MATH. SCAND. 66 (1990), 185–196

FIXED POINTS OF ACTIONS OF P-GROUPS ON PROJECTIVE VARIETIES

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Abstract.

Let G be a finite p-group acting on a complex projective variety V^n and suppose the degree of V is prime to p. Does G have a fixed point on V? We will always assume G preserves the hyperplane class in $H^2(V)$, or even that it acts projectively on the ambient projective space. In [3] we showed that if $G \cong \prod Z/p$ and in addition $n \not\equiv -1 \pmod p$ then G does have a fixed point, while for non-abelian G this is not true as shown in [4].

Both of the above papers used algebraic topology exclusively and proved fixed point theorems under certain topological assumptions. In this paper, we combine some of these methods with simple geometrical arguments in projective space to get more delicate results.

1. Introduction.

We first consider topological actions of a finite p-group G on finite dimensional spaces X with the $Z_{(p)}$-cohomology of CP^n, acting trivially on $H^2(X; Z_{(p)})$. In [1] it was shown that for $G \cong Z/p$, the fixed set $X^G = X_0 \cup \ldots \cup X_{p-1}$ with each X_i a $Z_{(p)}$ cohomology CP^{n_i} and $\sum (n_i + 1) = n + 1$.

We deduce that for any finite p-group G, $X^G \neq \emptyset$ if $n \not\equiv -1 \bmod p$, and for $G \cong Z/p^r$, Bredon's formula generalizes.

We study the action of G on $\pi_0(X^C)$ where C is a central subgroup of G isomorphic to Z/p and show, for any p-group G, G acts trivially if $\tau_G \alpha = 0$ where τ_G is the transgression in the spectral sequence of the Borel construction and $\alpha \in H^2(X; Z_{(p)})$ is the hyperplane class. For abelian G, $\tau_G \alpha = 0$ if and only if $X^G \neq \emptyset$.

All this is done in § 2.

In § 3, we recall some facts about actions on projective varieties in a cohomological setting, and deduce some technical results from § 2.

In § 4, we consider projective actions of a finite p-group G, and prove that with certain extra hypotheses, invariant subvarieties V^n with $\deg V \not\equiv 0 \bmod p$ must

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* Research partly supported by an NSF grant, and also by the Danish Natural Sciences Research Council.

Received October 10, 1987

contain points fixed under G. This happens when:

(4.5) $G \cong Z/p^r$.
(4.6) G abelian and
(i) $\dim V \not\equiv -1 \bmod p$, or
(ii) $\tau_G \alpha = 0$ or
(iv) codim V is not divisible by p, or
(v) G has a fixed point on the ambient projective space.
(4.7) G finite p-group acting on CP^m, $m < 2p - 1$, $V^n \subset CP^m$, $n < m$.
(4.12) G finite p-group, V^m an irreducible hypersurface, unless V^m is the plane perpendicular to a unique fixed point of CP^{m+1}.

We use a process of intersection with invariant hyperplanes to prove these results, or invariant planes of codimension p. The arguments break down into several cases for (4.12) (namely (4.8) and (4.10)) where we must keep track of the residue class mod p of the dimension, and use different arguments.

§ 2. Actions on $Z_{(p)}$-cohomology projective spaces.

In this section we will study a finite p-group G acting on a mod p cohomology projective space, i.e. a finite dimensional space X with $H^*(X; Z_{(p)}) \cong H^*(CP^n; Z_{(p)})$, where $Z_{(p)} = Z$ localized at p, and $G \times X \to X$ an action such that G acts trivially on $H^*(X; Z_{(p)})$ (equivalently on $H^2(X; Z_{(p)})$). We will fix a generator $\alpha \in H^2(X; Z_{(p)})$ called the hyperplane class.

Recall first the theorem of Bredon [1, VII (3.1)].

(2.1) THEOREM (Bredon). *If $G \cong Z/p$, then the fixed set $X^G = X_0 \cup \ldots \cup X_{p-1}$ with each X_i an $Z_{(p)}$-cohomology CP^{n_i} amd*

$$\sum (n_i + 1) = n + 1.$$

Further j_k: $X_k \to X$ induces surjection on $H^*(\ ; Z_{(p)})$.

We note that as in [3 § 4] we may define X_i as follows: Choose $\bar{\alpha} \in H^2(\hat{X}_G; Z_{(p)})$ such that $j^*\bar{\alpha} = \alpha$, where $\hat{X}_G = X \times_G E_G$ is the Borel construction, $j: X \to \hat{X}_G$, since we know j^* is onto where $G \cong Z/p$. Then with respect to the choice of $\bar{\alpha}$ and β generating $H^2(B_G)$ we define:

(2.2) DEFINITION. X_S is the unique component of X^G such that $\hat{\imath}_S^* \bar{\alpha} - S\beta$ is nilpotent on $X_S \times B_G \subset X^G \times B_G = (X^G)_G \xrightarrow{\hat{\imath}} \hat{X}_G$ (the inclusion) $\hat{\imath}_S = \hat{\imath} \mid (\hat{X}_S)_G$.

We will study general finite p-groups G acting on X by taking a central subgroup $C \cong Z/p$ in G and considering the action of G and G/C on $X^C \subset X$, using Bredon's Theorem. We note that G acts on $\pi_0(X^C)$, permuting components, and since $|\pi_0(X^C)| \leqq p$ and G is a p-group, either G acts trivially on $\pi_0(X^C)$ or G acts transitively on $\pi_0(X^C)$, and $|\pi_0(X^C)| = p$.

(2.3) THEOREM. *Let G be a p-group acting effectively on X. If $n \not\equiv -1 \bmod p$, then $X^G \neq \emptyset$.*

PROOF. Let $C \subset G$ be a central subgroup, $C \cong \mathbf{Z}/p$ as above. Then $X^C = X_0 \cup \ldots \cup X_{p-1}$, with $H^*(X_i; \mathbf{Z}_{(p)}) \cong H^*(\mathbf{C}P^{n_i}; \mathbf{Z}_{(p)})$ and $n + 1 = \sum_{i=0}^{p-1} (n_i + 1)$ by (2.1). If G permutes the X_i's non-trivially, then G is transitive on $(0, \ldots, p-1)$ and $n_0 = n_1 = \ldots = n_{p-1}$, so that $n + 1 = \sum (n_i + 1) = pn_0 + p$, and $n \equiv -1 \bmod p$.

If G sends each X_i into itself, then by induction on dimension, $X_i^{G/C} \neq \emptyset$ unless $n_i \equiv -1 \bmod p$, or $n_i + 1$ is divisible by p. So if $\bigcup_i X_i^{G/C} = X^G = \emptyset$ then $n + 1 = \sum (n_i + 1)$ is divisible by p.

Further, we may extend Bredon's Theorem (2.1) to cyclic groups:

(2.4) THEOREM. *Let $G \cong \mathbf{Z}/p^l$, acting on X, as above. Then $X^G = \bigcup_{i=0}^{p^l-1} X_i$, each X_i is a $\mathbf{Z}_{(p)}$-cohomology $\mathbf{C}P^{n_i}$ and*

$$\sum (n_i + 1) = n + 1.$$

PROOF. For $l = 1$ this is just Bredon's result (2.1). We proceed by induction on l, and let $C \subset G$, $C \cong \mathbf{Z}/p$. The singular set of the action is just X^C, so it follows that $H^*((\hat{X}^C)_G) \cong H^*(\hat{X}_G)$ in large dimensions. If G permutes $\pi_0(X^C)$ non-trivially, then all X_i's are homeomorphic and $H^*(X^C) \cong H^*(X_0) \otimes \Lambda$ as a $\mathbf{Z}[G]$ module, where $\Lambda = \mathbf{Z}[G/G_0]$, G_0 acting trivially on $\pi_0(X^C)$, so each X_i is a G_0-space. Then $(\hat{X}^C)_G = (\hat{X}_0)_{G_0}$. Now whenever X has

$$H^*(X) \cong \mathbf{Z}_{(p)}[\alpha]/\alpha^{n+1} \quad \text{and} \quad G \cong \mathbf{Z}/p^l$$

acting on X (trivially on $H^*(X)$), in the Cartan-Leray spectral sequence (i.e. the spectral sequence for the Borel construction), we have (with $\mathbf{Z}_{(p)}$ coefficients)

$$E_2 \cong H^*(X) \otimes H^*(B_G) \cong (\mathbf{Z}_{(p)}[\alpha]/\alpha^{n+1}) \otimes (\mathbf{Z}/p^l[\beta]).$$

Since $E_2 = 0$ in odd degrees, it follows that $E_2 = E_\infty$, and from this we deduce that

$$H^*(\hat{X}_G) \cong (\mathbf{Z}_{(p)}[\alpha]/\alpha^{n+1}) \otimes (\mathbf{Z}/p^l[\beta]).$$

Applying this to X_0, we get that:

$$H^*((\hat{X}_0)_{G_0}) \cong (\mathbf{Z}_{(p)}[\alpha_0]/\alpha_0^{n_0+1}) \otimes (\mathbf{Z}/p^{l-1}[\beta_0]),$$

while for X we get the contradicting result:

$$H^*(\hat{X}_G) \cong (\mathbf{Z}_{(p)}[\alpha]/\alpha^{n+1}) \otimes (\mathbf{Z}/p^l[\beta]),$$

where α (resp. α_0) generates $H^2(X)$ (resp. $H^2(X_0)$) and β (resp. β_0) generates $H^2(B_G)$ (resp. $H^2(B_{G_0})$). Hence G acts trivially on $\pi_0(X^C)$ (this could also be deduced from the Lefschetz Fixed Point Theorem, if X were a finite complex), and each component X_i of X^C is a $G/C \cong \mathbf{Z}/p^{l-1}$ space. Since X_i is a $\mathbf{Z}_{(p)}$-cohomology projective space and the inclusion $X_i \to X$ maps $H^*(\ ;\mathbf{Z}_{(p)})$ surjectively by (2.1), it follows that G (hence G/C acts trivially on $H^*(X_i;\mathbf{Z}_{(p)})$ and X_i satisfies the hypothesis of the theorem. Hence $X_i^{G/C} = \bigcup (X_i)_j$ etc., and the theorem follows.

Let G be a general finite p-group $C \cong \mathbf{Z}/p$ a central subgroup. Then G acts on $\pi_0(X^C)$, i.e. permuting components of X^C. Let τ_G: $H^2(X;\mathbf{Z}_{(p)}) \to H^3(B_G;\mathbf{Z}_{(p)})$ be the "transgression", i.e. d_3 in the spectral sequence for $\hat{X}_G$ over B_G, where it is easy to see $E_2 = E_3$.

(2.5) LEMMA. *If $G = C \times L$, $C \cong \mathbf{Z}/p$ and $L \cong \mathbf{Z}/p^l$, acting effectively on X, as above, then the following statements are equivalent*:
(i) $\tau_G\alpha = 0$
(ii) *G acts trivially on $\pi_0(X^C)$*
(iii) $X^G \neq \emptyset$.

PROOF. Clearly (iii) implies (i) because (iii) implies $\hat{X}_G \to B_G$ has a section so that $\tau_G \equiv 0$. Also if $X^G \neq \emptyset$, since $X^G \subset X^C$, some component of X^C is invariant, so that all components are invariant, since G acts either trivially or transitively on $\pi_0(X^C)$, so (iii) implies (ii).

If (ii), then G/C acts on each component of X^C, which are again $\mathbf{Z}_{(p)}$-cohomology $\mathbf{CP}^m$'s by (2.1), and G/C is cyclic so that G/C has a fixed point by (2.4) on X^C, so $X^G \neq \emptyset$, and (ii) implies (iii).

We complete the proof by showing (i) implies (iii). Let $H = C \times L_1 \subset G$, where $L_1 \cong \mathbf{Z}/p$. Since $\tau_G\alpha = 0$, it follows that $\tau_H\alpha = i^*\tau_G\alpha = 0$ so that $X^H \neq \emptyset$ since H is elementary abelian (see [2] or [3]). Hence C preserves components of X^{L_1}. Since L is cyclic, $X^L \neq \emptyset$ by (2.4), so L preserves components of X^{L_1}. Hence G preserves components of X^{L_1}, and let Y be such a component, so that $K = G/L_1$ acts effectively on Y, and Y is again a $\mathbf{Z}_{(p)}$ cohomology complex projective space by (2.1). By (2.1), j: $Y \to X$ induces surjection on $H^*(\ ;\mathbf{Z}_{(p)})$, so that G acts trivially on $H^*(Y;\mathbf{Z}_{(p)})$, since G acts trivially on $H^*(X;\mathbf{Z}_{(p)})$ and hence K acts trivially on $H^*(Y;\mathbf{Z}_{(p)})$.

Now if $\alpha_0 = j^*\alpha \in H^2(Y;F_p)$, then $\tau_G\alpha_0 = j^*\tau_G\alpha = 0$ in the spectral sequence for $\hat{Y}_G$. If we can show that $\tau_K\alpha_0 = 0$ in the spectral sequence for $\hat{Y}_K$ over B_K, the result (iii) will follow by induction on $|G|$ (or $\dim X$).

Let r: $G \to K$ be the quotient map so that $r^*\tau_K\alpha_0 = \tau_G\alpha_0 = 0$. But by (2.6) below r^* is injective, completing the proof of (2.5).

(2.6). PROPOSITION *Let $f: G_1 \to G_2$ be a surjection of finite abelian groups. Then $f^*: H^3(G_2; \mathsf{Z}) \to H^3(G_1; \mathsf{Z})$ is an injection.*

PROOF. A map of finite abelian groups is an injection if and only if it is an injection on the elements of prime order. From the exact homology sequence

$$\ldots \longrightarrow H^2(G_2; \mathsf{Z}/p) \xrightarrow{\delta} H^3(G_2; \mathsf{Z}) \xrightarrow{xp} H^3(G_2; \mathsf{Z}) \longrightarrow \ldots$$

associated to the coefficient sequence $0 \longrightarrow \mathsf{Z} \xrightarrow{p} \mathsf{Z} \longrightarrow \mathsf{Z}/p \longrightarrow 0$, it suffices to prove $f^* | \mathrm{im}\,\delta$ is injective. Now $H^*(G_i; \mathsf{Z}/p) = \Lambda(V_i) \otimes \mathsf{Z}/p[\bar{V}_i]$ where $\bar{V}_i$ is the mod p reduction of $H^2(G; \mathsf{Z})$, so that $\delta \bar{V}_i = 0$, and δ: $\Lambda^2 V_i \to \mathrm{im}\,\delta$ is an isomorphism. Now a surjection of groups induces a surjection on H_1 and $H_1(; \mathsf{Z}/p)$, so f^*: $V_2 \to V_1$ and $f^*: \Lambda^2 V_2 \to \Lambda^2 V_1$ are injections. Hence $f^* | \mathrm{im}\,\delta$ is an injection, so that $f^*: H^3(G_2; \mathsf{Z}) \to H^3(G_1; \mathsf{Z})$ is an injection.

Note that (2.6) fails for the simplest non-abelian examples, such as 3×3 matrices over F_p which are the identity on the diagonal and below.

From (2.5) we immediately deduce:

(2.7) PROPOSITION. *Let G be a finite p-group (not necessarily abelian) $C \subset G$, a central subgroup of order p, and suppose G acts effectively on X, a $\mathsf{Z}_{(p)}$-cohomology complex projective space, trivially on $H^*(X; \mathsf{Z}_{(p)})$. If $\tau_G \alpha = 0$ (α the hyperplane class in $H^2(X; \mathsf{Z}_{(p)})$ then G preserves components of X^C,* i.e. *G acts trivially on $\pi_0(X^C)$.*

PROOF. The proof goes by reducing to the commutative case and using (2.5). Let $g \in G$ permute components of X^C non-trivially and let K be the subgroup generated by g and C, so that we have a central extension:

$$0 \to C \to K \to L \to 0$$

and L is cyclic $\subset G/C$ so that $L \cong \mathsf{Z}/p^l$. Since K permutes components of X^C transitively, $X^K = \emptyset$. Now K must split as $C \times L$, for otherwise K would be cyclic and therefore would have fixed points, by (2.4).

Since $K \cong C \times L$, and K acts non-trivially on X^C, applying (2.5) we get that $\tau_K \alpha = \mathrm{j}^* \tau_G \alpha \neq 0$ so that $\tau_G \alpha \neq 0$ and (2.7) follows.

Returning to the case of commutative p-groups, we have:

(2.8) THEOREM. *Let G be a commutative p-group acting on a $\mathsf{Z}_{(p)}$-cohomology complex projective space X. Then $X^G \neq \emptyset$. if and only if $\tau_G \alpha = 0$ in the spectral sequence for $\hat{X}_G$ over B_G with $\mathsf{Z}_{(p)}$* coefficients.

We first prove:

(2.9) LEMMA. *If $X_0 \subset X$ is invariant under G and $G/H = K$ acts effectively on X_0, then $\tau_G \alpha = 0$ in the spectral sequence for $\hat{X}_G$ implies $\tau_K \alpha_0 = 0$ in the spectral sequence for $(\hat{X}_0)_K$, where $\alpha_0 = i^* \alpha \in H^2(X_0, \mathsf{Z}_{(p)})$,* i: *$X_0 \subset X$.*

PROOF. Since X_0 is a G-space $\tau_G\alpha = 0$ implies $\tau_G\alpha_0 = 0$ in the spectral sequence for $(\hat{X}_0)_G$. Then we have

$$\begin{array}{ccc} (\hat{X}_0)_G & \longrightarrow & (\hat{X}_0)_K \\ \downarrow & & \downarrow \\ B_G & \xrightarrow{r} & B_K \end{array}$$

and $r^*\tau_K\alpha_0 = \tau_G\alpha_0 = 0$, and r^* is injective by (2.6).

PROOF OF (2.8). By (2.7), $\tau_G\alpha = 0$ implies that G preserves components of X^C, where $C \cong Z/p \subset G$. Then G/C acts on a component X_0 and $\tau_K\alpha_0 = 0$ by (2.9). By induction on $|G|$, $X_0^K \neq \emptyset$, and $X_0^K \subset X^G$, where K is the quotient of G/C which acts effectively on X_0.

If $X^G \neq \emptyset$, then $\hat{X}_G \to B_G$ has a section, so that $\tau_G \equiv 0$.

(2.10) COROLLARY. *If G is an abelian p-group acting on $X \supset X_0$ with X_0 invariant, such that (X, X_0) is $Z_{(p)}$ cohomology equivalent to* (CP^m, CP^n) *and if $X^G \neq \emptyset$ then $X_0^G \neq \emptyset$.*

PROOF. Apply (2.8) and (2.9).

(2.11) THEOREM. *Let G be a finite p-group (not necessarily abelian) acting on X, a $Z_{(p)}$-cohomology complex projective n-space.*

(a) *If $n < p - 1$, $X^G \neq \emptyset$*

(b) *If $n = p - 1$, $X^G \neq \emptyset$ if and only if $\tau_G\alpha = 0$*, i.e. *G fixes $\pi_0(X^C)$, $C \cong Z/p$ a central subgroup of G.*

Further $X^G = \bigcup_k Y_k$, where Y_k is a $Z_{(p)}$-cohomology complex projective n_k space and $n + 1 = \sum (n_k + 1)$.

PROOF. If $n < p - 1$ it follows that $\tau\alpha = 0$, so it remains to prove (b) with $n \leqq p - 1$. *By* (2.1), *a non-empty component X_i* of X^C is a $Z_{(p)}$-cohomology CP^m, $0 \leqq m < n \leqq p - 1$, so by induction G/C has fixed points on X_i, and (2.11) follows by induction.

§ 3. Actions on $Z_{(p)}$-cohomology projective varieties of degree prime to p.

Recall that in [3], we showed that for an elementary abelian p-group G acts on a complex projective variety V^n preserving the hyperplane class $\alpha \in H^2(V)$, with $H^1(V) = 0$, degree V prime to p, and $n \not\equiv -1 \bmod p$, then $V^G \neq \emptyset$. In this section we will continue to study this situation for more complicated p-groups G.

We shall consider an algebraic topological abstraction of a complex projective variety in a complex projective space.

Let X be a $Z_{(p)}$-cohomology complex projective space of complex dimension $n + k$ and let i: $V^n \subset X$ be the inclusion of a polarized $2n$-$Z_{(p)}$ homologynear-manifold (following notation of [3]) which means the following: V is compact, $V - S$ is an oriented connected $2n$-manifold, $\dim S \leqq 2n - 2$, and $\alpha \in H^2(V)$ such that $\alpha^n[V] \equiv \deg V > 0$, where $[V]$ generates $H_{2n}(V; \mathbf{Z}_{(p)}) \cong \mathbf{Z}_{(p)}$. We suppose that $i^*(\beta) = \alpha$ where $\beta \in H^2(X; \mathbf{Z}_{(p)})$ is a generator.

Let G be a p-group acting on X such that $g^*\beta = \beta$, for all $g \in G$, and suppose $V \subset X$ is a G-subspace, and $gS \subset S$ for all $g \in G$.

(3.1) DEFINITION. We call the above situation "G-$\mathbf{Z}_{(p)}$-cohomology complex projective variety V".

An actual projective action on $\mathbf{C}P^{n+k}$ leaving a complex projective variety V invariant is an example.

In [3] it was shown:

(3.2) LEMMA. *In the spectral sequence for $\hat{V}_G$ over B_G with $\mathbf{Z}_{(p)}$ coefficients, if* degree $V \not\equiv 0 \bmod p$, *then $E_2^{3,0} \cong E_3^{3,0} \cong H^3(B_G; \mathbf{Z}_{(p)})$ and α is transgressive, so $d_3\alpha \in H^3(B_G; \mathbf{Z}_{(p)})$.*

PROOF. If $x \in E_2^{1,1}$ so that $d_2 x \in E_2^{3,0}$, then $d_2(\alpha^n x) = \alpha^n(d_2 x)$, but $\alpha^n x \in E_2^{1,2n+1} = 0$ and α^n: $E_2^{3,0} \to E_2^{3,2n} \cong H^3(B_G; H^{2n}(V))$ is an isomorphism.

(3.3) THEOREM. *Suppose G permutes $\pi_0(X^C)$ non-trivially ($C \cong \mathbf{Z}/p \subset G$ a central subgroup), (so that $n + k \equiv -1 \bmod p$). If $V^n \subset X$ is a G-$\mathbf{Z}_{(p)}$-cohomology complex subvariety of X and* deg $V \not\equiv 0 \bmod p$ *then $n \equiv -1 \bmod p$, so that codimension V in X is divisible by p.*

PROOF. If G acts non-trivially on $\pi_0(X^C)$, then $\tau_G\beta \neq 0$ by (2.7). Hence $d_3\alpha = \tau_G\alpha \neq 0$ by (3.2), and $d_3\alpha^{n+1} = (n+1)\alpha^n(d_3\alpha) = 0$ so that $(n+1) \equiv 0 \bmod p$.

Now we recall [3, (1.8)].

(3.4) THEOREM. *Let V^n be a G-polarized $\mathbf{Z}_{(p)}$-homology $2n$-near-manifold with $\alpha \in H^2(V; \mathbf{Z}_{(p)})$ the polarization. Suppose $G \cong \mathbf{Z}/p$ and* $\deg V = \alpha^n[V] \not\equiv 0 \bmod p$. *Then $V^G = V_0 \cup \ldots \cup V_{p-1}$ where V_i = largest union of components such that $\tilde{\alpha} - i\gamma$ is nilpotent in $H^*((\hat{V}_i)_G, \mathbf{Z}_{(p)})$, for γ a generator of $H^2(B_G; \mathbf{Z}_{(p)})$, $(\hat{V}_i)_G = V_i \times B_G$ the Borel construction, and $\tilde{\alpha} \in H^2(\hat{V}_G; \mathbf{Z}_{(p)})$ is an element such that $j^*\tilde{\alpha} = \alpha, j$: $V \to \hat{V}_G$. If i_k: $V_k \subset V$, and $\alpha_k = i_k^*(\alpha)$, then $\alpha_k^{h_k} \neq 0$ where $\sum_{k=0}^{p-1}(h_k + 1) \geqq n + 1$, so that* $\dim V_k \geqq 2h_k$.

We note that for (X^{n+k}, V^n) a G-$\mathbf{Z}_{(p)}$ cohomology complex projective variety, $G \cong \mathbf{Z}/p$ and $\deg V \not\equiv 0 \bmod p$, $X^G = X_0 \cup \ldots \cup X_{p-1}$ by (2.1) and $V_i = V \cap X_i$.

§ 4. Actions on complex projective varieties.

We will combine results of previous sections with a little bit of the geometry of projective space to prove stronger theorems.

We adapt a standard method to the equivariant context: intersection with an invariant hyperplane.

Since we use G-hyperplanes we will not be able to perturb them, so that we cannot ensure that they are generic for the given G-variety.

(4.1) LEMMA. *Let $V^n \subset \mathrm{CP}^m$ be a variety, and $H^{m-1} \subset \mathrm{CP}^m$ a hyperplane. Then* $\deg(V \cap H)$ *divides* $\deg V$.

PROOF. This is a standard elementary fact for generic H, taking a generic $(m-n)$-hyperplane $H' \subset H$, and counting $H' \cap V = H' \cap (H \cap V)$.

We now proceed as follows: In a small tubular neighborhood T of H, we can find a generic hyperplane $H_0 \subset T$, so that $\deg(V \cap H_0) = \deg V$. Now H_0 is homologous to H in T, and the retraction of T into H will carry $V_0 \cap H$ inside a regular neighborhood S of $V \cap H$ in H. Now $i_0^*(\alpha_0^{m-1})[V \cap H_0] = \deg(V \cap H_0)$, i_0: $V \cap H_0 \subset H_0$, $\alpha_0 \in H^2(H_0)$ the hyperplane class, and $\alpha_0 = j^* r^* \alpha$, $\alpha \in H^2(H)$, j: $H_0 \subset T$, the inclusion r: $T \to H$ the retraction. It follows that the inclusion i_0': $V \cap H_0 \subset \mathrm{CP}^m$ factors through the inclusion of $S \subset H \subset CP^m$ and $S \subset \mathrm{CP}^m$ has homology image $\deg(V \cap H)$(generator). It follows that $\deg(V \cap H)$ divides $\deg(V \cap H_0) = \deg V$.

We note an analogous fact for projections, which we will not use here:

(4.2) Let H^p and H_2^q be complementary planes in CP^m, $m = p + q + 1$, i.e. $H_1 \cap H_2 = \emptyset$, and let ρ: $\mathrm{CP}^m - H_1 \to H_2$ be the projection. If $V \cap H_1 = \emptyset$, then $\deg(\rho(V))$ divides $\deg(V)$.

PROOF. The argument is similar to (4.1), for the inclusion $V \subset \mathrm{CP}^m$ factors up to homotopy through the inclusion of $\rho(V) \subset \mathrm{CP}^m$.

(4.3) LEMMA. *Let G be a finite p-group $\subset \mathrm{PU}(m+1)$, V^n a G-subvariety of* CP^m *with* $\deg V \not\equiv 0 \bmod p$, *and let H^{m-t} be a G plane $\subset \mathrm{CP}^m$. If* $\dim V \cap H = n - t$, *then $V \cap H$ has an irreducible component V' with $GV' = V'$ and* $\deg V' \not\equiv 0 \bmod p$.

PROOF. By (4.1), $\deg V \cap H \not\equiv 0 \bmod p$. For each component V_i of $V \cap H$, such that $GV_i \not\subset V_i$ then GV_i has (p^t) components $t \geqq 1$. If no component were invariant, $\deg(V \cap H) = \sum_i \deg(GV_i)$ would be divisible by p.

We note the following obvious 'G-duality'.

(4.4) PROPOSITION. *Let G be a closed subgroup of $PU(m+1)$, and let $H \subset \mathrm{CP}^m$ be a G-invariant plane. Then the complementary plane $H^\perp \subset \mathrm{CP}^m$ is G-invariant.*

(4.5) THEOREM. *Let* $G \cong \mathbf{Z}/p^r \subset \mathrm{PU}(m+1)$, V^n *a G-subvariety of* $\mathbf{C}P^m$ *with* $\deg V \not\equiv 0 \bmod p$. *Then* $V^G \neq \emptyset$.

PROOF. By (2.4), there exist fixed points $x_0 \in \mathbf{C}P^m$, $Gx_0 = x_0$. Then $H^{m-1} = (x_0^\perp)$ is G-invariant, so $V \cap H$ is a G-subvariety of H^{m-1}, and either $V \subset H$ so $V = V \cap H$ or $\dim V \cap H = (\dim) V) - 1$ and there is a G invariant component V_1 of $V \cap H$ with $\deg V_1 \not\equiv 0 \bmod p$, by (4.3) and (4.1). The proof is completed by induction on dimension and codimension, where codimension 0 is (2.4) and dimension 0 is the usual counting argument. Namely if S is a finite G-set, and $|G| = p^t$, and if $S^G = \emptyset$ then $|S| = \sum |\text{orbits}|$ is divisible by p. But a variety V of dimension 0 is a finite set, and $\deg V = |V|$.

(4.6) THEOREM. *Let G be an abelian p-group* $\subset \mathrm{PU}(m+1)$ *and suppose* $V^n \subset \mathbf{C}P^m$ *is a G-invariant subvariety with* $\deg V \not\equiv 0 \bmod p$. *Then under any of the following conditions* $V^G \neq \emptyset$:

(i) $n \not\equiv -1 \bmod p$.

(ii) $\tau_G \alpha = 0$, $\alpha \in H^2(V)$ *the hyperplane class,* τ_G: $H^2(V) \to H^3(B_G)$ *the transgression.*

(iii) $\tau_G \beta = 0$, $\beta \in H^2(\mathbf{C}P^m)$ *the hyperplane class.*

(iv) $m - n \not\equiv 0 \bmod p$.

(v) *G has a fixed point in* $\mathbf{C}P^m$.

PROOF. Conditions (i), (ii), (iii) and (iv) all imply (v), (see (2.8)), so it suffices to show (v) implies $V^G \neq \emptyset$. If $x_0 \in \mathbf{C}P^m$ is fixed by G, then $H = (x_0^\perp)$ is a G-hyperplane. By (4.1) and (4.3) we get $V_1 \subset V \cap H$ with $\deg V_1 \not\equiv 0 \bmod p$ and by (2.10), $H^G \neq \emptyset$, so the proof goes by induction.

(4.7) THEOREM. *Let G be a p-group* $\subset \mathrm{PU}(m+1)$ *with* $0 < m < 2p-1$, *and let* $V^n \subset \mathbf{C}P^m$, $n < m$, *be a G-invariant subvariety with* $\deg V \not\equiv 0 \bmod p$. *Then* $V^G \neq \emptyset$

PROOF. Since $m < 2p - 1$, if $m \neq p-1$, G has a fixed point on $\mathbf{C}P^m$ by (2.3). If $m = p - 1$, $\tau_G x = 0$ by (3.3), so that G has a fixed point on $\mathbf{C}P^m$ by (2.11). Let x_0 be a fixed point, $H = (x_0^\perp)$. There is a G-invariant component V_1 of $V \cap H$ with degree $V_1 \not\equiv 0 \bmod p$ by (4.3), and we proceed by induction on m.

(4.8) THEOREM. *Let G be a finite p-group in* $\mathrm{PU}(m+2)$, V^m *a G-invariant hypersurface in* $\mathbf{C}P^{m+1}$ *with degree* $V \not\equiv 0 \bmod p$ *and suppose* $m \not\equiv -2$ *or* $-1 \bmod p$. *Then* $V^G \neq \emptyset$.

PROOF. Let C be a central $\mathbf{Z}/p \subset G$ so that by (2.1) $(\mathbf{C}P^{m+1})^C = X_0^{m_0} \cup \ldots \cup X_{p-1}^{m_{p-1}}$ with $m + 2 = \sum_{i=0}^{p-1} (m_i + 1)$. By (3.4) $V^C = V_0^{n_0} \cup \cdots \cup V_{p-1}^{n_{p-1}}$, with $m + 1 \leq \sum_{i=0}^{p-1} (n_i + 1)$ and each $V_i \subset X_i$. It follows that for at most one i, (say

$i = 0$), $n_0 \leqq m_0$, while $n_i = m_i$ for $i > 0$, so that $X_i = V_i$, $i > 0$. By (3.3), each X_i is G-invariant, i.e. G acts on X_i, each i, so $V_i = V \cap X_i$ is G invariant. If $X_i^G \neq \emptyset$, we are done so that we may assume $X_i^G = \emptyset$ for $i > 0$ so that $p \mid m_i + 1$ for $i > 0$. Hence $m \equiv m_0 \bmod p$, by (2.1), Bredon's Formula. Since $m + 1 \not\equiv -1 \bmod p$, $X_0^G \neq \emptyset$, by (2.3) so if $V_0 = X_0$ we are done, i.e. if $\dim V_0 = \dim X_0$. If $\dim V_0 < m_0 = \dim X_0$, then $\dim V_0 = m_0 - 1$, $m_0 \equiv m \bmod p$, and by (4.3) V_0 has irreducible component V' invariant under G with $\deg V' \not\equiv 0 \bmod p$. The result follows by induction on $|G|$ or m.

(4.9) LEMMA. *Let $G \subset \mathrm{PU}(n+1)$ act projectively on $X = \mathbf{C}P^n$, G a p-group $G_0 \subset G$ such that $G/G_0 \cong \mathbf{Z}/p$ and suppose $X^{G_0} \neq \emptyset$ and $X^G = \emptyset$. Then for any point $x_0 \in X^{G_0}$, Gx_0 consists of p orthogonal points.*

PROOF. The components of X^{G_0} are again complex projective spaces, so if $X^G = \emptyset$, then G/G_0 must freely permute components of X^{G_0}, for otherwise $G/G_0 \cong \mathbf{Z}/p$ would act on a single component X_i, and we would have $X_i^{\mathbf{Z}/p} = X_i^G \neq \emptyset$, by (2.1).

But different components of X^{G_0} are orthogonal (by induction on $|G_0|$), so that for any $x_0 \in X^{G_0}$, Gx_0 consists of p orthogonal points, each in different component of X^{G_0}.

(4.10) THEOREM. *Let G be a p-group $\subset \mathrm{PU}(m+2)$ leaving an irreducible $V^m \subset \mathbf{C}P^{m+1}$ $(= X^{m+1})$ invariant, with $\deg V \not\equiv 0 \bmod p$ and suppose p divides $m + 2$. Then $V^G \neq \emptyset$.*

PROOF. We proceed by induction on $|G|$ and on m, the cases $|G| = p$ and $m < 2p - 2$ being proved.

Let C be a central $\mathbf{Z}/p \subset G$ and let $C \subset G_0 \subset G$ where G_0 has index p *in* G.

Let $V^C = V_0 \cup \ldots \cup V_{p-1}$, $X^C = X_0^{m_0} \cup \cdots \cup X_{p-1}^{m_{p-1}}$, $V_i \subset X_i$ and $V_i = X_i$ for $i > 0$, as before. We consider two cases:

(a) $\dim V_0 = m_0 - 1$ so $V_0 \neq X_0$, and

(b) $V_0 = X_0$, so $V^C = X^C$.

By (3.3) each X_i is G invariant.

We may assume $p \mid m_i + 1$ for each $i > 0$, for otherwise $X_i^G \neq \emptyset$ for some $i > 0$, and $V_i = X_i$, so $V^G \neq \emptyset$.

In case (a), $\deg V_0 \not\equiv 0 \bmod p$, so V_0 has a G irreducible component with degree prime to p, *and* $p \mid (m_0 + 1)$, since $p \mid (m + 2)$, $p \mid (m_i + 1)$ for $i > 1$, and $\sum (m_i + 1) = m + 2$. Hence $V_0^G \neq \emptyset$ by induction.

Now consider case (b) so $V^C = X^C$ and assume $X_0^{G_0} \neq \emptyset$, $X^G = \emptyset$.

We would now like to make an inductive argument by intersecting V with a codimension p plane invariant under G. The difficulty in the argument is to be sure that the intersection has codimension p, rather than lower codimension.

Let $x_0 \in X^{G_0}$ so that $Gx_0 = \{x_0, gx_0, \ldots, g^{p-1}x_0\}, g \notin G_0$, is an orthogonal orbit (using (4.9)), and let $H = (x_0)^\perp$, so $G_0H = H$, $G_0g^iH = g^iH$ for each i and G permutes the g^iH's transitively. Then $K = \bigcap_{i=0}^{p-1} g^iH$ has codimension p in X and is a G-invariant complex projective subspace. It remains to show that $V \cap K$ has codimension 1 in K so that we may proceed by induction on dimension.

Let $H_0 = H$, $H_{i+1} = H_i \cap g^{i+1}H = \bigcap_{j=0}^{i+1} g^jH$, so that $K = \bigcap_{i=0}^{p-1} H_i = H_{p-1}$ and $X \supset H_0 \supset H_1 \supset \ldots \supset H_{p-2} \supset K$ is a descending sequence of codimension 1 hyperplanes. Since $H = x_0^\perp$ and $Gx_0 = \{x_0, gx_0, \ldots, g^{p-1}x_0\} \subset X_0$ is an orthogonal orbit it follows that each H_i meets X_0 transversally.

(4.11) LEMMA. *Suppose V^m is irreducible in CP^{m+1} and $V^m \supset A^k$, a projective subspace. Let $H^m \subset CP^{m+1}$ be a projective hyperplane and suppose H meets A transversally, so* $\dim A \cap H = k - 1$. *Then every irreducible component of $V \cap H$ contains $A \cap H$.*

PROOF. Since V is irreducible and H does not contain A, it follows that $H \neq V$ and $H \not\supset V$. Hence $V \cap H$ has dimension $m - 1$, $V \cap H \subset H^m$. Hence any irreducible component U of $V \cap H$ has dimension $m - 1$. It follows that $U \cap A$ has codimension at most 1 in A and $U \cap A \subset A \cap H$, which has codimension one in A, and $A \cap H$ is a projective space so $U \cap A = A \cap H$ since $\dim U \cap A \geq \dim A \cap H$ and $A \cap H$ is irreducible. This proves the lemma.

Since each H_i meets X_0 transversally and $V \supset X_0$, it follows by induction, using (4.11), that each irreducible component of $V \cap H_i$ contains $X_0 \cap H_i$. Since H_{i+1} does not contain $X_0 \cap H_i$, H_{i+1} cannot be a component of $V \cap H_i$, so $V \cap H_{i+1}$ has codimension 1 in H_{i+1}. Hence $V \cap K$ has codimension 1 in K, and (4.10) follows by induction on dimension.

Now we can derive the general result on hypersurfaces.

(4.12) THEOREM. *Let G be a finite p-group $\subset$ PU$(m + 1)$, leaving an irreducible variety $V^m \subset CP^{m+1}$ invariant, with* degree $V \not\equiv 0 \bmod p$. *Then either $V^G \neq \emptyset$ or $X^G = \{x_0\}$ and $V = (x_0)^\perp$.*

PROOF. By (4.8) and (4.10), it remains to consider the case where $m \equiv -1 \bmod p$, so that $X^G \neq \emptyset$ by (2.3). Let $x_0 \in X^G$ and let $H = (x_0)^\perp$. If $V \neq H$, then $V \cap H$ has an irreducible G-component with degree prime to p, $\dim V \cap H = m - 1$ and (4.10) applies.

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