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AN ABSTRACT CHARACTERIZATION OF ω-CONDITIONAL EXPECTATIONS

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Abstract.

An abstract characterization of ω-conditional expectations on von Neumann algebras is given as dual maps of canonical state extensions perturbed with convenient partial isometries.

1. Introduction, prelimnaries and notations.

The notion of an ω-conditional expectation (ω-c.e.) from a von Neumann algebra $\mathscr{M}$ to a von Neumann subalgebra $\mathscr{M}_0$ with respect to a faithful normal state ω on $\mathscr{M}$ (cfr. [4] for the generalization to states having only faithful restriction ω_0 to $\mathscr{M}_0$) was introduced in [1] as a generalization of ω-preserving norm one projections; there ω-c.e. were constructed explicitely for a general triple ($\mathscr{M}$, $\mathscr{M}_0$, ω) as above and their properties studied. The ω-c.e. have been characterized in the framework of modular theory for von Neumann algebras, [1] and of the noncommutative theory of integration on von Neumann algebras [2], but no abstract characterization of ω-c.e. exists in the literature. The main purpose of this paper is to give such a characterization.

We shall always consider a von Neumann algebra $\mathscr{M}$ with predual $\mathscr{M}_*$ acting on a separable Hilbert space $\mathscr{H}$, with a von Neumann subalgebra $\mathscr{M}_0$. The set of normal states on $\mathscr{M}$ will be denoted by $\mathscr{S}(\mathscr{M})$, and the set of faithful normal states on $\mathscr{M}_0$ by $\mathscr{S}_f(\mathscr{M}_0)$. We shall indicate by ε_ω the ω-c.e. from $\mathscr{M}$ to $\mathscr{M}_0$, and refer for the related theory to [1] and [4]. We shall only recall that ε_ω is an ω preserving completely positive linear weak-operator continuous contraction from $\mathscr{M}$ to $\mathscr{M}_0$. In [4] a notion of equivalence for ω-c.e. was studied. Namely, let $\mathscr{M}$ act in standard form on the Hilbert space $\mathscr{H}_{\mathscr{M}}$, Ω be a cyclic and separating vector in $\mathscr{H}_{\mathscr{M}}$, $\mathscr{H}_{\mathscr{M}_0}$ be the closure of the set $\{a_0\,\Omega : a_0 \in \mathscr{M}_0\}$. The $\mathscr{M}_0$ acts in standard form on the Hilbert subspace $\mathscr{H}_{\mathscr{M}_0}$ of $\mathscr{H}_{\mathscr{M}}$. Two ω-c.e. ε_ω and ε_φ are equivalent ($\varepsilon_\omega \sim \varepsilon_\varphi$) if the corresponding states ω and φ admit representative vectors in the natural positive cone $V_\Omega^{\mathscr{M}_0}$ of $\mathscr{H}_{\mathscr{M}_0}$. In this case there is a partial isometry $u \in \mathscr{M}$ such that $\varepsilon_\omega = \varepsilon_\varphi(u^+ \cdot u)$ (and symmetrically), which is explicitly given in [4]. There is

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a bijection between the set of the elements of $\mathcal{S}(\mathcal{M})$ with restriction to $\mathcal{M}_0$ in $\mathcal{S}_f(\mathcal{M}_0)$ and $\mathcal{S}_f(\mathcal{M}_0) \times E$ if we indicate by E the set of all equivalence classes $\mathcal{E}$ of ω-c.e. from $\mathcal{M}$ to $\mathcal{M}_0$. It is obtained by associating to each ω in the former set the couple $(\omega_0, \mathcal{E})$, with $\omega_0 = \omega|_{\mathcal{M}_0}$ and $\mathcal{E}$ the equivalence class of ω-c.e. to which ε_ω belongs. With the above notation, we can associate to $\mathcal{E}$ the operator $T_\mathcal{E}$ defined by setting $T_\mathcal{E}(\omega_0) = \omega$. If $\mathcal{E}$ is a singleton i.e. it has only one element), then its unique element is a norm one projection, and, conversely, if $\mathcal{E}$ contains a norm one projections then it is a singleton. For a comprehensive treatment of this theory see [4].

In [6] it is proved that if $\omega, \varphi \in \mathcal{S}(\mathcal{M})$ and there is an $\alpha > 0$ such that $\varphi \leq \alpha\,\omega\,(\varphi \ll \omega)$ then the Connes cocycle $(D\varphi : D\omega)_t$ (cfr. [5]) admits a bounded and continuous $\mathcal{M}$-valued extension to the strip $S = \{z\ \mathrm{C} : 0 \leq \operatorname{Im} z \leq \frac{1}{2}\}$, which is analytic in its interior, and that if we denote by $\Delta(\varphi, \omega)$ its value at $z = \frac{i}{2}$, we have

$$\varphi(a) = \omega(\Delta(\varphi, \omega)^+ \, a \, \Delta(\phi, \omega))$$

for all $a \in \mathcal{M}$. It follows easily from the modular theory of von Neumann algebras that if $\mathcal{M}$ acts standardly on $\mathcal{H}$ and Ω is cyclic and separating vector in $\mathcal{H}_\mathcal{M}$ representing ω (i.e. $\omega(a) = \langle \Omega, \Omega \rangle$ for $a \in \mathcal{M}$), then $\Delta(\varphi, \omega)\,\Omega$ is the vector representing φ in the natural positive cone $V_\Omega^\mathcal{M}$.

2. The main result.

2.1 DEFINITION. A mapping $\rho: \mathcal{S}(\mathcal{M}_0) \to \mathcal{S}(\mathcal{M})$ will be called a state extension if $\rho(\omega_0)|_{\mathcal{M}_0} = \omega_0$ for all $\omega_0 \in \mathcal{S}(\mathcal{M}_0)$.

2.2 PROPOSITION. *The following conditions are equivalent if ρ is a state extension from $\mathcal{S}(\mathcal{M}_0)$ to $\mathcal{S}(\mathcal{M})$:*

a) *For all $\omega_0, \varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$ such that $\varphi_0, \ll \omega_0$ we have, for all $a \in \mathcal{M}$:*

$$[\rho(\varphi_0)](a) = [\rho(\omega_0)](\Delta(\varphi_0, \omega_0)^+ \, a \, \Delta(\varphi_0, \omega_0))$$

b) *Let $\mathcal{M}$ act standardly on $\mathcal{H}_\mathcal{M}$, Ω be a vector (separating for $\mathcal{M}_0$) representing in $\mathcal{H}_\mathcal{M}$ the state $\rho(\omega_0)$, with $\omega_0 \in \mathcal{S}_f(\mathcal{M}_0)$, and $\mathcal{H}_{\mathcal{M}_0}$ and $V_\Omega^{\mathcal{M}_0}$ as defined in the preliminaries. Then each vector in $V_\Omega^{\mathcal{M}_0}$ is a representative vector for some state $\rho(\varphi_0)$ with $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$ and, conversely, for each $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$ the state $\rho(\varphi_0)$ has a (unique) representative vector $V_\Omega^{\mathcal{M}_0}$.*

c) *The mapping ρ is norm continuous and there is an $\omega_0 \in \mathcal{S}_f(\mathcal{M}_0)$ such that if $\varphi_0 \ll \omega_0$ then for all $a \in \mathcal{M}$:*

$$[\rho(\varphi_0)](a) = [\rho(\omega_0)](\Delta(\varphi_0, \omega_0)^+ \, a \, \Delta(\varphi_0, \omega_0))$$

d) *$\rho = T_\mathcal{E}$ for some equivalence class of ω-c.e. from $\mathcal{M}$ to $\mathcal{M}_0$.*

PROOF. a $\Rightarrow$ b. Let $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0), \Omega_{\frac{1}{2}(\omega_0+\varphi_0)})$ be the representative vector for $\varphi_0(\frac{1}{2}(\omega_0 + \varphi_0))$ in $V_\Omega^{\mathcal{M}_0}$. It suffices to prove that Ω_{φ_0} is a representative vector for $\rho(\varphi_0)$. We have

$$\Omega = \Delta(\omega_0, \tfrac{1}{2}(\omega_0 + \varphi_0))\,\Omega_{\frac{1}{2}(\omega_0+\varphi_0)}$$

$$\Omega_{\varphi_0} = \Delta(\varphi_0, \tfrac{1}{2}(\omega_0 + \varphi_0))\,\Omega_{(\omega_0+\varphi_0)}$$

Condition a) applied to the pair of states ω and $\frac{1}{2}(\omega_0 + \varphi_0)$ gives that $\Omega_{\frac{1}{2}(\omega_0+\varphi_0)}$ is the vector representative of $\rho(\frac{1}{2}(\omega_0 + \varphi_0))$ in $V_\Omega^{\mathcal{M}_0}$, and then applied to φ_0 and $\frac{1}{2}(\omega_0 + \varphi_0)$ yields that Ω_{φ_0} is the vector representative of $\rho(\varphi_0)$ in $V^{\mathcal{M}_0}_{\Omega\frac{1}{2}(\omega_0+\varphi_0)} = V_\Omega^{\mathcal{M}_0}$, thus proving our claim.

b $\Rightarrow$ c. Let ω_0 and Ω be as in the statement. If $\varphi_0 \ll \omega_0$, the only vector representing an extension of φ_0 in $V_\Omega^{\mathcal{M}_0}$ is $\Omega_{\varphi_0} = \Delta(\varphi_0, \omega_0)\,\Omega$, so by b) it must represent $\rho(\varphi_0)$. Therefore

$$[\rho(\varphi_0)](a) = \langle \Omega_{\rho(\varphi_0)}, a\,\Omega_{\rho(\varphi_0)}\rangle = \langle \Omega\varphi_0, a\,\Omega\varphi_0\rangle =$$

$$= \langle \Delta(\varphi_0, \omega_0)\,\Omega, a\,\Delta(\varphi_0, \omega_0)\,\Omega\rangle = [\rho(\omega_0)]\,(\Delta(\varphi_0, \omega_0)^+\, a\,\Delta(\varphi_0, \omega_0))$$

Let now $(\varphi_0)_n$ be a sequence in $\mathcal{S}_f(\mathcal{M}_0)$ converging in norm to $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$, and $\Omega_{(\varphi_0)_n}$, $\Omega\varphi_0$ their representative vectors in $V_\Omega^{\mathcal{M}_0}$. Because of the properties of the natural positive cone $V_\Omega^{\mathcal{M}_0}$ we have

$$\|\rho((\varphi_0)_n) - \rho(\varphi_0)\| \leq \|\Omega_{(\varphi_0 n_n} - \Omega\varphi_0\|^2 \leq \|(\varphi_0)_n - \varphi_0\| \to 0$$

and get the continuity of ρ.

c $\Rightarrow$ d. Let $\mathcal{E}$ be the equivalence class of ω-e.c. to which ε_ω belongs. Then if $\varphi_0 \ll \omega_0$, by [4] $\Delta(\varphi_0 + \omega_0)\,\Omega$ is a representative vector for $T_{\mathcal{E}}(\varphi_0)$, and so $T_{\mathcal{E}}(\varphi_0) = \rho(\varphi_0)$. The density of the states φ_0 such that $\varphi_0 \ll \omega_0$ and the norm continuity of $T_{\mathcal{E}}$ proved in [4] give $T_{\mathcal{E}} = \rho$.

d $\Rightarrow$ a. Let ω_0, φ_0 be as in a), and Ω a vector representative of $\rho(\omega_0) = T_{\mathcal{E}}(\omega_0)$ $(\varepsilon_{\rho(\omega_0)} \in \mathcal{E})$. Then by [4] $\Delta(\varphi_0 + \omega_0)\,\Omega$ is a vector representing $T_{\mathcal{E}}(\varphi_0) = \rho(\varphi_0)$, and a) follows.

2.3. DEFINITION. A state extension $\rho: \mathcal{S}(\mathcal{M}_0) \to \mathcal{S}(\mathcal{M})$ will be called *canonical* if it satisfies the equivalent conditions in proposition 2.2. If $\mathcal{M}$ is abelian the above conditions are equivalent to the preservation by ρ of Radon-Nikodym derivates. Conditions a) (and c)) in 2.2 give an abstract of canonical state extensions, while condition d) is constructive and allows us to establish a bijection between canonical state extensions and equivalence classes of ω-c.e., since $T_{\mathcal{E}_1} = T_{\mathcal{E}_2}$ if and only if $\mathcal{E}_1 = \mathcal{E}_2$ (cfr. [4]). We shall denote by $\mathcal{E}_p$ the equivalence class of ω-c.e. corresponding to ρ and $\rho_{\mathcal{E}}$ the canonical state extension corresponding to $\mathcal{E}$. In other words if ρ is a canonical state extension all the ω-c.e. $\varepsilon_{\rho(\omega_0)}$ for $\omega_0 \in \mathcal{S}_f(\mathcal{M}_0)$ are equivalent and form the equivalence class $\mathcal{E}_p$.

2.4. Theorem. *Let ρ be a canonical state extension from $\mathscr{S}(\mathscr{M}_0)$ to $\mathscr{S}(\mathscr{M})$. Then ρ can be extended to a bounded linear mapping $\tilde{\rho}$ from $(\mathscr{M}_0)_*$ to M_* if and only if $\mathscr{E}_\rho$ is a singleton.*

Proof. Let $\mathscr{E}_\rho$ be a singleton. By [4] its only element ε_ρ is a norm one projection, and

$$\rho(\omega_0) = T_{\mathscr{E}_\rho}(\omega_0) = \omega_0 \circ \varepsilon_\rho.$$

The mapping $\tilde{\rho}: \varphi_0 \to \varphi_0 \circ \varepsilon_\rho$ from $(\mathscr{M}_0)_*$ to $\mathscr{M}_*$ is clearly a bounded extension of ρ. The converse implication is a consequence of prop. 2.8.

In the general situation the notion of bounded linear ω-up to a phase extension to be defined below can be used as a substitute of the (nonexisting) bounded linear extension of a canonical state extension ρ.

2.5. Definition. Let $\mathscr{N}, \mathscr{M}$ be von Neumann algebras, λ a mapping from $\mathscr{S}_f(\mathscr{N})$ to $\mathscr{S}(\mathscr{M})$, $\omega_0 \in \mathscr{S}_f(\mathscr{N})$. We say that a bounded linear mapping $\lambda_\omega: \mathscr{N}_* \to \mathscr{M}_*$ is a bounded linear *ω_0-up to a phase extension of* λ if for each $\varphi \in \mathscr{S}_f(\mathscr{N})$ there is a partial isometry (phase) $u_{\omega_0}(\varphi)$ in $\mathscr{M}$ such that

a) $u_{\omega_0}(\omega_0) = I$

b) $[\lambda_{\omega_0}(\varphi)](a) = [\lambda(\varphi)](u_\omega(\varphi)^+ \, a \, u_\omega(\varphi))$

for all $a \in \mathscr{M}$.

Obviously the notion defined above reduces to a usual bounded linear extension if $u_\omega(\varphi) = I$ for all $\varphi \in \mathscr{S}_f(\mathscr{N})$.

2.6. Theorem. *For any canonical extension $\rho: \mathscr{S}(\mathscr{M}_0) \to \mathscr{S}(\mathscr{M})$ and all $\omega_0 \in \mathscr{S}_f(\mathscr{M}_0)$ there is a unique bounded linear ω_0-up to a phase extension ρ_{ω_0} of ρ. Its explicit form is $\rho_{\omega_0}(\varphi_0) = \varphi_0 \circ \varepsilon_\rho(\omega_0)$.*

Proof. The mapping $\varphi_0 \to \varphi_0 \circ \varepsilon_{\rho(\omega)} (\varphi_0 \in \mathscr{S}_f(\mathscr{M}_0))$ is clearly linear and bounded. In [4] the explicit form of a family of partial isometries $u_{\omega_0}(\varphi_0)$ in $\mathscr{M}$ is given, such that, for $a \in \mathscr{M}$, we have

$$\varepsilon_{\rho(\omega_0)}(a) = \varepsilon_{\rho(\varphi_0)}(u_{\omega_0}(\varphi_0)^+ \, a \, u_{\omega_0}(\varphi_0)),$$

because, as already remarked, $\varepsilon_{\rho(\omega_0)} \sim \varepsilon_{\rho(\varphi_0)}$. So, for $a \in \mathscr{M}$:

$$[\varphi_0 \circ \varepsilon_{\rho(\omega_0)}](a) = \varphi_0(\varepsilon_{\rho(\omega_0)}(u_{\omega_0}(\varphi_0)^+ \, a \, u_{\omega_0}(\varphi_0))) =$$

$$= [\rho(\varphi_0)](u_{\omega_0}(\varphi_0)^+ \, a \, u_{\omega_0}(\varphi_0)),$$

and $u_{\omega_0}(\omega_0) = I$. This implies that $\rho_{\omega_0} \equiv \varphi_0 \circ \varepsilon_{\rho(\omega_0)}$ is a bounded linear ω_0 up to a phase extension of ρ.

Let now, for each $\varphi_0 \in \mathscr{S}_f(\mathscr{M}_0)$, $u_{\omega_0}(\varphi_0)$ be a partial isometry in $\mathscr{M}$, $u_{\omega_0}(\omega_0) = I$

and $\varepsilon_{\rho(\varphi_0)}$ the ω-c.e. from $\mathcal{M}$ to $\mathcal{M}_0$ relative to $\rho(\varphi_0)$. A necessary condition for the mapping

$$\varphi_0 \to [\rho(\varphi_0)](u_{\omega_0}(\varphi_0)^+ \cdot u_{\omega_0}(\varphi_0))$$

to have an extension to a bounded linear mapping from $(\mathcal{M}_0)_*$ to $\mathcal{M}_*$ is that for each $a \in \mathcal{M}$ the mapping

$$\varphi_0 \to [\rho(\varphi_0)](u_{\omega_0}(\varphi_0)^+ \, a \, u_{\omega_0}(\varphi_0)) =$$
$$= \varphi_0(\varepsilon_{\rho(\varphi_0)}(u_{\omega_0}(\varphi_0))$$

must have an extension to a bounded linear functional on $(\mathcal{M}_0)_*$. This implies that $\varepsilon_{\rho(\varphi_0)}(u_{\omega_0}(\varphi_0)^+ \, a \, u_{\omega_0}(\varphi_0))$ must be independent from φ_0, or, more precisely, for all $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$ the condition

$$\varepsilon_{\rho(\varphi_0)}(u_{\omega_0}(\varphi_0)^+ \, a \, u_{\omega_0}(\varphi_0)) = \varepsilon_{\rho(\omega_0)}(a)$$

must be satisfied, which can happen only for the mapping $\varphi_0 \to \varphi_0 \circ \varepsilon_{\rho(\omega_0)}$.

Note that in general the mapping ρ_{ω_0} and ρ_{φ_0} by [4] and their explicit form given above differ only by a partial isometry in $\mathcal{M}$. The above proposition gives us immediately the following main theorem.

2.7 THEOREM (abstract characterization of ω-conditional expectations). *An ω-c.e. from $\mathcal{M}$ to $\mathcal{M}_0$ is the dual mapping (in the sense of Banach spaces duality) of the unique bounded linear ω_0 up to a phase extension ρ_{ω_0} of a canonical state extension ρ from $\mathcal{S}(\mathcal{M}_0)$ to $\mathcal{S}(\mathcal{M})$ relative to an $\omega_0 \in \mathcal{S}_f(\mathcal{M}_0)$.*

PROOF. We have for all $a \in \mathcal{M}$ and $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$

$$(\rho^*{}_{\omega_0}(a)) = [\rho_{\omega_0}(\varphi_0)](a) = [\varphi_0 \circ \varepsilon_{\rho(\omega_0)}](a),$$

which implies $\rho^*{}_{\omega_0} = \varepsilon_{\rho(\omega_0)}$.

2.8. PROPOSITION. $\rho_{\omega_0} = \rho_{\varphi_0}$ *if and only if* $\varepsilon_{\rho(\omega_0)} = \varepsilon_{\rho(\varphi_0)}$

PROOF. If $\varepsilon_{\rho(\varphi_0)}$, we have, for $\psi_0 \in \mathcal{S}_f(\mathcal{M}_0)\, \rho_{\omega_0}(\psi_0) = \psi_0 \circ \varepsilon_{\rho(\omega_0)} = \psi_0 \circ \varepsilon_{\rho(\varphi_0)} = \rho_{\omega_0}(\psi_0)$, or $\rho_{\omega_0} = \rho_{\omega_0}$.

Conversely, for all $a \in \mathcal{M}, \psi_0 \in \mathcal{S}_f(\mathcal{M}_0)$:

$$\psi_0(\varepsilon_{\rho(\omega_0)}(a)) = [\rho_{\omega_0}(\psi_0)](a) = [\rho_{j_0}(\psi_0)](a) =$$
$$= \psi_0(\varepsilon_{\rho(\varphi_0)}(a)), \text{ which implies } \varepsilon_{\rho(\varphi_0)} = \varepsilon_{\rho(\omega_0)}.$$

For a study of the condition $\varepsilon_{\rho(\varphi_0)} = \varepsilon_{\rho(\omega_0)}$ see [7].

PROOF OF THEOREM 2.4 (end) Let ω_0, $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$, and the canonical state extension ρ admit a bounded linear extension $\hat{\rho}\,(\mathcal{M}_0)_* \to \mathcal{M}_*$. By the unicity of the bounded linear ω_0 up to a phase extension we have $\rho_{\omega_0} = \hat{\rho} = \rho_{\varphi_0}$. By prop. 2.8 this implies $\varepsilon_{\rho(\omega_0)} = \varepsilon_{\rho(\varphi_0)}$. As this holds for all ω_0, $\varphi_0 \in \mathcal{S}_f(\mathcal{M}_0)$, $\mathcal{E}_\rho$ is a singleton.

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