that $\sum F_j^T = 1$ and $CF_j^T = F_j^TC$ for each j. By Proposition 2.9, for each F_j there exists a vector $x_j \in F_j\tilde{H}$ such that $C_T(x_j) = F_j$ and a family $\{x_{jk}\}$ of m_j vectors in $F_j\tilde{H}$ such that $F_j\tilde{H} = \oplus_k Z_{x_{jk}}\tilde{H}$ and such that, for each k, $\rho_{x_{jk}}^T = \rho_{x_j}^T$ and $Cx_{jk} = x_{jk}$. Then, by Proposition 2.3, there exist isomorphisms U_{jk} from $L^2(\sigma_T, \rho_{x_j}^T)$ onto $Z_{x_{jk}}\tilde{H}$ such that $U_{jk}^*\tilde{T}U_{jk}[f] = [\mathrm{id} \cdot f]$ and $U_{jk}^*CU_{jk}[f] = [\overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}]$. The result follows by letting $U_j = \oplus_k U_{jk}$.

Theorem 2.10 is the basic tool in the classification of real-linear normal operators up to orthogonal equivalence; the extra step needed is to show that the measures $\rho_{x_j}^T$ can be defined solely in terms of the orthogonal equivalence class of T. To do this we use the *multiplicity function* associated with $\tilde{T}$, which is a function from finite measures on σ_T to cardinal numbers defined by

$$m_T(\mu) = \min\{m_T(C_T(\nu)) : 0 \neq \nu \ll \mu\},$$

where $C_T(\nu)$ is the projection (in $\{\tilde{T}\}''$) onto the subspace $\{x : \rho_x^T \ll \nu\}$. In an analogous manner to the situation for projections, a measure μ is said to be of *uniform multiplicity* if $m_T(\mu) = m_T(\nu)$ for each $0 \neq \nu \ll \mu$.

Note that the notation $C_T(\mu)$ (taken from [1]) is consistent with the notation $C_T(P)$ for P a projection in $\{\tilde{T}\}'$ in the sense that $C_T(\rho_x^T) = C_T(Z_x)$ for each $x \in \tilde{H}$ (as shown in 66.2 of [1]).

LEMMA 2.11. *For each finite measure μ on σ_T,*

(i) $CC_T(\mu)C = C_T(\mu \circ \overline{\mathrm{id}})$,

(ii) $m_T(C_T(\mu)) = m_T(C_T(\mu \circ \overline{\mathrm{id}}))$,

(iii) $m_T(\mu) = m_T(\mu \circ \overline{\mathrm{id}})$.

PROOF. (i) Note that $CC_T(\mu)Cx = x$ if and only if $\rho_{Cx}^T \ll \mu$. By Proposition 2.3 (ii) this holds if and only if $\rho_x^T \ll \mu \circ \overline{\mathrm{id}}$ and hence if and only if $C_T(\mu \circ \overline{\mathrm{id}})x = x$.

(ii) This is a direct consequence of (i) (using the result $m_T(F) = m_T(CFC)$ for each projection F in $\{\tilde{T}\}''$, established in the proof of Proposition 2.5).

(iii) From (ii), $m_T(\mu) = \min\{m_T(C_T(\nu)): 0 \neq \nu \ll \mu\} = \min\{m_T(C_T(\nu \circ \overline{\mathrm{id}})): 0 \neq \nu \ll \mu\} = m_T(\mu \circ \overline{\mathrm{id}})$.

The notion of the multiplicity function m_T associated with $\tilde{T}$ can be abstracted to an axiomatically defined multiplicity function m defined on the finite measures on a compact subset X of C. (See § 49 of [1] for details.) The following lemma shows that if m has the property in Lemma 2.11 (iii), then any measure μ with $\mu = \mu \circ \overline{\mathrm{id}}$ can be decomposed into measures of uniform multiplicity, each having this property. The decomposition is described in terms of the lattice operations $\vee$ *and* $\wedge$ on the set of finite measures on X, ordered by absolute continuity. (See § 48 of [1] for details.)

LEMMA 2.12. *Let X be a compact subset of* C, *invariant under complex conjugation, and let m be a multiplicity function on the finite measures on X such that $m(\mu) = m(\mu \circ \overline{\mathrm{id}})$ for each μ. Then there exists an orthogonal family $\{\mu_j\}$ of finite measures of uniform multiplicity on X such that $\mu_j = \mu_j \circ \overline{\mathrm{id}}$ for each j and such that, for each finite measure μ on X satisfying $\mu = \mu \circ \overline{\mathrm{id}}$, μ is equivalent to $\vee_j(\mu \wedge \mu_j)$.*

PROOF. Let $X^+ = \{\lambda \in X: \operatorname{Im}\lambda > 0\}$, $X^0 = \{\lambda \in X: \operatorname{Im}\lambda = 0\}$ and $X^- = \{\lambda \in X: \operatorname{Im}\lambda < 0\}$. The restriction of m to the finite measures supported on $X^+ \cup X^0$ is a multiplicity function and hence, by 49.3 of [1], there exists an orthogonal family $\{\nu_j\}$ of non-zero finite measures of uniform multiplicity on $X^+ \cup X^0$ such that ν is equivalent to $\vee_j(\nu \wedge \nu_j)$ for each finite measure ν on $X^+ \cup X^0$. Let μ_j^+ be the restriction of ν_j to X^+, let μ_j^0 be the restriction of ν_j to X^0 and let $\mu_j = \mu_j^+ + \mu_j^0 + \mu_j^+ \circ \overline{\mathrm{id}}$. By construction, $\{\mu_j\}$ is an orthogonal family of finite measures on X such that $\mu_j = \mu_j \circ \overline{\mathrm{id}}$ for each j and such that μ is equivalent to $\vee_j(\mu \wedge \mu_j)$ whenever $\mu = \mu \circ \overline{\mathrm{id}}$. To see that each μ_j is of uniform multiplicity, let $0 \neq \nu \ll \mu_j$ and let $\nu = \nu^+ + \nu^0 + \nu^-$ with $\nu^+ \ll \mu_j^+$, $\nu^0 \ll \mu_j^0$, $\nu^- \ll \mu_j^+ \circ \overline{\mathrm{id}}$. Note that at least one summand ν^+, ν^0, ν^- is non zero; in the following argument terms involving zero arguments are to be omitted. By hypothesis, $m(\mu_j^+) = m(\mu_j^+ \circ \overline{\mathrm{id}})$ and $m(\nu^-) = m(\nu^- \circ \overline{\mathrm{id}})$. Therefore, since ν_j (and hence μ_j^+ and μ_j^0) are of uniform multiplicity,

$$m(\nu^+) = m(\mu_j^+), m(\nu^0) = m(\mu_j^0), m(\nu^-) = m(\nu^- \circ \overline{\mathrm{id}}) = \mathrm{m}(\mu_j^+) = \mathrm{m}(\mu_j^+ \circ \overline{\mathrm{id}})$$

from which it follows that

$$m(\nu) = \min\{m(\nu^-), m(\nu^0), m(\nu^+)\} = \min\{m(\mu_j^+ \circ \overline{\mathrm{id}}), m(\mu_j^0), m(\mu_j^+)\} = m(\mu_j)$$

as required.

The next lemma applies Lemma 2.12 to the multiplicity function m_T on σ_T.

LEMMA 2.13. *There exists a family* $\{\mu_j\}$ *of orthogonal measures of uniform multiplicity on* σ_T, *defined solely in terms of the multiplicity function* m_T, *such that* $\mu_j = \mu_j \circ \overline{\mathrm{id}}$ *for each j and such that* $\{C_T(\mu_j)\}$ *forms an orthogonal family of projections of uniform multiplicity with sum* 1 *and* $CC_T(\mu_j) = C_T(\mu_j)C$ *for each j. Furthermore, for each j there exists* $x_j \in C_T(\mu_j)\tilde{H}$ *such that* $Cx_j = x_j$ *and* $\mu_j = \rho^T_{x_j}$.

PROOF. Let $\{\mu_j\}$ be the family of orthogonal measures on σ_T given by Lemma 2.12. By 66.3 of [1] $\{C_T(\mu_j)\}$ is an orthogonal family, by 67.3 of [1] each $C_T(\mu_j)$ has uniform multiplicity and, by Lemma 2.11 (i), $CC_T(\mu_j)C = C_T(\mu_j \circ \overline{\mathrm{id}}) = C_T(\mu_j)$ for each j. To see that $\sum C_T(\mu_j) = 1$ let $\sigma_T^+, \sigma_T^0, \sigma_T^-, \mu_j^+, \mu_j^0$ be as defined in Lemma 2.12 and note that, by §68 of [1] applied to the restriction of $\tilde{T}$ to $P_T(\sigma_T^+ \cup \sigma_T^0)\tilde{H}$, $\sum C_T(\mu_j^0) = P_T(\sigma_T^0)$ and $\sum C_T(\mu_j^+) = P_T(\sigma_T^+)$, from which it follows that $\sum C_T(\mu_j^+ \circ \overline{\mathrm{id}}) = CP_T(\sigma_T^+)C = P_T(\sigma_T^-)$ and hence $\sum C_T(\mu_j) = 1$.

By Proposition 2.7 and Lemma 2.11 (i) there exists $y_j \in C_T(\mu_j)\tilde{H}$ with $Cy_j = y_j$ and $C_T(y_j) = C_T(\mu_j)$. By the second paragraph of the proof of 67.3 of [1], μ_j is equivalent to $\rho^T_{y_j}$ and hence, by Lemma 2.8, there exists x_j as required.

We can now restate Theorem 2.10 in a way which makes clear the invariance of the decomposition under orthogonal equivalence.

THEOREM 2.14. (i) *Let T be a normal real-linear operator on a real Hilbert space H, let* $\tilde{T}$ *be the complexification of T on* $\tilde{H}$, *the complexification of H, and let* σ_T *denote the spectrum of* $\tilde{T}$. *Then there exists an orthogonal family* $\{\mu_j\}$ *of finite measures of uniform multiplicities* m_j *on* σ_T, *defined solely in terms of the multiplicity function* m_T *of* $\tilde{T}$, *such that T is orthogonally equivalent to* $\oplus_j \oplus_k M_{\mathrm{id}}$ *on* $\oplus_j \oplus_k \{[f] \in L^2(\sigma_T, \mu_j): [f] = [\overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}]\}$, *where k ranges over an index set of cardinally* m_j.

(ii) *Two normal real-linear operators T, S on a real Hilbert space H are orthogonally equivalent if and only if their complexifications* $\tilde{T}$, $\tilde{S}$ *are unitarily equivalent.*

PROOF. (i) Let $\{\mu_j\}$ be the family given by Lemma 2.13 and let $x_j \in C_T(\mu_j)\tilde{H}$ satisfy $Cx_j = x_j$ and $\mu_j = \rho^T_{x_j}$. Then, by Proposition 2.9, there exists a family $\{x_{jk}\}$ of m_j vectors with $Cx_{jk} = x_{jk}$ and $\rho^T_{x_{jk}} = \rho^T_{x_j}$ for each k, such that the corresponding cyclic projections form an orthogonal family of sum $C_T(\mu_j)$. The result follows from Proposition 2.3, on identifying

$$\oplus_j \oplus_k \{U_j h: h \in Z_{x_{jk}}\tilde{H}, Ch = h\} \text{ with}$$

$$\oplus_j \oplus_k \{f \in L^2(\sigma_T, \mu_j): [f] = [\overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}]\}.$$

(ii) It is clear that if T, S are orthogonally equivalent then $\tilde{T}$, $\tilde{S}$ are unitarily equivalent. Conversely, if $\tilde{T}$, $\tilde{S}$ are unitarily equivalent then, by §68 of [1], $m_T = m_S$ so that, by part (i), T is orthogonally equivalent to S.

THEOREM 2.15. *Let X be a compact subset of the complex plane which is invariant under complex conjugation and let m be a multiplicity function on the finite measures on X such that $m(\mu) = m(\mu \circ \overline{\mathrm{id}})$ for each finite measure μ on X. Then there exists a normal real-linear operator T on a real Hilbert space for which $\sigma_T = X$ and $m = m_T$, where σ_T and m_T denote the spectrum and multiplicity functions of the complexification $\tilde{T}$ of T.*

PROOF. By Lemma 2.12 there exists an orthogonal family $\{\mu_j\}$ of finite measures of uniform multiplicities m_j on X such that μ is equivalent to $\vee_j(\mu \wedge \mu_j)$ for each finite measure μ on X with $\mu = \mu \circ \overline{\mathrm{id}}$. Define $H = \oplus_j \oplus_k \{[f] \in L^2(X, \mu_j): [f] = [\overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}]\}$, where k ranges over an index set of cardinality m_j, and define $T = \oplus_j \oplus_k M_{\mathrm{id}}$. Then $\tilde{T}$ can be identified with $\oplus_j \oplus_k M_{\mathrm{id}}$ on $\oplus_j \oplus_k L^2(X, \mu_j)$, from which the result follows.

3. Unitary equivalence of antilinear isometries.

Let J be an antilinear isometry on a complex Hilbert space H and let $\tilde{J}$ be the (complex-linear) complexification of J on $\tilde{H}$ (for which the complex scalar multiplication does not extend that on H). Let I be the multiplication operator by i on H (with respect to the original scalar multiplication), let $\tilde{I}$ be its complexification on $\tilde{H}$ and, as in § 2, let C be the antilinear involution on $\tilde{H}$ associated with H. Note that both $\tilde{J}$ and $\tilde{I}$ commute with C but, by the antilinearity of J, $\tilde{J}\tilde{I} + \tilde{I}\tilde{J} = 0$.

LEMMA 3.1. (i) *σ_J is symmetric* (i.e. $\sigma_J = \{-\lambda: \lambda \in \sigma_J\}$).
(ii) *For each Borel set E in σ_J, $\tilde{I}P_J(E) = P_J(-E)\tilde{I}$.*

PROOF. (i) An operator S in $B(\tilde{H})$ satisfies $(\tilde{J} - \lambda 1)S = 1$ if and only if $\tilde{I}(\tilde{J} - \lambda 1)S\tilde{I} = -1$ and hence if and only if $(\tilde{J} + \lambda 1)\tilde{I}S\tilde{I} = 1$.

(ii) The map Φ: $f \mapsto f \circ (-\mathrm{id})$ defines a sequentially normal $*$-automorphism of the Borel functions on σ_J and, by considering its effect on polynomials, it can be seen that $(\Phi f)(\tilde{J}) = -\tilde{I}f(\tilde{J})\tilde{I}$ for each Borel function f. Hence $P_J(-E) = (\Phi\chi_E)(\tilde{J}) = -\tilde{I}\chi_E(\tilde{J})\tilde{I} = -\tilde{I}P_J(E)\tilde{I}$, as required.

In a similar manner to that used in the proof of Lemma 2.12 where a subset X of $\mathbf{C}$ was divided into parts X^-, X^0, X^+ according to the sign of the imaginary part, we will now divide the spectrum σ_J of $\tilde{J}$ into parts σ^J_+, σ^J_0 and σ^J_- corresponding to the sign of the real parts. We define $\tilde{H}^J_+ = P_J(\sigma^J_+)\tilde{H}$ and $H^J_+ = P_J(\sigma^J_+)H$ (where H is identified with $\{h \in \tilde{H}: Ch = h\}$), which is a subspace of H by Lemma 2.2, with similar definitions of $\tilde{H}^J_0$, $\tilde{H}^J_-$, H^J_0 and H^J_-. Note that, since $\sigma_J \subseteq S^1$, $\sigma^J_0 \subseteq \{i, -i\}$ so $\tilde{H}^J_0$ is either zero or is the direct sum of the eigenspaces of $\tilde{J}$ associated with the eigenvalues $\pm i$. Note also that H^J_0 is a complex subspace of H but, by Lemma 3.1, $\tilde{I}H^J_\pm = H^J_\mp$.

LEMMA 3.2. *There exists a unitary map V from H^J_0 onto a direct sum $\oplus_k \mathbf{C}^2$ (taken to be zero if the index set is empty) such that $(VJV^*)((\lambda_k, \mu_k)) = ((-\bar{\mu}_k, \bar{\lambda}_k))$ for each $\lambda_k, \mu_k \in \mathbf{C}$.*

PROOF. Since $\tilde{J}h = \pm ih$ for each $h \in \tilde{H}_0^J$, $J^2 = -1$ on $H_J{}^0$. Then, as proved in Lemma 7.5.6 of [2], H_0^J is a direct sum of complex subspaces $\{\lambda e_\alpha + \mu J e_\alpha: \lambda, \mu \in \mathbf{C}\}$ on which the action of J is given by $J(\lambda e_\alpha + \mu J e_\alpha) = -\bar{\mu} e_\alpha + \bar{\lambda} J e_\alpha$, as required.

PROPOSITION 3.3. *There exists an orthogonal family $\{\mu_j\}$ of measures of uniform multiplicities m_j on σ_+^J, constructed in terms of the multiplicity function m_J^+ of the restriction of $\tilde{J}$ to $\tilde{H}_+^J$, and a unitary map U from $H_+^J \oplus H_-^J$ onto $\oplus_j \oplus_k L^2(\sigma_+^J, \mu_j)$, where k ranges over an index set of cardinality m_j, such that $UJU^*([f_{jk}]) = ([\mathrm{id} \cdot \overline{\mathrm{id}} \circ f_{jk} \circ \overline{\mathrm{id}}])$ for each $([f_{jk}]) \in \oplus_j \oplus_k L^2(\sigma_+^J, \mu_j)$.*

PROOF. By Theorem 2.14 applied to the restriction of J to H_+^J there exists an orthogonal family $\{\mu_j\}$ of measures on σ_+^J, defined in terms of m_J^+, and an orthogonal isomorphism U_+ from H_+^J onto $\oplus_j \oplus_k \{[f] \in L^2(\sigma_+^J, \mu_j): [f] = [\overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}]\}$ such that $U_+ J U_+^* = \oplus_j \oplus_k M_{\mathrm{id}}$. The equations $\tilde{I} P_J(\sigma_\pm^J) = P_J((\sigma_\mp^J)\tilde{I}$ imply that $IH_\pm^J = H_\mp^J$; hence the orthogonal map U_+ can be extended to a unitary map $U: H_+^J \oplus H_-^J \to \oplus_j \oplus_k L^2(\sigma_+^J, \mu_j)$ by $U(h_1 + Ih_2) = U_+ h_1 + iU_+ h_2$. A simple calculation then shows that $UJU^*([f_{jk}]) = ([\mathrm{id} \cdot \overline{\mathrm{id}} \circ f_{jk} \circ \overline{\mathrm{id}}])$.

COROLLARY. *Let $(\sigma_+^J)^2 = \{\lambda^2 : \lambda \in \sigma_+^J\}$ and let v_j be defined on $(\sigma_+^J)^2$ by $v_j(E) = \mu_j(\{\lambda \in \sigma_+^J : \lambda^2 \in E\})$. Then $\{v_j\}$ is an orthogonal family of measures of uniform multiplicity such that J^2 is unitarily equivalent to $\oplus_j \oplus_k M_{\mathrm{id}}$ on $\oplus_j \oplus_k L^2((\sigma_+^J)^2, v_j)$.*

PROOF. From Proposition 3.3 it follows that J^2 is unitarily equivalent to $\oplus_j \oplus_k M_{\mathrm{id}^2}$ on $\oplus_j \oplus_k L^2(\sigma_+^J, \mu_j)$. However $V: L^2((\sigma_+^J)^2, v_j) \to L^2((\sigma_+^J)^2, \mu_j)$ defined by $V[f] = [f \circ \mathrm{id}^2]$ is a unitary with $VM_{\mathrm{id}}V^* = M_{\mathrm{id}^2}$, yielding the required result.

THEOREM 3.4. *Let J, K be antilinear isometries on a complex Hilbert space H. Then J is unitarily equivalent to K if and only if J^2 is unitarily equivalent to K^2.*

PROOF. If J^2 is unitarily equivalent to K^2, then their eigenspaces associated with the eigenvalue -1 have the same dimension (possibly 0). Hence, by Lemma 3.2, the restriction of J to H_0^J is unitarily equivalent to the restriction of K to H_0^K.

Let $\{\mu_j\}$ be a family given by Proposition 3.3, let μ be a measure on σ_+^J and, for each Borel set E in $(\sigma_+^J)^2$, let $v(E) = \mu((\mathrm{id}^2)^{-1}(E))$ and $v_j(E) = \mu_j((\mathrm{id}^2)^{-1}(E))$. Then, using Proposition 3.3 and its corollary, $m_J^+(\mu) = \min\{m_j : \mu \wedge \mu_j \neq 0\} = \min\{m_j : v \wedge v_j \neq 0\} = m_{J^2}(v)$. Hence, if J^2 is unitarily equivalent to K^2 then $m_J^+ = m_K^+$ and therefore, by Proposition 3.3, the restriction of J to $H_-^J \oplus H_+^J$ is unitarily equivalent to the restriction of K to $H_-^K \oplus H_+^K$.

THEOREM 3.5. *Let X be a closed subset of S^1 invariant under complex conjugation and let m be a multiplicity function on the finite measures on X such that $m(\mu) = m(\mu \circ \overline{\mathrm{id}})$ for each finite measure μ on X and $m(\delta_{-1})$ is even or infinite (but* *may be zero). Then there exists an antilinear isometry J on a Hilbert space H such that X is the spectrum of J^2 and m is its multiplicity function.*

PROOF. Let $Y = \{\lambda \in S^1: \operatorname{Re}\lambda \geqq 0, \lambda^2 \in X\}$ and define a multiplicity function m_+ on the finite measures on Y by $m_+(\delta_i) = m_+(\delta_{-i}) = \frac{1}{2}m(\delta_{-1})$ and, for ν singular to δ_i and δ_{-i}, by $m_+(\nu) = m(\mu)$ where $\mu(E) = \nu(\{\lambda \in S^1: \operatorname{Re}\lambda \geqq 0, \lambda^2 \in E\})$. Let $\{\nu_j\}$ be an orthogonal family of finite measures of uniform multiplicities m_j on Y (with respect to m_+), such that $\nu_j = \nu_j \circ \overline{\mathrm{id}}$ for each j and ν is equivalent to $\vee_j(\nu \wedge \nu_j)$ for each finite measure ν on Y with $\nu = \nu \circ \overline{\mathrm{id}}$. (Such a family exists by Lemma 2.12.) Define $H = \oplus_j \oplus_k L^2(Y, \nu_j)$, where k ranges over an index set of cardinality m_j, and define $J = \oplus_j \oplus_k M_{\mathrm{id}} \circ C_j$, where $C_j f = \overline{\mathrm{id}} \circ f \circ \overline{\mathrm{id}}$ for each f. Then J is an antilinear isometry with the properties required.

4. Conjugacy of antiautomorphisms.

Let Φ_J and Φ_K be $*$-antiautomorphisms of $B(H)$, where J and K are antilinear isometries on H. Then, by Proposition 1.1 and Theorem 3.4, Φ_J is conjugate to Φ_K if and only if J^2 is unitarily equivalent to K^2 (henceforth written $J^2 \sim K^2$). Moreover, by Theorem 3.5, isometries of the form J^2 are characterized up to unitary equivalence by a closed subset X of S^1 which is invariant under complex conjugation and by a multiplicity function m such that $m(\mu) = m(\mu \circ \overline{\mathrm{id}})$ for each finite measure μ on X and such that $m(\delta_{-1})$ is even or infinite. This, in one sense, provides a complete answer to the conjugacy problem but it does not establish the number of non-conjugate pairs Φ_J, Φ_K for which Φ_J^2 is conjugate to Φ_K^2. We will now tackle this problem, bearing in mind Proposition 1.2, and then illustrate our results by some simple examples.

LEMMA 4.1. *Let J, K be antilinear isometries with $J^2 \sim \lambda K^2$. Then $J^2 \sim \lambda^2 J^2$.*

PROOF. Let U be a unitary with $UJ^2U^* = \lambda K^2$. Then

$$\begin{aligned}(JU^*KU)J^2(U^*K^*UJ^*) &= JU^*K(\lambda K^2)K^*UJ^* \\ &= \lambda JU^*K^2UJ^* \\ &= \lambda J(\bar{\lambda}J^2)J^* \\ &= \lambda^2 J^2.\end{aligned}$$

LEMMA 4.2. *Let J be an antilinear isometry of finite order. Then $G = \{\lambda: J^2 \sim \lambda^2 J^2\}$ is a finite cyclic subgroup of S^1.*

PROOF. Let J have order n (which must be even since J is antilinear). Then G is a subgroup of the group of nth roots of unity.

THEOREM 4.3. *Let Φ_J be a periodic $*$-antiautomorphism of $B(H)$. Then there is, up* *to conjugacy, at most one *-antiautomorphism Φ_K of $B(H)$ with Φ_K^2 conjugate to Φ_J^2 but Φ_K not conjugate to Φ_J.*

PROOF. Let λ be a generator of the group G of Lemma 4.2. If Φ_K^2 is conjugate to Φ_J^2 then there exists $m \geqq 0$ such that $K^2 \sim \lambda^m J^2$ (by Proposition 1.2 and Lemmas 4.1 and 4.2). Thus $K^2 \sim \lambda^{m+2k} J^2$ for each k and hence either $K^2 \sim J^2$ or $K^2 \sim \lambda J^2$. In the first case Φ_K is conjugate to Φ_J; the second case gives the unique possibility for which Φ_K may not be conjugate to Φ_J.

The following example shows that the conclusion of Theorem 4.3 does not persist for aperiodic antiautomorphisms.

EXAMPLE 4.4. Let $\{r_j: j \in \mathbb{N}\}$ be a family of distinct irrationals in (0, 1) such that $\{1\} \cup \{r_j: j \in \mathbb{N}\}$ is linearly independent over $\mathbb{Q}$, for each $j \geqq 0$ let $X_j = \{\pm e^{\pi i r_j} e^{2\pi i(m_1 r_1 + \ldots + m_n r_n)}: n \in \mathbb{N},\ m_1, \ldots, m_n \in \mathbb{Z}\} \cap \{\lambda: \operatorname{Re}\lambda > 0, \operatorname{Im}\lambda > 0\}$ (where $r_0 = 0$) and, for each $j \geqq 0$, let J_j be defined on $\oplus_{x \in X_j} \mathbb{C}^2$ by $J_j = \oplus_x J_j^x$, where $J_j^x(\lambda, \mu) = (x\bar{\mu}, \bar{x}\bar{\lambda})$. Then each J_j can be regarded as an antilinear isometry on a separable Hilbert space H. We will show that, for each j, k, the point spectrum of J_j^2 is distinct from that for J_k^2, so that J_j^2 and J_k^2 are not unitarily equivalent and hence correspond to distinct antiautomorphisms Φ_{J_j} and Φ_{J_k}. We will also show that, nevertheless, $J_j^2 \sim e^{2\pi i r_j} J_0^2$ so that, for each j, $\Phi_{J_j}^2$ is conjugate to $\Phi_{J_0}^2$.

By construction, the point spectrum (or set of eigenvalues) of J_j^2 is equal to $Y_j = \{e^{2\pi i r_j} e^{4\pi i(m_1 r_1 + \ldots + m_n r_n)}: n \in \mathbb{N}, m_1, \ldots, m_n \in \mathbb{Z}\}$ and, from the independence of $\{1, r_1, r_2, \ldots\}$ over $\mathbb{Q}$, it follows that $e^{2\pi i r_j}$ does not belong to the point spectrum of J_k^2 for $j \neq k$ but does belong to the point spectrum of J_j^2. To see that $J_0^2 \sim e^{2\pi i r_j} J_j^2$, define $\beta: Y_0 \to Y_j$ by $\beta(t) = e^{2\pi i r_j} t$ and define $U: \oplus_{x \in Y_0} \mathbb{C} \to \oplus_{y \in Y_j} \mathbb{C}$ by $(Uf)_{\beta(x)} = f_x$. Then $J_j^2 \sim \oplus_{y \in Y_j} M_y \sim U^*(\oplus_{y \in Y_j} M_y)U$. However, for $f \in \oplus_{x \in Y_0} \mathbb{C}, (U^* \oplus_{y \in Y_j} M_y Uf)_x = (\oplus_y M_y Uf)_{\beta(x)} = M_{\beta(x)}(Uf)_{\beta(x)} = \beta(x)(Uf)_{\beta(x)} = \beta(x) f_x = e^{2\pi i r_j} x f_x = (e^{2\pi i r_j} \oplus_{x \in Y_0} M_x f)_x$, where $M_x \lambda = x\lambda$ for each $\lambda \in \mathbb{C}$. Hence $J_j^2 \sim U^*(\oplus_{y \in Y_j} M_y)U \sim e^{2\pi i r_j} \oplus_{x \in Y_0} M_x \sim e^{2\pi i r_j} J_0^2$ and so $\Phi_{J_j}^2$ is conjugate to $\Phi_{J_o}^2$, as required.

EXAMPLE 4.5. The simplest example (which is, of course, well-understood) occurs when $\Phi_J^2 = \mathrm{id}$, for which $J^2 = \pm 1$ and the set X of Theorem 3.5 is either $\{1\}$ or $\{-1\}$. The multiplicity $m(\delta_1)$ in the first case can be any number (giving the dimension of the Hilbert space on which J operates) but $m(\delta_{-1})$ must be even or infinite in the second case. In both cases the group G of Lemma 4.2 is $\{1, -1\}$, reflecting the fact that, for any J, K with $\Phi_J^2 = \Phi_K^2 = \mathrm{id}$, $K^2 = \pm J^2$.

EXAMPLE 4.6. Another simple example occurs when $\Phi_J^4 = \mathrm{id}$ (but $\Phi_J^2 \neq \mathrm{id}$), for which $J^4 = \pm 1$ and the set X of Theorem 3.5 is either $\{1, -1\}$ (in the case $J^4 = 1$) or $\{i, -i\}$ (in the case $J^4 = -1$). In the first case the possible multiplicities are an arbitrary non-zero $m(\delta_1)$ and an arbitrary even or infinite non-zero $m(\delta_{-1})$, with the sum of these giving the Hilbert space dimension of the space on which J operates. In the second case $m(\delta_{-i})$ must equal $m(\delta_i)$ which can be an arbitrary non-zero cardinal, giving half the Hilbert space dimension of the space on which J operates.

When $X = \{i, -i\}$ or $X = \{1, -1\}$ and $m(\delta_{-1}) = m(\delta_1)$ then $J^2 \sim -J^2$ and the group G of Lemma 4.2 is equal to $\{1, -1, i, -i\}$. The isometry J^2 is either a symmetry $S = 1 - 2p$, where p is a projection of dimension $2m$ on a space of dimension $4m$ for some m, or $J^2 = iS' = i(1 - 2p')$ where p' is a projection of dimension m' on a space of dimension $2m'$. If m' is even or infinite then J^2 and iJ^2 are both squares of antilinear isometries and two conjugacy classes correspond to Φ_J^2; otherwise Φ_J^2 corresponds to a single antiautomorphism.

When $X = \{1, -1\}$ and $m(\delta_1) \neq m(\delta_{-1})$ (but both are non-zero) then the group G of Lemma 4.2 is $\{1, -1\}$. The isometry J^2 is a symmetry $S = 1 - 2p$. If the dimensions of both p and $1 - p$ are even or infinite (corresponding to both $m(\delta_1)$ and $m(\delta_{-1})$ being even or infinite) then both J^2 and $-J^2$ are squares of antilinear isometries and two conjugacy classes correspond to Φ_J^2; otherwise Φ_J^2 corresponds to a single antiautomorphism.

If, for example, H is of odd finite dimension $2k + 1$ then $B(H)$ possesses k conjugacy classes of $*$-antiautomorphisms of period 4, corresponding to $X = \{1, -1\}$ with $m(\delta_{-1}) \in \{2, \ldots, 2k\}$, $m(\delta_1) = 2k + 1 - m(\delta_{-1})$; the corresponding squares are of the form $\mathrm{Ad}(1 - 2p)$, where p is a projection of dimension $2, 4, \ldots,$ or $2k$, and the projections are pairwise non-conjugate.

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