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AMENABILITY FOR DISCRETE CONVOLUTION SEMIGROUP ALGEBRAS

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Summary.

For any semigroup S we show that if the convolution algebra $\ell^1(S,\omega)$ is amenable for some weight ω then S is a regular semigroup with a finite number of idempotents; in particular, for the case of an inverse semigroup S, we have $\ell^1(S)$ amenable if and only if S has a finite number of idempotents and every subgroup of S is amenable. Various known results on amenability are shown to be easy corollaries of our results.

Let S be a semigroup and let $\ell^1(S)$ be the usual discrete convolution semigroup algebra. The condition that $\ell^1(S)$ be an amenable Banach algebra imposes a very strong constraint on the structure of S. When S is a group, $\ell^1(S)$ is amenable iff S is an amenable group (Johnson [5]). When S is commutative, it is amenable iff S is a finite semilattice of abelian (and hence amenable) groups (Groenbaek [3]). When S is a cancellative semigroup with identity, it is amenable iff S is an amenable group (Groenbaek [2]). When S is an E-unitary inverse semigroup it is amenable iff the set of idempotents in S is finite and every subgroup of S is amenable (Duncan-Namioka [1]). In this note, by a rather simple argument, we prove a general result which gives most of the above results as almost immediate corollaries. We also complete the story for inverse semigroups by showing that the above result holds without the restriction that S be E-unitary. In general we show that if $\ell^1(S)$ is amenable then S must be a regular semigroup with a finite number of idempotents. It is easy to describe the latter class of semigroups in terms of a finite composition series, but we have been unable to classify exactly which of these semigroups has $\ell^1(S)$ amenable. We give some examples to indicate the scope of the difficulties. Our argument uses the appproximate diagonal characterization of amenability and adapts to deal with weighted semigroup algebras.

We recall some general ideas. For a Banach algebra A the projective tensor product $A \hat{\otimes} A$ is a bimodule with products determined by $(a \otimes b)c = a \otimes bc$, $c(a \otimes b) = ca \otimes b$. Let $\pi : A \hat{\otimes} A \to A$ be the canonical homomorphism deter-

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mined by $\pi(a \otimes b) = ab$. Then A is amenable iff it has an approximate diagonal ([6]), i.e. a bounded net $m^{(\alpha)}$ in $A \hat{\otimes} A$ such that, for each a in A,

$$\lim_{\alpha} m^{(\alpha)}a - am^{(\alpha)} = 0, \lim_{\alpha} \pi(m^{(\alpha)})a = a.$$

When $A = \ell^1(S)$ we may identify $A \hat{\otimes} A$ with $\ell^1(S \times S)$. We may write $m^{(\alpha)} = \sum_{S \times S} \beta^{(\alpha)}_{s,t}(s, t)$, and we see that $m^{(\alpha)}$ is an approximate diagonal iff, for each v in S,

$$\text{(D)} \ \lim_{\alpha} \textstyle\sum_{S \times S} \beta^{(\alpha)}_{s,t} [(s, tv) - (vs, t)] = 0, \lim_{\alpha} \textstyle\sum_{S \times S} \beta^{(\alpha)}_{s,t}\, stv = v.$$

Similar remarks apply for weighted convolution semigroup algebras. To be precise, let ω be a weight on S, i.e. $\omega(s) > 0$ and $\omega(st) \leqq \omega(s)\omega(t)$ for all s, t in S. Then $\ell^1(S, \omega)$ consists of all elements $x = \sum_S v_s s$ with $\|x\| = \sum_S |v_s| \omega(s) < \infty$, and it is a Banach algebra under convolution. We may identify $\ell^1(S, \omega) \otimes \ell^1(S, \omega)$ with $\ell^1(S \times S, \omega \otimes \omega)$ where

$$(\omega \otimes \omega)(s, t) = \omega(s)\omega(t).$$

The description of an approximate diagonal is exactly as in (D) with the convergence in the weighted norm, of course.

We recall also that S is a regular semigroup if for each $s \in S$ there exists $s^* \in S$ with $ss^*s = s$, $s^*ss^* = s^*$. We write E_S for the set of idempotents in S. A regular semigroup S is an inverse semigroup provided the element s^* exists uniquely in S for each $s \in S$; in this case E_S is a semilattice. We follow Groenbaek in writing $[uu^{-1}]$ for $\{x \in S : xu = u\}$, where S is any semigroup. Similarly we write $[u^{-1}u] = \{x \in S : ux = x\}$. We write $X(u) = uS \cap [uu^{-1}]$.

THEOREM 1. *Let S be a semigroup that contains an infinite pairwise disjoint sequence of sets $X(u_n)$. Then $\ell^1(S, \omega)$ is not amenable for any weight function ω.*

PROOF. Suppose $\ell^1(S, \omega)$ has approximate diagonal $m^{(\alpha)}$ and fix $v \in S$. The sum below is the norm of the first sum in (D) with the summands for $s \in vS$ removed. We thus have

$$\lim_{\alpha} \textstyle\sum_{s \in S \setminus vS} \sum_x |\sum_{tv = x} \beta^{(\alpha)}_{s,t}|\, \omega(s)\omega(tv) = 0$$

and so

$$\lim_{\alpha} \textstyle\sum_x \sum_{s \in S \setminus vS,\, sx = v} |\sum_{tv = x} \beta^{(\alpha)}_{s,t}\, \omega(stv) = 0.$$

For each s, t in the summation we have $stv = v$. It follows that

$$\lim_{\alpha} \sum_{s \in S \setminus vS,\, stv = v} \beta^{(\alpha)}_{s,t} = 0.$$

Property (D) also gives

$$\lim_{\alpha} \sum_{stv = v} \beta^{(\alpha)}_{s,t} = 1$$

and so we have

$$\lim_{\alpha} \sum_{s \in vS,\, stv = v} \beta^{(\alpha)}_{s,t} = 1.$$

Let $Z(v) = \{(s,t) \in S \times S\colon s \in vS \text{ and } stv = v\}$. If $(s,t) \in Z(v)$, then $st \in vS$ and hence $st \in vS \cap [vv^{-1}] = X(v)$. Since the sets $X(u_n)$ are pairwise disjoint it follows that the sets $Z(u_n)$ are pairwise disjoint. Choose distinct $u_1, \ldots, u_N \in S$ with $N > M = \sup\|m^{(\alpha)}\|$. It follows that

$$N = \lim_{\alpha} \sum_{i=1}^{N} \sum_{Z(v_i)} \beta^{(\alpha)}_{s,t} \leqq \sup \sum_{S \times S} |\beta^{(\alpha)}_{s,t}| \leqq \mathrm{M}.$$

This contradiction completes the proof.

COROLLARY 1. *Let S be an inverse semigroup with E_S infinite. Then $\ell^1(S, \omega)$ is not amenable for any weight ω.*

PROOF. Let $e \in E_S$ and let $x \in X(e)$. Then $x = es$, $xe = e$. The second equation gives $exex = ex$ and the first gives $ex = x$. Since the idempotents commute in an inverse semigroup we have $x = ex = xe = e$ and so $X(e) = \{e\}$. Theorem 1 now applies.

REMARK. Corollary 1 combines with the result of [1] to show that, for an inverse semigroup S, $\ell^1(S)$ is amenable iff E_S is finite and every subgroup of S is amenable. The results in [1] were insufficient to establish the non-amenability of $\ell^1(\mathscr{F})$ where $\mathscr{F}$ is the important inverse semigroup of all finite partial one-one maps on N. For $n \in \mathsf{N}$ let $\mathscr{F}_n$ be the ideal in $\mathscr{F}$ of all maps of rank at most n. Then $\mathscr{F} = \cup \mathscr{F}_n$ and $\ell^1(\mathscr{F})$ is the closure of the union of the ideals $\ell^1(\mathscr{F}_n)$. None of the ideals $\ell^1(\mathscr{F}_n)$ has a bounded approximate identity though each has an approximate identity. On the other hand $\ell^1(\mathscr{F})$ has a bounded approximate identity. Since $\mathscr{F}$ has a zero element (the empty map) it is the antithesis of being E-unitary (i.e. $e, es \in E_S \Rightarrow s \in E_S$). The resolution of our amenability question for $S = \mathscr{F}$ led to a general argument for Corollary 1 and thence to the even more general argument in Theorem 1.

COROLLARY 2. *Let S be a semigroup with $\ell^1(s, \omega)$ amenable for some weight ω. Then S is a regular semigroup.*

PROOF. Let $u \in S$. The argument in Theorem 1 gives $X(u) \neq \emptyset$. Note first that $X(u) \subset E_S$. Let $x \in X(u)$. Then $x = us$ and $xu = u$. Hence $usu = u$ and so $usus = us$. We have $u(sx)u = u$ and also $(sx)u(sx) = susx = sxx = sx$. This proves that S is regular.

COROLLARY 3. (Groenbaek [2]) *Let S be a left cancellative semigroup with identity with $\ell^1(S, \omega)$ amenable for some weight ω. Then S is a group.*

PROOF. S has exactly one idempotent, and is regular by Corollary 2. Hence S is a group.

COROLLARY 4. (Groenbaek [3]) *Let S be a commutative semigroup with $\ell^1(S, \omega)$ amenable for some weight ω. Then S is a finite semilattice of abelian groups.*

PROOF. A commutative regular semigroup is an inverse semigroup which is a semilattice of abelian groups. Apply Corollaries 1 and 2.

THEOREM 2. *Let S be a semigroup with $\ell^1(S, \omega)$ amenable for some weight ω. Then S is a regular semigroup with E_S finite.*

PROOF. By Corollary 2 it is enough to prove that E_S is finite. We recall from above that $X(u) \subset E_S$ for each $u \in S$. We define a relation R on E_S by uRv if $v \in X(u)$. Clearly uRu. Let $v \in X(u)$. Then $vu = u$, $v = ux$, so that $uv = uux = ux = v$ and hence $u \in X(v)$. It is easy to verify that R is transitive. Thus R is an equivalence relation and E_S is partitioned into the sets $X(u)$. By Theorem 1 there exist only finitely many $X(u)$. It is easy to check that $X(u)$ is a subsemigroup of E_S and hence it is a band (a semigroup of idempotents). It follows that $X(u)$ is a semilattice of rectangular bands (see [4, Theorem IV.3.1]). For $v \in X(u)$ we have $vu = u$, $uv = v$. Since uv, vu must belong to the same rectangular band, it follows that $X(u)$ is a rectangular band, say $I \times \Lambda$, with multiplication $(i, \lambda)(j, \mu) = (i, \mu)$. Since $vu = u$ for $v \in X(u)$ it follows that I is a singleton set. Suppose that Λ is infinite for some u. Amenability is preserved under reversed multiplication. Reverse the multiplication in S. The band $\Lambda \times I$ now gives an infinite pairwise disjoint family of sets $X(v)$. This contradiction completes the proof.

We now discuss further implications for the structure of S when S is a regular semigroup with $\ell^1(S)$ amenable. We are able to give only a partial analysis except in special cases. The case when S is a regular semigroup with E_S a rectangular band is rather easy; then S is isomorphic to $E_S \times G$ for some group G (see [4, page 211]). It follows that $\ell^1(S)$ is amenable iff E_S is a singleton and G is amenable. Now let S be a regular semigroup with E_S finite. We may apply the standard kernel theory (see, for example [4, Chapter 3]) to obtain a chain of ideals

$$S = S_1 \supset S_2 \supset \ldots \supset S_n$$

such that each Rees quotient S_k/S_{k+1} is a completely 0-simple regular semigroup with a finite number of idempotents, and S_n is a completely simple regular semigroup with a finite number of idempotents. With $\ell^1(S)$ amenable, S_n is in fact a group as noted above. For the class of inverse semigroups the above composition series reduces the amenability problem to the completely 0-simple case (see [1]). This reduction is enabled by the fact that each ideal $\ell^1(S_k)$ has an identity when E_S is finite; this is not the case for regular semigroups. Accordingly we begin our disucussion for the case of completely 0-simple regular semigroups with E_S finite, i.e. we suppose that S is a Rees matrix semigroups as below ([4, Chapter 3]).

Let I, Λ be finite index sets, let G be a group and let $G^o = G \cup \{0\}$. Let P be a $\Lambda \times I$ matrix over G^o with no row or column consisting entirely of zeros. For $g \in G$ let $(g)_{i\lambda}$ be the $I \times \Lambda$ matrix with g in position (i, λ) and zero elsewhere. The Rees matrix semigroup $\mathcal{M}^o[G; I, \Lambda; P]$ is the set of all matrices $(g)_{i\lambda}$ with product

$$(g)_{i\lambda} \circ (h)_{j\mu} = (g)_{i\lambda} P (h)_{j\mu}$$

the products on the right being the usual matrix product. For brevity we shall denote this Rees matrix semigroup by $\mathcal{M}$. We may elaborate the argument of Munn [7] to see that $\ell^1(\mathcal{M})$ has an identity iff the sets I, Λ have the same cardinality and P is invertible as a matrix over $\ell^1(G)$; the map $T \to TP$ then gives an isomorphism of the reduced semigroup algebra $\ell^1(\mathcal{M})/\mathbf{C}\theta$ with the algebra of all $n \times n$ matrices over $\ell^1(G)$. It follows as in [1] that $\ell^1(\mathcal{M})$ will then be amenable iff G is amenable. We note at this point that very different semigroups $\mathcal{M}$ can produce isomorphic reduced semigroup algebras; $\mathcal{M}$ could be an inverse semigroup or equally it might not even be an orthodox semigroup, i.e. the idempotents need not form a subsemigroup.

Let S be a regular semigroup with a finite number of idempotents. The method of [1] now shows that $\ell^1(S)$ is amenable if the kernel of S is an amenable group and each quotient Rees matrix semigroup has invertible sandwich matrix and amenable group. To prove the converse result by the method of [1] we need to show that if $\ell^1(S)$ is amenable and J is an ideal of S then $\ell^1(J)$ has a bounded approximate identity. We have been unable to achieve this technical step. Nonetheless, we conjecture that the above sufficient condition for $\ell^1(S)$ to be amenable is in fact also necessary.

REFERENCES

1. J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proc. Roy. Soc. Edinburgh Sect A 80 (1978), 309–321.

2. N. Groenbaek, *Amenability of weighted discrete convolution algebras on cancellative semigroups*, Proc. Roy. Soc. Edinburgh Sect A 110 (1988), 351–360.
3. N. Groenbaek, *Amenability of discrete convolution algebras, the commutative case*, preprint.
4. J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press, London, 1976.
5. B. E. Johnson, *Cohomology in Banach algebras*, Mem. Amer. Math. Soc. 127 (1972).
6. B. E. Johnson, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math. 94 (1972), 685–698.
7. W. D. Munn, *On semigroup algebras*, Proc. Camb. Phil. Soc. 51 (1955), 1–15.

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