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AN EXTREMAL PROBLEM FOR POLYNOMIALS WITH A PRESCRIBED VALUE AT A GIVEN POINT OF THE REAL AXIS

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Let $n \in \mathbf{N}$, $z_0 \in \mathbf{C} \setminus \{1\}$, and $\mathscr{C}_n(z_0)$ be the class of polynomials P of degree $\leqq n$ which satisfy

$$(1) \qquad \max\{|P(e^{i\varphi})| \,|\, \phi \in \mathbf{R}\} = |P(1)| > 0$$

and $P(z_0) = 0$. It is easily seen that there exists an open set $G_n \in \mathbf{C}$ containing the point $z = 1$ such that $\mathscr{C}_n(z_0)$ is empty if and only if $z_0 \in G_n$. Naturally, it is much more difficult to determine these sets G_n explicitly. P. Turán (see [3]) proved some partial results in this direction and C. Hyltén-Cavallius (see [1]) gave the exact result:

THEOREM A. *For* $n \in \mathbf{N}$

$$G_n = \{\rho e^{i\varphi} | \cos(\phi/2) > \tfrac{1}{2}(\sqrt{\rho} + 1/\sqrt{\rho})\cos(\pi/2n),\ -\pi < n\phi < \pi\}.$$

The present paper is dedicated to a generalization of the above observation. Let $\delta \in \mathbf{R}^+ := \{x \,|\, x \geqq 0\}$ and consider the class $\mathscr{C}_n(z_0, \delta)$ of all polynomials P of degree $\leqq n$ which fulfill (1) and

$$(2) \qquad |P(z_0)| = \delta\,|P(1)|.$$

Again it is obvious that for $\delta \neq 1$ there exists an open set $G_n(\delta)$ containing the point $z = 1$ such that $\mathscr{C}_n(z_0, \delta)$ is empty if and only if $z_0 \in G_n(\delta)$. In the sequel we shall determine those pairs $(z_0, \delta) \in \mathbf{R} \times \mathbf{R}^+$ for which $\mathscr{C}_n(z_0, \delta)$ is empty.

To reach this aim we use the following equivalence:

PROPOSITION 1. *Let* $(z_0, \delta) \in \mathbf{C} \times \mathbf{R}^+$ *and* $\mathscr{P}_n(z_0, \delta)$ *be the class of all polynomials* P *of degree* $\leqq n$ *which fulfill* $P(z_0) = \delta$ *and*

$$(3) \qquad \max\{|P(e^{i\varphi})| \,|\, \phi \in \mathbf{R}\} = 1.$$

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Furthermore we define

(4) $$M_n(z_0, \delta) := \max\{|P(1)|\,|P \in \mathscr{P}_n(z_0, \delta)\}.$$

Then (z_0, δ) *is empty if and only if* $M_n(z_0, \delta) < 1$.

PROOF. The existence of the maximum (4) is proved by compactness arguments which use the fact that $\mathscr{P}_n(z_0, \delta)$ is locally uniformly bounded on C. If $M_n(z_0, \delta) = 1$ obviously $\mathscr{C}_n(z_0, \delta)$ is not empty. If $\mathscr{C}_n(z_0, \delta)$ is not empty there exists a $P \in \mathscr{C}_n(z_0, \delta)$ such that $|P(1)| = 1 = \max\{|P(e^{i\varphi})|\,|\phi \in \mathrm{R}\}$ and $P(z_0) = \delta$, hence $M_n(z_0, \delta) = 1$.

The task of determining $M_n(z_0, \delta)$ may be reduced by the following two simple facts:

PROPOSITION 2. a) *If* $n \in \mathrm{N}$, $0 < |z_0| < 1$ *and* P *is a polynomials of degree* $\leqq n$ *such that* (3) *is fulfilled then* $|P(1/z_0)| \leqq 1/|z_0|^n$.

b) *For* $n \in \mathrm{N}$, $0 < |z_0| \leqq 1$, $\delta \in [0, 1]$ $M_n(1/z_0, \delta/|z_0|^n) = M_n(z_0, \delta)$.

PROOF. a) Assume that there exists a polynomial P of degree $\leqq n$ such that (3) is fulfilled and $|P(1/z_0)| > 1/|z_0|^n$. Then the polynomials $\tilde{P}$ defined by

$$\tilde{P}(z) = z^n P(1/z)$$

is of degree $\leqq n$ and satisfies (3), too. But $|\tilde{P}(z_0)| > 1$ which is impossible according to the maximum principle.

b) Let $E_n(z_0, \delta; \cdot) \in \mathscr{P}_n(z_0, \delta)$ be a polynomial such that $|E_n(z_0, \delta; 1)| = M_n(z_0, \delta)$. It is easily seen that the polynomial D defined by

$$D(z) := (z_0/|z_0|)^n z^n E_n(z_0, \delta; 1/z)$$

belongs to the class $\mathscr{P}_n(1/z_0, \delta/|z_0|^n)$. Now assume that there exists a polynomial $P \in \mathscr{P}_n(1/z_0, \delta/|z_0|^n)$ such that $|P(1)| > |D(1)|$ and consider the polynomial P^* defined by

$$P^*(z) := (|z_0|/z_0)^n z^n P(1/z).$$

$P^* \in \mathscr{P}_n(z_0, \delta)$ and $|P(1)| > M_n(z_0, \delta) = |D(1)|$, a contradiction to the definition of $M_n(z_0, \delta)$. Therefore $M_n(1/z_0, \delta/|z_0|^n) = |D(1)| = M_n(z_0, \delta)$. This means that we may restrict our considerations to the pairs $(z_0, \delta) \in \{z\,|\,|z| \leqq 1\} \times [0, 1]$.

A further reduction of the considered pairs is due to Theorem A:

PROPOSITION 3. *Let* $(z_0, \delta) \in (\{z\,|\,|z| \leqq 1\} \backslash G_n) \times [0, 1]$. *Then* $M_n(z_0, \delta) = 1$.

PROOF. According to Theorem A there exists a polynomial $H \in \mathscr{C}_n(z_0)$ such that

$$H(1) = 1 = \max\{|H(e^{i\varphi})|\,|\phi \in \mathrm{R}\}.$$

Hence the polynomial P defined by

$$P(z) = \delta + (1 - \delta)H(z)$$

belongs to the class $\mathcal{P}_n(z_0, \delta)$ and $P(1) = 1$ which implies $M_n(z_0, \delta) = 1$.

If we only consider pairs $(\rho, \delta) \in \mathrm{R} \times \mathrm{R}^+$ as we shall do in the sequel we conclude from Theorem A and Propositions 1–3 that we only have to determine $M_n(\rho, \delta)$ for

$$\rho \in (t_n, 1), t_n := \frac{1 - \sin(\pi/2n)}{1 + \sin(\pi/2\pi)} \quad \text{and} \quad \delta \in [0, 1].$$

The central part in these determinations is played by a class of bounded polynomials which were used by St. Ruscheweyh and the present author in [2] to solve a similar extremum problem.

We summarize the facts proved in [2]:

THEOREM B. *Let* $\mu \in [0, 1]$ *and let*

$a, b, c \colon \mathrm{C} \backslash \{0\} \to \mathrm{C}$

be functions of the form

$$z \mapsto \varepsilon z + \frac{\bar{\varepsilon}}{z}, \quad \varepsilon \in \mathrm{C} \backslash \{0\},$$

such that

$$a(z)^2 + b(z)^2 + c(z)^2 = 1, \quad z \in \mathrm{C} \backslash \{0\},$$

and

$$f(z) := a(z) - i\sqrt{1 - \mu^2}\, b(z) \neq 0 \quad \text{for} \quad |z| = 1.$$

For $n \in \mathrm{N}$ *let*

$$P(z^2) := z^n[\mu T_n(c(z)) - (\sqrt{1 - \mu^2}\, a(z) - ib(z))U_{n-1}(c(z))],$$

where T_n, U_{n-1} *denote the Chebyshev polynomials of the first and the second kind. Then*

$$P \colon z \mapsto P(z)$$

is a polynomial of degree $\leqq n$ *and for* $\phi \in \mathrm{R}$ *the following identity is valid*

$$(5) \qquad e^{-in\varphi/2} \frac{\overline{f(e^{i\varphi/2})}}{|f(e^{i\varphi/2})|} P(e^{i\varphi}) = \cos \Theta(\phi) + i \frac{\mu b(e^{i\varphi/2})}{\sqrt{1 - c(e^{i\varphi/2})^2}} \sin \Theta(\phi),$$

where

$$\Theta(2\phi) := n \arccos c(e^{i\varphi}) + \arccos \frac{\mu a(e^{i\varphi})}{|f(e^{i\varphi})|}.$$

(5) *implies*

a) $|P(e^{i\varphi})| \leqq 1$ *for* $\phi \in \mathsf{R}$,

b) $|P(e^{i\varphi})| = 1$ *if and only if* $\sin\Theta(\phi) = 0$,

c) *If there exists a* $\tilde{\phi}$ *with* $\Theta(\tilde{\phi}) < \pi$ *then there are n points* $e^{i\varphi_j}$, $j = 1, \ldots, n$, *with* $\tilde{\phi} < \phi_1 < \phi_2 < \ldots < \phi_n < \tilde{\phi} + 2\pi$ *such that* $|P(e^{i\varphi_j})| = 1$.

The existence of such a value $\tilde{\phi}$ *further implies*

$$\Theta(\phi_j) = j\pi, \cos\Theta(\phi_j) = (-1)^j, \Theta'(\phi_j) > 0, \tag{6}$$

and

$$P(e^{i\varphi_j}) = (-1)^j \frac{f(e^{i\varphi_j/2})}{|f(e^{i\varphi_j/2})|} e^{in\varphi_j/2}$$

for $j = 1, \ldots, n$.

Among the polynomials described in Theorem B we now consider those which serve as extremal polynomials for the present problem:

THEOREM 1. *Let* $n \in \mathsf{N}$, $\rho \in (t_n, 1)$, *and* $\delta \in [0, 1]$.
Suppose that there exists $r \geqq 1$ *such that*

$$\frac{\delta^2(1-\rho)^2}{\rho^{n+1}} = \frac{(1-r)^2}{r^{n+1}} \tag{7}$$

and

$$\mu := \delta\sqrt{(r/\rho)^n} \in [0, 1]. \tag{8}$$

In this case let

$$a(z) := -\tfrac{1}{2}\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}(z + 1/z),$$

$$b(z) := \frac{1}{2i}(z - 1/z),$$

$$c(z) := \frac{\sqrt{4\rho + \mu^2(1-\rho)^2}}{2(1+\rho)}(z + 1/z);$$

and

$$E_n(\rho, \delta; z^2) := z^n[\mu T_n(c(z)) - (\sqrt{1-\mu^2}\,a(z) - ib(z))U_{n-1}(c(z))].$$

Then

a) *the polynomial* $E_n(\rho, \delta; \cdot): z \mapsto E_n(\rho, \delta; z)$ *belongs to the class* $\mathscr{P}_n(\rho, \delta)$,

b) $E_n(\rho, \delta; 1) = \sin\left(n \arcsin\left(\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}\right) + \arcsin\mu\right)$,

c) *if*

$$n \arcsin\left(\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}\right) + \arcsin\mu \leqq \pi/2 \tag{9}$$

then $E_n(\rho,\delta;1) = M_n(\rho,\delta)$.

PROOF. The proof runs along similar lines as the proof of Theorem 4 in [2].

It is no problem to verify that the functions a, b, c defined in Theorem 1 satisfy the conditions of Theorem B, the function f in this case is of the simple form

$$f(z) = -\frac{\sqrt{1-\mu^2}}{1+\rho}(z-\rho/z).$$

So it is an immediate consequence of Theorem B that $E_n(\rho,\delta;\cdot)$ is a polynomial of degree $\leqq n$ and that $|E_n(\rho,\delta;e^{i\varphi})| \leqq 1$ for $\phi \in \mathrm{R}$. To prove a) it remains to show that $E_n(\rho,\delta;\rho) = \delta$. As in [2] we derive from $f(\sqrt{\rho}) = 0$

$$E_n(\rho,\delta;\rho) = \mu\sqrt{\rho^n}(c(\sqrt{\rho}) + i\mu b(\sqrt{\rho}))^n.$$

If we insert the explicit expressions for $c(\sqrt{\rho})$ and $b(\sqrt{\rho})$ and use (7) and (8) to get

$$\mu\frac{1-\rho}{\sqrt{\rho}} = \frac{r-1}{\sqrt{r}}$$

we arrive at

$$E_n(\rho,\delta;\rho) = \mu\sqrt{(\rho/r)^n} = \delta.$$

To prove b) we use (5) for $\phi = 0$. We get

$$-E_n(\rho,\delta;1) = \cos\left(n\arcsin\left(\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}\right) + \arccos(-\mu)\right)$$

$$= -\sin\left(n\arcsin\left(\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}\right) + \arcsin\mu\right).$$

The proof of c) is trivial in the case of equality in (9) since then $E_n(\rho,\delta;1) = 1$ implies $M_n(\rho,\delta) = 1$. The inequality

$$n\arcsin\left(\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}\right) + \arcsin\mu < \pi/2$$

is equivalent with $\Theta(0) < \pi$. Hence the conditions of Theorem B c) are fulfilled and we get n points $e^{i\varphi_j}, j = 1,\ldots,n$, such that $0 < \phi_1 < \phi_2 < \ldots < \phi_n < 2\pi$ and

$$E_n(\rho,\delta;e^{i\varphi_j}) = -(-1)^j e^{in\varphi_j/2}\frac{e^{i\varphi_j/2} - \rho e^{-i\varphi_j/2}}{|e^{i\varphi_j/2} - \rho e^{-i\varphi_j/2}|}.$$

Now let

$$Q(z) = -(z-\rho)\sqrt{W(z)},\ W(z) := z^n(1 - E_n(\rho,\delta;z)\overline{E_n(\rho,\delta;1/\bar{z})}).$$

We may choose the square root such that $Q'(\rho) < 0$ since W is a polynomial of degree $2n$ which has double zeros in the n points $e^{i\varphi_j}$. So this polynomial has no real zeros. Since all coefficients are real, W is real on the real axis. Hence $W(1) > 0$ implies $W(\rho) > 0$. Observing (6) we get from (5)

$$Q(e^{i\varphi}) = -(e^{i\varphi} - \rho)e^{in\varphi/2}g(\phi)\sin\Theta(\phi),$$

$$g(\phi) = \left(1 - \frac{\mu^2 b(e^{i\varphi/2})^2}{1 - c(e^{i\varphi/2})^2}\right)^{1/2} > 0.$$

Again taking notice of (6) we conclude

$$Q'(e^{i\varphi_j}) = -\frac{1}{i}(e^{i\varphi_j/2} - \rho e^{-i\varphi_j/2})e^{i(n-1)\varphi_j/2}g(\phi_j)(-1)^j\Theta'(\phi_j),$$

$$\text{(10)}\qquad \frac{E_n(\rho,\delta;e^{i\varphi_j})}{Q'(e^{i\varphi_j})(1 - e^{i\varphi_j})} = -\frac{1}{2\,|e^{i\varphi_j/2} - \rho e^{-i\varphi_j/2}|\,g(\phi_j)\Theta'(\phi_j)\sin\phi_j/2} < 0$$

for $j = 1,\ldots,n$, and

$$\text{(11)}\qquad \frac{E_n(\rho,\delta;\rho)}{Q'(\rho)(1-\rho)} = \frac{\delta}{Q'(\rho)(1-\rho)} \leqq 0.$$

(10) and (11) are decisive for the proof which consists in the use of Lagrange's interpolation formula for a polynomial $R \in \mathscr{P}_n(\rho,\delta)$ with the interpolation points $\rho, e^{i\varphi_j}, j = 1,\ldots,n$. This yields

$$\text{(12)}\qquad |R(1)| \leqq |Q(1)|\left(\sum_{j=1}^{n}\frac{1}{|Q'(e^{i\varphi_j})(1 - e^{i\varphi_j})|} + \frac{\delta}{|Q'(\rho)(1-\rho)|}\right).$$

Since $|E_n(\rho,\delta;e^{i\varphi_j})| = 1, j = 1,\ldots,n$, and $E_n(\rho,\delta;\rho) = \delta$ the inequalities (10) and (11) imply that the right side of (12) is just $|E_n(\rho,\delta;1)|$ and therefore $|E_n(\rho,\delta;1)| = M_n(\rho,\delta)$. To complete the proof one remarks that $E_n(\rho,\delta;1) > 0$.

In the rest of the paper we shall show that Theorem 1 allows to compute $M_n(\rho,\delta)$ for every pair $(\rho,\delta) \in (t_n, 1) \times [0,1]$.

Since the discussions of Theorem 1 in the case $n = 1$ differ a little from those for $n \geqq 2$ we first mention the result in this case. We omit the simple proof which may be performed by elementary extremum computations as well.

THEOREM 2. a) *Let $\rho \in (0,1)$ and $\delta \in [0,\rho)$. Then*

$$M_1(\rho,\delta) = E_1(\rho,\delta;1) = \frac{2\delta + 1 - \rho}{1+\rho} < 1,$$

$$E_1(\rho,\delta;z) = \frac{\delta+1}{1+\rho}z + \frac{\delta-\rho}{1+\rho},\quad E_1(\rho,\delta;\cdot) \in \mathscr{P}_n(\rho,\delta).$$

b) *Let* $\rho \in (0,1)$ *and* $\delta \in [\rho, 1]$. *Then*

$$M_1(\rho,\delta) = \tilde{E}_1(\rho,\delta;1) = 1$$

$$\tilde{E}_1(\rho,\delta;z) = \frac{1-\delta}{1-\rho}z + \frac{\delta-\rho}{1-\rho}, \quad \tilde{E}_1(\rho,\delta;\cdot) \in \mathscr{P}_n(\rho,\delta).$$

For $n \geqq 2$ we get more complicated formulae. Therefore we need some preparations.

PROPOSITION 4. *Let* $n \in \mathsf{N}\backslash\{1\}$, $\delta \in [0,1]$ *and*

$$g_n\colon (0,\infty) \to \mathsf{R},\ x \mapsto g_n(x) := (1-x)^2/x^{n+1}$$

a) *If* $\rho \in \left(0, \frac{n-1}{n+1}\right)$ *and* $\delta^2 g_n(\rho) < g_n\left(\frac{n+1}{n-1}\right)$, *then*

$$\delta^2 g_n(\rho) = g_n(r) \tag{13}$$

has a unique solution $r \in \left[1, \frac{n+1}{n-1}\right)$.

b) *If* $\rho \in \left[\frac{n-1}{n+1}, 1\right)$, $\delta \in [0, \rho^n)$, *then* (13) *has a unique solution* $r \in [1, 1/\rho)$.

PROOF. It is immediately seen that g_n is strictly increasing and continuous on $\left[1, \frac{n+1}{n-1}\right]$, $g_n\left(\left[1, \frac{n+1}{n-1}\right)\right) = \left[0, g_n\left(\frac{n+1}{n-1}\right)\right)$. This implies a).

b) follows from the same facts and the additional observations

$$1/\rho \leqq \frac{n+1}{n-1} \quad \text{and} \quad g_n(1/\rho) = (1-\rho)^2\rho^{n-1} > \delta^2 g_n(\rho).$$

PROPOSITION 5. *Let* $n \in \mathsf{N}\backslash\{1\}$,

$$q_n\colon [0,1) \to (0, \pi/2n],\ \mu \mapsto q_n(\mu) = \frac{\pi}{2n} - \frac{\arcsin\mu}{n},$$

$$\rho_n\colon [0,1) \to \left[t_n, \frac{n-1}{n+1}\right),\ \mu \mapsto \rho_n(\mu) = \frac{\sqrt{1-\mu^2} - \sin q_n(\mu)}{\sqrt{1-\mu^2} + \sin q_n(\mu)}. \tag{14}$$

a) $\rho_n(\mu) < \rho$ *is equivalent with*

$$n \arcsin\left(\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}\right) + \arcsin\mu < \pi/2, \tag{15}$$

$\rho_n(\mu) = \rho$ *with*

$$n \arcsin\left(\sqrt{1-\mu^2}\,\frac{1-\rho}{1+\rho}\right) + \arcsin\mu = \pi/2.$$

b) ρ_n *is strictly increasing and bijective.*

PROOF. a) is trivial.

To prove b) we see that $\rho_n(0) = t_n$, $\lim_{\mu \to 1-0} \rho_n(\mu) = \frac{n-1}{n+1}$ and ρ_n is continuous and $\frac{d\rho_n}{d\mu}(\mu) > 0$ for $\mu \in [0, 1)$. Only the last statement needs a little explanation. Write

$$\frac{d\rho_n}{d\mu}(\mu) = \frac{2Z_n(\mu)}{\sqrt{1-\mu^2}(\sqrt{1-\mu^2} + \sin q_n(\mu))^2},$$

$$Z_n(\mu) = -\mu \sin q_n(\mu) + \frac{\sqrt{1-\mu^2}}{n} \cos q_n(\mu).$$

Then

$$\frac{dZ_n}{d\mu}(\mu) = (-1 + 1/n^2) \sin q_n(\mu) < 0 \quad \text{and} \quad \lim_{\mu \to 1-0} Z_n(\mu) = 0$$

prove this assertion.

PROPOSITION 6. *Let $n \in \mathsf{N} \setminus \{1\}$, $\mu \in [0, 1)$, ρ_n as above. The equation*

$$\mu^2 \frac{(1-\rho_n(\mu))^2}{\rho_n(\mu)} = \frac{(1-r)^2}{r} \tag{16}$$

has a unique solution $r = r_n(\mu) \geqq 1$. The function

$$r_n\colon [0, 1) \to \left[1, \frac{n+1}{n-1}\right), \ \mu \mapsto r_n(\mu) = 1 + \frac{2\mu \sin q_n(\mu)}{\sqrt{1-\mu^2} \cos q_n(\mu) - \mu \sin q_n(\mu)} \tag{17}$$

is strictly increasing.

PROOF. Since (16) is a quadratic equation in r it is easy to get the explicit expression for $r = r_n(\mu)$ given in (17). $\lim_{\mu \to 1-0} r_n(\mu) = \frac{n+1}{n-1}$ is an exercise (l'Hospital's rule).

To show the monotonicity of r_n which here implies $r_n([0, 1)) = \left[1, \frac{n+1}{n-1}\right)$ we prove that

$$K_n\colon (0, 1) \to \mathsf{R}, \ \mu \mapsto K_n(\mu) = \frac{\sqrt{1-\mu^2} \cos q_n(\mu)}{\mu \sin q_n(\mu)}$$

is strictly decreasing:

$$\frac{dK_n}{d\mu}(\mu) = \frac{\frac{1}{n} \sin(2 \arcsin \mu) - \sin 2q_n(\mu)}{2\mu^2 (\sin q_n(\mu))^2 \sqrt{1-\mu^2}} < 0.$$

The last inequality is equivalent with

$$0 < \sin\left(\frac{\pi - y}{n}\right) - \frac{\sin y}{n} \quad \text{for} \quad y \in (0, \pi)$$

which is easily verified by differentiation.

We now define

(18) $$d_n\colon [0,1) \to [0,1)$$

$$\mu \mapsto d_n(\mu) = \mu\sqrt{(\rho_n(\mu)/r_n(\mu))^n} = \mu\left[\frac{\sqrt{1-\mu^2} - \sin q_n(\mu)}{\sqrt{1-\mu^2}\cos q_n(\mu) + \mu \sin q_n(\mu)}\right]^n$$

It follows from Proposition 5 and Proposition 6 that $d_n(\mu) < 1$ for $\mu \in [0, 1)$. According to Proposition 5 there exists

$$\rho_n^{-1}\colon \left[t_n, \frac{n-1}{n+1}\right) \to [0,1),\ \rho \mapsto \rho_n^{-1}(\rho),$$

the inverse function of ρ_n.

The decisive part in the solution of our problem is played by the function

$$\delta_n\colon \left[t_n, \frac{n-1}{n+1}\right) \to [0,1),\ \rho \mapsto \delta_n(\rho) = d_n(\rho_n^{-1}(\rho)).$$

As we have seen above a parametrisation of this function is given by (14) and (18). The announced solution is as follows:

THEOREM 3. *Let* $n \in \mathbb{N}\backslash\{1\}$, $\rho \in (t_n, 1)$ *and*

$$\Delta_n(\rho) = \begin{cases} \delta_n(\rho) & \rho \in \left(t_n, \dfrac{n-1}{n+1}\right) \\ \rho_n & \rho \in \left[\dfrac{n-1}{n+1}, 1\right). \end{cases}$$

$M_n(\rho, \delta) < 1$ *if* $\delta \in [0, \Delta_n(\rho))$ *and* $M_n(\rho, \delta) = 1$ *if* $\delta \in [\Delta_n(\rho), 1]$.

PROOF. Let $\rho \in \left(t_n, \frac{n-1}{n+1}\right)$. According to Proposition 5 there exists an unique $\mu_0 \in (0, 1)$ such that $\rho_n(\mu_0) = \rho$. Let $r_n(\mu_0)$ and $d_n(\mu_0)$ be defined as above and $0 \leqq \delta < \delta_n(\rho) = d_n(\mu_0)$. Then

$$\frac{\delta^2(1-\rho)^2}{\rho^{n+1}} < d_n(\mu_0)^2 \frac{(1-\rho)^2}{\rho^{n+1}} = \frac{\mu_0^2(1-\rho_n(\mu_0))^2}{r_n(\mu_0)^n \rho_n(\mu_0)} =$$

$$\frac{(1-r_n(\mu_0))^2}{r_n(\mu_0)^{n+1}} = g_n(r_n(\mu_0)) < g_n\left(\frac{n+1}{n-1}\right).$$

Since $r_n(\mu_0) \in \left[1, \frac{n+1}{n-1}\right)$ there exists an unique $r \in [1, r_n(\mu_0)) \subset \left[1, \frac{n+1}{n-1}\right)$ such that $\delta^2 g_n(\rho) = g_n(r)$. From the above we see that

$$\mu := \delta\sqrt{(r/\rho)^n} < d_n(\mu_0)\sqrt{(r_n(\mu_0)/\rho_n(\mu_0))^n} = \mu_0.$$

This implies $\rho_n(\mu) < \rho_n(\mu_0) = \rho$ and hence as we have seen in Proposition 5 the inequality (15). Using Theorem 1 we see that for $\delta \in [0, \delta_n(\rho))$

$$M_n(\rho, \delta) = E_n(\rho, \delta; 1) = \sin\left(n \arcsin\left(\sqrt{1-\mu^2}\frac{1-\rho}{1+\rho}\right) + \arcsin\mu\right) < 1.$$

We notice that for $\delta = \delta_n(\rho) = d_n(\mu_0)$, $\mu_0 \in (0, 1)$, we get

$$\delta^2 g_n(\rho) = g_n(r_n(\mu_0))$$

such that

$$\mu_0 = \delta\sqrt{(r_n(\mu_0)/\rho_n(\mu_0))^n} \in (0, 1)$$

and

$$n \arcsin\left(\sqrt{1-\mu_0^2}\frac{1-\rho}{1+\rho}\right) + \arcsin\mu_0 = \pi/2.$$

According to Theorem 1 this implies

$$M_n(\rho, \delta_n(\rho)) = E_n(\rho, \delta_n(\rho); 1) = 1.$$

If $\delta \in (\delta_n(\rho), 1]$ we consider

$$\tilde{E}_n(\rho, \delta; z) := \frac{1-\delta}{1-\delta_n(\rho)} E_n(\rho, \delta_n(\rho); z) + \frac{\delta - \delta_n(\rho)}{1-\delta_n(\rho)}.$$

and verify $\tilde{E}_n(\rho, \delta; \cdot) \in \mathscr{P}_n(\rho, \delta)$ and $\tilde{E}_n(\rho, \delta; 1) = 1 = M_n(\rho, \delta)$.

Now let $\rho \in \left[\frac{n-1}{n+1}, 1\right)$. If $\delta \in [0, \rho^n)$ the existence of an unique solution $r \in [1, 1/\rho)$ of $\delta^2 g_n(\rho) = g_n(r)$ is guaranteed by Proposition 4. $\mu = \delta\sqrt{(r/\rho)^n} < 1$ follows from $r < 1/\rho$. (15) is fulfilled since $\rho \geqq \frac{n-1}{n+1} > \rho_n(\mu)$ for all $\mu \in [0, 1)$. Again we conclude from Theorem 1

$$M_n(\rho, \delta) = E_n(\rho, \delta; 1) = \sin\left(n \arcsin\left(\sqrt{1-\mu^2}\frac{1-\rho}{1+\rho}\right) + \arcsin\mu\right) < 1.$$

For $\delta \in [\rho^n, 1]$ we consider

$$\tilde{E}_n(\rho, \delta; z) = \frac{1-\delta}{1-\rho^n} z^n + \frac{\delta - \rho^n}{1-\rho^n}$$

and verify $\tilde{E}_n(\rho, \delta; \cdot) \in \mathscr{P}_n(\rho, \delta)$ and $\tilde{E}_n(\rho, \delta; 1) = 1 = M_n(\rho, \delta)$.

REMARK. As we have seen in the proof of Theorem 3 for any pair (ρ, δ) with $M_n(\rho, \delta) < 1$ there exists a polynomial $E_n(\rho, \delta; \cdot)$ from the class described in Theorem B such that $M_n(\rho, \delta) = E_n(\rho, \delta; 1)$.

One should conjecture that the same holds true for all paars (z_0, δ) with $M_n(z_0, \delta) < 1$. The author wants to express his hope that the contents of the present article may be helpful for the identifying of the "right" members of the said class if this conjecture is true.

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