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ON THE NUMBER OF SEMIGROUPS OF NATURAL NUMBERS

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Introduction.

This paper consists of two parts. In part I two problems concerning subsemigroups S of $\mathbb{N}$ are treated:

- How many S with a given Frobenius number g (or a given conductor $c = g + 1$) are there?
- How many *maximal* S with a given Frobenius number g are there?

(For g odd, the maximal semigroups are precisely he symmetric semigroups.) The first question was raised by H. S. Wilf in [3]. The second question is considered in [2, proposition 5], where it is proved that (for g odd)

$$\#\,\text{symmetric } S \geq 2^{[g/8]}.$$

The answers given in the theorem below are, roughly: $C \cdot 2^{g/2}$ and $C' \cdot 2^{g/6}$, respectively, where C and C' vary within finite bounds. For the answer of the second question it was necessary to investigate a question of some interest in itself, namely:

- How many subsets X of $\{1,2,\ldots,n\}$, such that there are at most q different sums of pairs of elements from X, are there?

We did not find any treatment of this question in the literature. A sufficiently good answer for our applications is given as a "main lemma". Part II of this paper is devoted to the proof of that lemma. (It is quite independent of part I.)

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I. Semigroups.

I.0. Terminology. In this part, mainly the terminology of [2] is adopted:
N denotes the set of natural numbers (including 0).

A *semigroup* S will always denote a submonoid of $(\mathsf{N}, +)$, i.e. a subset of N which contains 0 and is closed under addition. Furthermore (discarding the trivial semigroups $\{0\}$ and N, and employing isomorphisms of monoid) we always assume that $\mathsf{N}\setminus S\,\{= n \in \mathsf{N}\colon n \notin S\})$ is a non-empty finite set.

If S is a semigroup, then the *Frobenius number* $g(S) := \max \mathsf{N}\setminus S$, while the *minimal generator* $m(S) := \min S\setminus\{0\}$.

If X is any subset of N, then $S\langle X\rangle$ is the minimal semigroup containing X, i.e. the set of all linear combinations of elements in X, with non-negative integer coefficients; while $2X := \{x + y\colon x, y \in X\}$.

If X is any finite set, then $|X|$ = number of elements in X.

If a is a real number, then $[a]$ = integer part of a; and if $a > 0$, $\ln a$ and $\lg a$ denote the logarithms of a with bases e and 2, respectively.

$\mathscr{S}_g := \{$semigroups S with $g(S) = g\}$, for $g = 1, 2, \ldots$. $\mathscr{S}_g$ is partially ordered by (set-theoretical) inclusion; let $\mathscr{M}_g$ be the set of the maximal elements in $\mathscr{S}_g$.

In [2] a number of equivalent conditions on elements $S \in \mathscr{S}_g$ are given, which are shown to be equivalent to the condition "S be maximal". They are slightly dependent on the parity of g. In particular, for g odd it is shown that $S \in \mathscr{S}_g$ is maximal iff S is *symmetric* in the sense that for any integer i we have that $i \in S$ iff $\mathrm{g} - \mathrm{i} \notin \mathrm{S}$. (There are some more precise conditions on symmetric semigroups than on "even-case" maximal ones, like this: S is symmetric iff the semigroup ring $k[S]$ is Gorenstein (k some field). These conditions are of no concern in this article.)

Below we shall concentrate on achieving estimates for the quantities $f(g) := |\mathscr{S}_g|$ and $e(g) := |\mathscr{M}_g|$.

I.1. The main results. The main object of this article is to prove.

Theorem. *With the terminology above,*

$$\text{(i)} \qquad 0 < \liminf_{g\to\infty} 2^{-g/2}\,|\mathscr{S}_g| < \limsup_{g\to\infty} 2^{-g/2}\,|\mathscr{S}_g| < \infty, \textit{ and}$$

$$\text{(ii)} \qquad 0 < \liminf_{g\to\infty} 2^{-g/6}\,|\mathscr{M}_g| < \limsup_{g\to\infty} 2^{-g/6}\,|\mathscr{M}_g| < \infty.$$

In fact, we will get stronger results, like

$$\text{(1)} \qquad 2^{[(g-1)/2]} \le |\mathscr{S}_g| < 4 \cdot 2^{[(g-1)/2]} \text{ for all positive integers } g, \text{ and}$$

PROPOSITION 1. $\lim_{g\,\mathrm{odd}} 2^{-g/2}|\mathscr{S}_g|$ *and* $\lim_{g\,\mathrm{even}} 2^{-g/2}|\mathscr{S}_g|$ *exist, and likewise* $\lim_{g:g\equiv i(\mathrm{mod}\ 6)} 2^{-g/6}|\mathscr{M}_g|$ *exists for* $i = 0, 1, 2, 3, 4, 5$.

The values of these limits are briefly discussed in section I.3. Furthermore, "almost all" S in $\mathscr{S}_g$ (in $\mathscr{M}_g$) have their minimal generators $m(S)$ "almost equal" to $g/2$ (to $g/3$, respectively), in the following strong sense:

PROPOSITION 2. *For any real number* $\varepsilon > 0$ *there is an integer* n_0 *such that for every positive integer* g *we have*

$$|\{S\in\mathscr{S}_g: |m(S) - g/2| > n_0\}| < \varepsilon\cdot 2^{g/2}, \textit{ and} \tag{2}$$

$$|\{S\in\mathscr{M}_g: |m(S) - g/3| > n_0\}| < \varepsilon\cdot 2^{g/6}. \tag{3}$$

In the next section, the theorem and the propositions are proved, except for the "lim inf $\neq$ lim sup"-parts of the theorem, the proof of which is postponed to I.3. The proofs are mainly by subdivision into different cases, depending on $m(S)$.

I.2. PROOFS. It will turn out that the sets $A := \{n\in \mathsf{N}: g/2 < n < g\}$ and $B := \{n\in\mathsf{N}: g/3 < n < g/2\}$ are of fundamental interest for the counting arguments, and we shall compute in terms of 2^a and 2^b, where $a := |A|$ and $b := |B|$, rather than in terms of $2^{g/2}$ and $2^{g/6}$. (A, B, a and b are functions of g, strictly speaking.) Clearly $a = [(g-1)/2]$ and $b = [(g-1)/2] - [g/3]$, and for all g we have $|a - g/2| \leq 1$ and $|b - g/6| \leq 1$; the exact differences depend only on the residue classes of g modulo 2 and modulo 6, respectively. For any positive integers g and m, let $\mathscr{S}_{g,m} := \{S\in\mathscr{S}_g: m(S) = m\}$ and $\mathscr{M}_{g,m} := \{S\in\mathscr{M}_g: m(S) = m\}$. Furthermore, let $f(g,m) := |\mathscr{S}_{g,m}|$ and let $e(g,m) := |\mathscr{M}_{g,m}|$. Clearly,

$$\sum_m f(g,m) = f(g) \quad \text{and} \tag{4}$$

$$\sum_m e(g,m) = e(g). \tag{5}$$

LEMMA 1.

(i) $f(g,m) = |\{X \subseteq \{m, m+1,\ldots,g-1\}: g\notin S\langle X\rangle \,\&\, m\in X\}|$ *for* $m < g$.

(ii) $e(g,m) = |\{X \subseteq \{m,m+1,\ldots,[(g-1)/2]\}: g\notin S\langle X\rangle \,\&\, m\in X\}|$ *for* $m < g/2$.

(iii) $\sum_{m\geq g} f(g,m) = f(g,g+1) = 1$.

(iv) $\sum_{m\geq [(g+1)/2]} e(g,m) = e(g,[(g+2)/2]) = 1$.

PROOF. If $S \in \mathscr{S}_{g,m}$ and $S \neq \{0, g+1, g+2, g+3, \ldots\}$, then $S \cap \{1, 2, \ldots, g-1\}$ is a set X fulfilling the conditions in (i). On the other hand, if X is such a set, then $\{0\} \cup X \cup \{g+1, g+2, \ldots\} \in \mathscr{S}_{g,m}$. (i) and (iii) follow.

(ii) and (iv) are proved similarily, using the following implicit facts from [2]: *If S is a maximal semigroup and n is any integer* ($\neq g(S)/2$), *then $n \in S \Leftrightarrow g(S) - n \notin S$; and if S is any semigroup, then there is exactly one maximal semigroup S_1 such that* $S \subseteq S_1$ & $g(S) = g(S_1)$ & $S \cap \{1, 2, \ldots, [(g-1)/2]\} = S_1 \cap \{1, 2, \ldots, [(g-1)/2]\}$.

If $X \subseteq A$ or $X \subseteq B$, then $g \notin S\langle X \rangle$. For $g \notin A$, while the sum of two or more elements in A is greater than $2 \cdot g/2 = g$; and likewise $g \notin B$, the sum of two elements in B is less than g, and the sum of three or more elements in B is greater than $3 \cdot g/3 = g$. Thus and by lemma 1,

$$f(g,m) = 2^{g-m-1} \quad \text{if} \quad g/2 < m < g, \quad \text{and} \tag{6}$$

$$e(g,m) = 2^{a-m} \quad \text{if} \quad g/3 < m < g/2; \quad \text{and furthermore,} \tag{7}$$

$$\sum_{m > g/2} f(g,m) = 2^a \quad \text{and} \tag{8}$$

$$\sum_{m > g/3} e(g,m) = 2^b. \tag{9}$$

(4), (8), (5) and (9) immediately prove the left inequalities of the theorem and of (1).

Of course, (8) is just a numerical consequence of the correspondence "S with 'upper' $m(S) \leftrightarrow$ subsets of A" (where "upper" obviously means "greater than $g(S)/2$"). The reason for analyzing it as a geometric sum by means of (6) is that this also yields the "upper case" part of (2). Similarly, the "upper case" of (3) follows from (7). (Here "upper" means: "greater than $g(S)/3$".)

With more effort we shall establish upper bounds on $f(g,m)$ and $e(g,m)$ for "lower" m, and thus be able to bound the sums of these with 2^a or 2^b times a geometrical sum, thus establishing both the right inequalities of the theorem and the rest of proposition 2.

From now on, fix g and always assume that $2 \leq m \leq a$. Let X be any subset of $D_m := \{m, m+1, \ldots, a\}$, such that $m \in X$. X is called *m-admissible* if there is an $S \in \mathscr{S}_{g,m}$, such that $X = S \cap D_m$. If X is m-admissible, then let $f(g,m,X) := |\{S \in \mathscr{S}_{g,m} : S \cap D_m = X\}|$.

LEMMA 2.

(i) *X is m-admissible iff $g \notin S\langle X \rangle$.*

(ii) $f(g,m) = \sum_{X \text{ m-adm.}} f(g,m,X)$.

(iii) $e(g,m) = |\{\textit{m-admissible } X\}|$.

This is just reformulations of the definitions and former results.

In order to get estimates for $f(g, m)$, we shall use different methods, depending on whether $m > g/4$ or not.

$f(g, m)$ for $m > g/4$: Fix an m with $g/4 < m < g/2$; we may exclude the trivial case $m = g/3$. Let $s := a + 1 - m$. Then we have:

$$(10) \qquad |\{m\text{-admissible X}: |X| = k\}| \le \binom{s-1}{k-1} \qquad (k = 1, \dots, s).$$

Let $A := \{[g/2] + 1, \dots, g - 1\}$ (as before). Fix an m-admissible X. Let $k := |X|$. We shall try to restrict the number of X-*admissible* sets Y, i.e. the subsets Y of A such that $S = \{0\} \cup X \cup Y \cup \{g + 1, g + 2, \dots\} \in \mathscr{S}_{g,m}$. To begin with, we note that such a Y must fulfill three conditions:

(a) If $x \in X$, then $g - x \notin Y$ (since $x + (g - x) \notin S$);

(b) If $x_1, x_2 \in X$, then $x_1 + x_2 \in Y$; and

(c) If $y_1, y_2 \in A$ and $y_2 - y_1 \in X$, then none of y_1 and y_2, or both of them, or y_2 but not y_1 belong to Y
(since $y_1 \in Y \Rightarrow y_2 = y_1 + (y_2 - y_1) \in S \cap A = Y$).

On the other hand, when $m > g/3$ any $Y \subseteq A$ fulfilling (a), (b) and (c) indeed is X-admissible: (b) and (c) guarantee that $S\langle X \cup Y\rangle \cap A = Y$, whence (a) implies that $g \notin S\langle X \cup Y\rangle$. Thus indeed $S\langle X \cup Y\rangle = S \in \mathscr{S}_{g,m}$.

The elements $g - x$ of (a) and $x_1 + x_2$ of (b) (all of which indeed lie strictly between $g/2$ and g) form two disjoint sets $g - X$ and $2X$, respectively, for which there is no freedom of choice when constructing an X-admissible Y. Thus Y is determined by $Y \cap L$, where $L := A \setminus ((g - X) \cup 2X)$. Now $|N| = a$ and $|g - X| = k$, while $|2X| \geqq 2k - 1$ by the following well-known argument (cf e.g. [1, 1.8] and note that $m = \min X$): $(m + X) \cup (X + \max X) \subseteq 2X$; and $|m + X| = |X + \max X| = \mathrm{k}$, while $|(m + X) \cap (X + \max X)| = |\{m + \max X\}| = 1$, whence $|(m + X) \cup (X + \max X)| = 2k - 1$. Thus

$$(11) \qquad |L| \le a - 3k + 1.$$

The condition (c) may be reformulated thus: Let us turn A into a directed graph by introducing an arrow $y_1 \to y_2$ $(y_1, y_2 \in A)$ iff $y_2 - y_1 \in X$. Then any X-admissible Y does not contain any *tail* (i.e., arrow-start) without containing the corresponding *head* (arrow-end). In order to get a computable upper bound, let us confine ourselves to the *principal* arrows, i.e. to $y_1 \to y_2$ given by $y_2 - y_1 = m$. In the entire set A there are exactly $s - 1 = a - m$ (disjoint) principal arrows, namely $g - a \to g - s + 1$, $g - a + 1 \to g - s + 2, \dots, g - m - 1 \to g - 1$. The other $a - 2(s - 1)$ points in A be called *isolated*.

As we saw above there are but three possibilities as for what part of a principal arrow may be contained in a given X-admissible Y; and for any isolated point there are two possibilities ("to belong or not to belong"). Thus only using (c) we get

$$f(g,m,X) \leq 3^{s-1} \cdot 2^{a-2(s-1)} = 2^a \cdot \left(\frac{3}{4}\right)^{s-1},$$

while (a) and (b) alone gave (11) and thus

$$f(g,m,X) \leq 2^{a-3k+1}.$$

Let us combine the results, by starting with (A, {principal arrows}) and removing the elements in $g - X$ and in $2X$ one by one, checking the effect of each step on the upper bound for the number of X-admissible sets!

The element $g - m$ is isolated, whence deleting it reduces the bound to 1/2 of its former value.

Deleting any other point in $g - X$ or in $\{y \in 2X: y > g - m\}$ means deleting either an isolated point or one of the points of a principal arrow (thus also deleting the arrow, turning its other end to a new isolated points). This reduces the bound to 1/2 or to 2/3 of the former value, respectively, and thus reduces it "at least" to 2/3 thereof.

Finally, deleting any remaining element $y \in 2X$ either deletes an isolated element or the tail of an arrow; in the latter case, however, we know that also $y + m \in Y$ for any X-admissible Y, whence *the whole arrow* (including both its points) may be deleted. Thus the reduction is at least $\max(1/2, 1/3) = 1/2$; and there are at least $|m + X| = k$ such Y.

Summing up, we find that

$$(12)\quad f(g,m,X) \leq 2^a \cdot \left(\frac{3}{4}\right)^{s-1} \cdot \left(\frac{1}{2}\right)^{k+1} \cdot \left(\frac{2}{3}\right)^{2k-2} = \frac{1}{4} \cdot 2^a \left(\frac{3}{4}\right)^{s-1} \cdot \left(\frac{2}{9}\right)^{k-1},$$

$$(k = |X| \text{ and } g/4 < m < g/2).$$

By (10), (12) and the binomial theorem we have:

$$(13)\qquad f(g,m) \leq \frac{1}{4} \cdot 2^a \cdot \left(\frac{11}{12}\right)^{s-1} \quad (\text{for } g/4 < m = a + 1 - s < g/2).$$

Thus indeed

$$(14)\qquad \sum_{g/4 < m < g/2} f(g,m) \leq \sum_{s=1}^{a-[g/4]} \frac{1}{4} \cdot 2^a \cdot \left(\frac{11}{12}\right)^{s-1} < 3 \cdot 2^a.$$

$f(g,m)$ for $m < g/4$: Fix an m with $1 < m < g/4$ and such that m does not divide g. Let $n := [g/m]$ and $r := g - mn$ ($=$ the remainder of g modulo m). Thus $n \geqq 4$ and $r > 0$.

Consider $S \in \mathscr{S}_{g,m}$. We shall partition that "candidates" for elements in

$S \cap (m+1,\dots,g-1\}$ into residue classes modulo m, and employ the fact that if one element in a class belongs to S, then so do all higher elements in this class. (This corresponds to the "principal arrows" above.) Furthermore, the condition $g \notin S$ will turn out to impose heavy restrictions.

For any integer i, let $r_i := \{j \in \mathrm{N}: m < j < g \,\&\, j \equiv i (mod\, m)\}$, and let $s_i = s_i(S) := r_i \cap S$. Then we have:

$$s_0 = r_0, \text{ and}$$

$$s_r = \emptyset; \text{ whence}$$

$$S \cap \{m+1,\dots,g-1\} = \left(\bigcup_{\substack{0<i<m \\ i \neq r}} s_i\right) \cup r_0.$$

Henceforth we only regard r_i and s_i for $i \neq 0, r$. If $x \in s_i$ and $x < y \in r_i$, then $y \in s_i$. Thus s_i is completely determined by $|s_i| \in \{0,\dots,|r_i|\}$. If $0 < i < r$ then $|r_i| = n$, while if $r < i < m$ then $|r_i| = n-1$.

Let us call $r_1,\dots,r_{r-1}$ and $r_{r+1},\dots,r_{m-1}$ the *lower* and the *upper* classes, respectively. If $r_i \neq r_j$ and $i+j \equiv r$, then we shall call $\{r_i, r_j\}$ a *class couple.* Clearly, the classes of a couple either both will be lower or both will be upper, whence we may talk of *lower class couples* or *upper class couples.* If $2i \equiv r$, then r_i is called a (lower or upper) *class singleton.*

Fix a lower class couple $\{r_i, r_j\}$ with $0 < i < j < r$, and assume that $|s_i| + |s_j| \geq n+2$. Then we have $i+j=r$; $r_i = \{m+i, 2m+i,\dots,nm+i\}$; $r_j = \{m+j, 2m+j,\dots,nm+j\}$; $s_i = \{\alpha m+i,\dots,nm+i\}$ (where $\alpha = n+1-|s_i|$); and $s_j = \{\beta m+j,\dots,nm+j\}$ (where $\beta = n+1-|s_j|$). By the assumption $\alpha+\beta = 2n+2-(|s_i|+|s_j|) \leq n$, whence

$$g = nm+i+j = (n-\alpha-\beta)m + (\alpha m+i) + (\beta m+j) \in S,$$

a contradiction. Thus, in fact

$$|s_i| + |s_j| \leq n+1.$$

This together with the conditions $|s_i|$, $|s_j| \leq n$ yields: there are (at most) $(n^2+5n+2)/2$ different possible pairs $(|s_i|, |s_j|)$ and thus different possible $s_i \cup s_j$.

For upper class couples $\{r_i, r_j\}$, lower class singletons r_l, and upper class singletons r_h, we similarly get the conditions

$$|s_i| + |s_j| \leq n,$$

$$|s_l| \leq [(n+1)/2], \quad \text{and}$$

$$|s_h| \leq [n/2],$$

yielding (at most) $(n^2+3n-2)/2$, $[(n+1)/2]+1$, and $[n/2]+1$ possibilities, respectively. Thus, if we count the numbers of the various couples and singletons (with due consideration to the parities of m and r), we get an upper bound for $f(g,m)$. To be precise,

$$f(g,m) \le \left(\frac{n^2+5n+2}{2}\right)^{[\frac12(r-1)]}\left(\frac{n^2+3n-2}{2}\right)^{[\frac12(m-r-1)]}\cdot \tag{15}$$
$$\left[\frac{n+5}{2}\right]^{r-1-2[\frac12(r-1)]}\left[\frac{n+4}{2}\right]^{m-r-1-2[\frac12(m-r-1)]} \qquad (\text{for } m<g/4).$$

The rest is an exercise in elementary calculus: Considering odd and even cases for $r-1$ and for $m-r-1$ separately, and using the assumption $n \ge 4$, we find that the right side of (15), and thus $f(g,m)$, is not greater than

$$4(13\sqrt{19})^{-1}(\tfrac12(n^2+5n+2))^{\frac12 r}(\tfrac12(n^2+3n-2))^{\frac12(m-r)} <$$
$$0.071h(n+1)^{\frac12 r}h(n)^{\frac12(m-r)}2^{-\frac12 m},$$

where $h(x) = x^2+3x-2$ (x real). Now $(h(n+1)^r\cdot h(n)^{m-r})^{1/m} = h(n+1)^w\cdot h(n)^{1-w}$ ($w := r/m$) is a weighted geometric mean of $h(x)$. Taking logarithms we pass to an *arithmetic* mean, and since

$$\frac{d^2}{dx^2}\ln h(x) < 0 \quad (x \ge 4),$$

we get

$$w\ln h(n+1) + (1-w)\ln h(n) < \ln h(n+w) = \ln h(g/m),$$

whence

$$f(g,m) < 0.071(\tfrac12 h(g/m))^{\frac12 m} = 0.071\left(\frac12\left(\left(\frac{g}{m}\right)^2 + \frac13\frac{g}{m} - \frac12\right)\right)^{\frac12 m} \tag{16}$$
$$\text{for} \quad m < g/4.$$

Now $(\frac12 h(4))^{\frac12 g/4} = 13^{g/8} = 2^{g/2}\cdot 2^{-g(\lg(16/13))/8}$,

$$\frac{d}{dy}(\tfrac12 y\ln(\tfrac12 h(g/y)))|_{y=g/4} = \tfrac12\ln 13 - 11/13 > 0.436, \quad \text{and}$$

$$\frac{d^2}{dy^2}(\tfrac12 y\ln(\tfrac12 h(g/y))) < 0 \quad \text{for} \quad 0<y<g/4.$$

Thus $(\frac12 h(g/(m-1)))^{\frac12(m-1)} < (\frac12 h(g/m))^{\frac12 m}\cdot e^{-0.436} < (\frac12 h(g/m))^{\frac12 m}2^{-0.628}$, for $2 \le m \le g/4$, and we get indeed

$$f(g,m) < 0.071\cdot 2^{g/2}\cdot(13/16)^{g/8}\cdot 2^{-0.628((g/4)-m)} \quad \text{for} \quad m < g/4 \tag{17}$$

and thus a way to express $\sum_{m<g/4} f(g,m)$ as a geometric sum. For uniformity, it may

be nicer to note that (since $(13/16)^{1/8} < (11/12)^{1/4}$ and $0.628 > \lg(12/11)$) we can extend (13) to the case $m < g/4$, too. That is, we have

$$f(g,m) \leq \frac{1}{4} \cdot 2^a \cdot \left(\frac{11}{12}\right)^{a-m} \quad \text{for} \quad m = 2,3,\ldots,a(=[(g-1)/2]). \tag{18}$$

This (and (8)) yield the upper inequality of (1), whence indeed $\limsup_g 2^{-g/2} f(g) \leqq 4 < \infty$. Moreover, (6) and (8) imply the (2)-part of proposition 2.

Finally, we shall derive bounds for the $e(g,m)$ in a similar way; we are through if $m := m(S) > g/3$, and otherwise we divide the analysis into two parts, depending on whether $m > g/6$ or not. However, just copying the methods from the $\limsup f(g)$ – proof is not enough; indeed that yields $2^b \cdot$ (geometrical sums), but the bases of the terms of the power expressions are greater than rather than less than 1. We overcome this by using the following *Main lemma* (instead of the crude estimate $|2X| \geq 2|X| - 1$) which essentially tell us that *on the average* the cardinality of $2X$ is *much* bigger than the cardinality of X. The explicit lemma deals with the inverse problem: Given $|2X|$, there are few possible X, and thus in particular few possible X of large cardinality:

$$(\forall \varepsilon > 0)(\exists C)(\forall n, q \in \mathbf{N})(|\{X \subseteq \{1,2,\ldots,n\} : |2X| \leq q\}| < C \cdot 2^{\varepsilon n + \frac{1}{2}q})$$

$e(g,m)$ for $m > g/6$: Fix m such that $g/6 < m < g/3$ (and $m \neq g/4, g/5$). Let $s := [(g+2)/3] - m$. This time, we shall only regard "small" m-admissible sets X, i.e. such that $m \in X \subseteq \{m,\ldots,[(g-1)/3]\}$ and $g \notin S\langle X \rangle$. Then

$$e(g,m) = \sum_X e(g,m,X), \tag{19}$$

where $e(g,m,X) := \# Y \subseteq B$, such that $g \notin S\langle X \cup Y\rangle$ and that $S\langle X \cup Y \rangle \cap B = Y$. (This clearly is equivalent to $X \cup Y$ being a *general* m-admissible set; cf. lemma 2 (iii).) Now, let $e(g,m,q) := \sum_{|2X| = q} e(g,m,X)$ for $q = 1,\ldots,2s-1$. Then

$$e(g,m) = \sum_{q=1}^{2s-1} e(g,m,q). \tag{20}$$

Let $\varepsilon > 0$, $\varepsilon < \lg(2/\sqrt{3}) = 1 - \frac{1}{2}\lg 3$ be given. By the main lemma there is a constant C such that for any fixed q we have

$$|\{\text{small } m\text{-admissible } X \text{ with } |2X| = q\}| \leq$$
$$|\{\text{small } m\text{-admissible } X \text{ with } |2X| \leq q\}| < C \cdot 2^{\varepsilon s} \cdot 2^{\frac{1}{2}q}.$$

Next, fix such an X (with $|2X| = q$) and let us estimate $e(g, m, X)$. The element m induces a structure of *undirected* graph on B, by the rule that there should be an edge between y and z ($y, z \in B$) iff $m + y + z = g$. These edges are disjoint. Their exact number depends on the residue class of g modulo 3 and on s, but there are not less than $(s - 1)/2$ of them within B. They are unordered and 'antipathic', where the principal arrows analyzed above were ordered and 'sympathic': if $\{y, z\}$ forms an edge, then none or one but not both of y and z may be contained in a Y such that $g \notin S\langle X \cup Y\rangle$. Thus the effect of their presence is the same as for the principal arrows: they reduce the number of possibilities to $(3/4)^{\#\text{edges}}$ of the a priori number.

Let $D' := 2X \cap \{1, \dots, a\}$, $D'' := g - (2X \setminus D')$. Then $D' \subseteq B$ (since $2m > g/3$), $D'' \subseteq B$, and for any Y as above, $D' \subseteq Y$ while $D'' \cap Y = \emptyset$. (In particular $D' \cap D'' = \emptyset$.) Thus Y is determined by $Y \cap C$ where $C := B \setminus (D' \cup D'')$; and as before we find that there are not less than $(s - 1)/2 - |D' \cup D''| = (s - 1)/2 - q$ edges in C. Thus

$$(21) \quad e(g,m,X) \leq 2^{|C|} \cdot \left(\frac{3}{4}\right)^{\frac{1}{2}(s-1)-q} = \sqrt{\frac{4}{3}} \cdot 2^{b} \cdot \left(\frac{2}{3}\right)^{q} \cdot \left(\frac{3}{4}\right)^{\frac{1}{2}s} \quad (m > g/6;\ |2X| = q).$$

Thus and by (20) and the main lemma

$$(22) \qquad e(g, m) < \left(\sum_{q \geq 1} \left(\frac{2}{3}\right)^{q} \cdot 2^{\frac{1}{2}q}\right) \cdot C \cdot 2^{\varepsilon s + \frac{1}{2}(\lg(3/4))s} \cdot 2^{b},$$

where C only depends on ε. The sum over q is $C_1 := \sqrt{8}/(3 - \sqrt{8})$; and $C_2 := 2^{\varepsilon + \frac{1}{2}\lg(3/4)} < 1$ by the choice of ε. Thus indeed

$$(23) \qquad \sum_{(g/6) < m < (g/3)} e(g, m) < \sum_{s \geq 1} C_2^s \cdot C_1 C \cdot 2^b = C' \cdot 2^b$$

for some C' (which is independent of g).

$e(g, m)$ *for* $m < g/6$: Fix m; for $S \in \mathscr{M}_{g,m}$ let $X(S) := S \cap \{m, m + 1, \dots, 2m\}$; for $X \subset \{m, \dots, 2m\}$ let $e'(g, m, X) := |\{S \in \mathscr{S}_{g,m}: X(S) = X\}|$; for $q = 3, 4, \dots, 2m - 1$ let $e(g, m, q) := \sum_{X:|2X| = q} e'(g, m, X)$; choose an ε with $0 < \varepsilon < 1 - 0.4 \lg 5$; and (by means of the main lemma) choose a C such that

$$(24) \qquad |\{X \subset \{m, \dots, 2m\}: |2X| = q\}| < C \cdot 2^{\varepsilon m + \frac{1}{2}q} < C \cdot 2^{\varepsilon g/6 + \frac{1}{2}q}.$$

We have also

$$(25) \qquad e(g, m) = \sum_{q} e(g, m, q).$$

Fix q and X (with $|2X| = q$). We are trying to decide in how many ways we may extend $2X \setminus \{2m\}$ to a set $Z \subset \{2m+1, \ldots, a\}$ such that $g \notin S\langle X \cup Z\rangle$. Thus, let $r_i := \{j : 2m+1 \leq j \leq a \,\&\, j \equiv i (\operatorname{mod} m)\}$; $s_i := r_i \cap S$ (whenever S is chosen); $n := [g/m]$ $(\geqq 6)$; and $r := g - mn$. As before, $r_1, \ldots, r_{r-1}$ and $r_{r+1}, \ldots, r_{m-1}$ be the *lower* and the *upper* classes, respectively; and lower or upper class couples and singletons be as before.

If $\{r_i, r_j\}$ is a class couple, $x \in r_i$, and $y \in r_j$, then $x + y < g$ and $x + y \equiv g (\operatorname{mod} m)$; thus $x + y \notin S$. I.e., if $\{r_i, r_j\}$ is a class couple, then $s_i = \emptyset$ or $s_j = \emptyset$. Should e.g. $s_i \neq \emptyset$, then s_i is determined by $\min s_i \in r_i$. Thus

$$\#\,\text{possible}(s_i, s_j) \leq |r_i| + |r_j| + 1 = \begin{cases} n-2 & \text{if } \{r_i, r_j\} \text{ is lower} \\ n-3 & \text{if } \{r_i, r_j\} \text{ is upper} \end{cases}.$$

(If r_i is a singleton, then $s_i = \emptyset$.) Thus (ignoring the restrictions imposed by having fixed $2X$) we get an upper bound

$$(n-2)^{\frac{1}{2}r} \cdot (n-3)^{\frac{1}{2}(m-r)}$$

on $e'(g, m, X)$. Arguing as before (taking logarithms, etc.) yields

$$e'(g, m, X) \leq \left(\frac{g}{m} - 3\right)^{\frac{1}{2}m}.$$

The influence of the $2X$-elements may be estimated thus: Let $\{r_i, r_j\}$ be a class couple. Assume that $x \in 2X$ and that $x \equiv i (\operatorname{mod} m)$; then we must have $x = 2m + i$ or $x = 3m + i$. Thus $g/2 + m > 3m + i \in S$. If $y \in s_j$, then $g + m > y + 3m + \mathrm{i} \equiv g (\operatorname{mod} m)$, whence $y \notin S$. Thus $s_j = \emptyset$ (even if $x \notin s_i$). Furthermore, if in addition $z \in 2X$ and $z \equiv j (\operatorname{mod} m)$, then we must have $x = 3m + i$, $z = 3m + j$, and $s_i = s_j = \emptyset$; while if $z \in 2X$, $z \equiv i (\operatorname{mod} m)$ and $z \neq x$, then $s_i = r_i$. To sum up, there are at most two $2X$-elements with remainders corresponding to a given class couple $\{r_i, r_j\}$, and *if* there are two such elemments, then they completely determine (s_i, s_j). If $\{r_i, r_j\}$ is a class couple with just one single element $x \in 2X$ which is congruent to i (and none congruent to j), then there may be one choice left: if $x > 3m$ then we may or we may may not have $x - m \in s_i$. There are at most four elements in $2X$ with zero or singleton remainders. Any other element will yield a reduction not "less" than $\max(2/(n-3), 1/2) \leq 2/3$, and we get

$$e'(g, m, X) < \left(\frac{2}{3}\right)^{q-4} \left(\frac{g}{m} - 3\right)^{\frac{1}{2}m},$$

whence by (24)

$$e(g, m, q) < \frac{81C}{16} \left(\frac{2\sqrt{2}}{3}\right)^{q} \cdot 2^{\varepsilon g/6} \cdot \left(\frac{g}{m} - 3\right)^{\frac{1}{2}m}.$$

Summing over q yields

$$e(g,m) < C''2^{\varepsilon g/6}\left(\frac{g}{m}-3\right)^{\frac{1}{2}m} \tag{26}$$

for some constant C''.

Now the continuous function $h(y) = \left(\frac{g}{y}-3\right)^{\frac{1}{2}y}$, $0 \leqq y \leqq g/6$, has a negative second derivative in the whole interval, and its firstderivative changes sign within the interval; thus h has a unique maximum, which in fact is attained for some y_0 such that $g/7.98 < y_0 < g/7.97$. In particular,

$$\left(\frac{g}{m}-3\right)^{\frac{1}{2}m} < \left(\frac{g}{g/8}-3\right)^{\frac{1}{2}g/7.5} = 5^{g/15} = 2^{(g/6)\cdot 0.4\lg 5}. \tag{27}$$

Thus and by (26)

$$\sum_{m<g/6} e(g,m) < \frac{C''g}{6}\cdot 2^{-C_3 g}\cdot 2^{g/6},$$

where $C_3 := \frac{1}{6}(1 - 0.4\lg 5 - \varepsilon) > 0$, whence $\lim_{g\to\infty} g2^{-C_3 g} = 0$. Proposition 2 and the last inequality of the theorem follow.

Proof of proposition 1: By proposition 2, as g tends to infinity it is enough to investigate $f(g,m)$ and $e(g,m)$ for m being close to $g/2$ and to $g/3$, respectively. In particular, we may confine ourselves to $m > g/3$ and to $m > g/4$, respectively. For concreteness, consider

$$C(g) := \sum_{m>g/3} f(g,m)\cdot 2^{-g/2}, \quad g \text{ even},$$

as g tends to infinity. By lemma 1, the "upper" part of the sum (i.e. the "$m > \frac{1}{2}g$"-part) equals $2^{a-\frac{1}{2}g} = 2^{-1}$. In the "lower" cases, we found that for each m-admissible set X we had three conditions (a), (b), (c), which were translatable to a graph structure on a certain subset of A; and this graph structure completely determined the quantity $f(g,m,X)$. Now since $m > g/3$, all elements in $g - X$ are strictly smaller than the elements in $2X$. Therefore, if we put $g' := g + 2$, $m' := m + 1$, and $X' := X + 1$, then the graph corresponding to $f(g',m',X')$ will be isomorphic to the graph corresponding to $f(g,m,X)$ *with one isolated vertex added*. Thus $f(g',m',X') = 2f(g,m,X)$, whence $f(g',m')\cdot 2^{-g'/2} = f(g,m)\cdot 2^{-g/2}$ and (using proposition 2) the existence of $\lim C(g)$ follows.

The case g odd and the $e(g)$-cases are similar. In the latter cases the graph-structures depend on $g(\mathrm{mod}\,3)$, while $b - g/6$ depends on $g(\mathrm{mod}\,6)$.

I.3. Approximate values of the limits. We have established the existence of the limits of proposition 1. In this section we consider the less important question: what might these limits be? Let us reformulate proposition 1 in terms of 2^a and 2^b, where as before

$$\begin{aligned} a &= a(g) = [(g-1)/2], \quad \text{and} \\ b &= b(g) = [(g-1)/2] - [g/3]. \end{aligned}$$

The proof of the proposition yields:

There are constants C_0, C_1, D_0, D_1 and D_2 such that

$$\lim_{g \equiv i (\mathrm{mod}\, 2)} 2^{-a} f(g) = C_i \quad (i = 0, 1) \quad \textit{and} \tag{28}$$

$$\lim_{g \equiv i (\mathrm{mod}\, 3)} 2^{-b} e(g) = D_i \quad (i = 0, 1, 2). \tag{29}$$

We also get methods for computing arbiitrarily good approximations for these constants:

$$C_i = 1 + \sum_{s=1}^{\infty} c_{i,s}, \text{ and} \tag{30}$$

$$D_i = 1 + \sum_{s=1}^{\infty} d_{i,s}, \tag{31}$$

where $c_{i,s} = f(g, [(g+1)/2] - s) \cdot 2^{-a}$ for any g such that $(g/3) < [(g+1)/2] - s$ and that $g \equiv i (\mathrm{mod}\, 2)$, and $d_{i,s} = e(g, [(g+2)/3] - s) \cdot 2^{-b}$ for any g such that $(g/4) < [(g+2)/3] - s$ and that $g \equiv i (\mathrm{mod}\, 3)$.

Fixing i and s we may calculate $c_{i,s}$ as follows: Pick appropriate g and $m (= [g+1)/2] - s)$ with $c_{i,s} = f(g,m) 2^{-a} = \sum_X f(g, m, X)$ (sum over m-admissible X). For any subset $T \subseteq \{1, \ldots, s-1\}$, let $X = X(T) := \{[(g+1)/2] - k \colon k \in T \cup \{s\}\}$; and let the directed graph $L = L(T) := A \setminus ((g - X) \cup 2X)$ with the arrows as introduced in section I.2. If (y_1, y_2) is an arrow in L we have $y_1 < g - [(g+1)/2] + s = [g/2] + s < 2([(g+1)/2] - s) < y_2$. Hence all elements in $g - X$ and all tails are $\leq g - m$, while all elements in $2X$ and all heads are $\geq 2m (g - m)$. Thus L consists of a bipartite graph $L' = L'(T)$ in disjoint union with $3m - g - 1 = a - 3s + i$ isolated vertices, where L' (up to isomorphisms) is independent of the choice of g; and we get:

$$f(g, m, X(T)) 2^{-a} = |\{\text{independent subsets of } L'\}| \cdot 2^{-3s+i}, \text{ whence}$$

$$c_{i,s} = \sum_T |\{\text{independent subsets of } L'(T)\}| \cdot 2^{-3s+i}.$$

By (13), $c_{i,s} \leq \frac{1}{4} \cdot \left(\frac{11}{12}\right)^{s-1}$. Thus, if we calculate $c_{i,1}, \ldots, c_{i,n}$ for some n, we find that

$$1 + \sum_{s=1}^{n} c_{i,s} < C_i \leq 1 + \sum_{s=1}^{n} c_{i,s} + 3\cdot\left(\frac{11}{12}\right)^n.$$

Choosing $n = 15$ and calculating (by computer) gave

$$\sum_{s}^{15} c_{0,s} = 1.472\ldots;\ \sum_{s}^{15} c_{1,s} = 1.502\ldots;\ 3\cdot\left(\frac{11}{12}\right)^{15} = 0.813\ldots;$$

thus

$$2.47 < c_0 < 3.3;\ 2.5 < c_1 < 3.32. \tag{32}$$

In particular,

$$\lim_{g\,\text{even}} (S_g|\cdot 2^{-\frac{1}{2}g}) = \tfrac{1}{2}\cdot C_0 < \sqrt{\tfrac{1}{2}}\,C_1 = \lim_{g\,\text{odd}} (|S_g|\cdot 2^{-\frac{1}{2}g}),$$

(Actually, we may get sharper results without increasing n, by employing non-principal arrows and more precise results than the crude inequality $|2X| \geq 2\,|X| - 1$. In this manner we may deduce that the value of the constants lie "fairly" close to the lower estimates given in (32).)

For the D_i's, the situation is slightly "better", but much "worse". "Better" in the sense that we may show the remaining middle inequality in the theorem directly, just by noting that the quantities $b(g) - g/6$ depend *strictly* on the residue class of g modulo 6, while D_* only depends on $g(\text{mod } 3)$. Thus e.g. $b(6g') - 6g'/6 = -1$, while $b(6g' + 3) - (6g' + 3)/6 = -\frac{1}{2}$, whence

$$\lim_{g\equiv 0(\text{mod } 6)} 2^{-g/6}\,|\mathcal{M}_g| = \tfrac{1}{2}D_0 < \sqrt{\tfrac{1}{2}}D_0 = \lim_{g\equiv 3(\text{mod } 6)} 2^{-g/6}\,|\mathcal{M}_g|.$$

On the other hand, although we may calculate any $d_{i,s}$ in a manner corresponding to the method above, and thus calculate any partial sum $\sum_{1}^{n} d_{i,s}$ of the sum in (31), we have no good estimate for the "tail" $\sum_{s=n+1}^{\infty} d_{i,s}$ of that sum. Indeed, we used the main lemma in order to ensure the convergence in (31); and, while realizations of the sought constants of that lemma may be deduced from its proof presented below, these yield ridicuously bad estimates for the *rate* of convergence in (32).

A computer calculation of $d_{0,s}$ for $s = 1,\ldots,21$ is given in Table 1. It shows that $9.36 < D_0$. The marked diffferences between the odd and the even cases are due to the "extra" forbidden elements $g/6 + s/2$ for s even. Inspecting the table (taking this into account) makes it seem less likely that D_0 would not be less than (say) 15.

TABLE 1.

s	$d_{0,s}$ (exact)	$d_{0,s}$ ($\approx$)	accumulated sum
1	2/4	0.5	0.5
2	5/16	0.3125	0.8125
3	37/64	0.578125	1.390625
4	100/256	0.390625	1.78125
5	615/1024	0.60058593	2.3818359
6	1491/4096	0.36401367	2.7458496
7	10058/16384	0.61389160	3.3597412
8	25080/65536	0.38269042	3.7424316
9	154485/262144	0.58931350	4.3317451
10	347956/1048576	0.33183670	4.6635818
11	2275341/4194304	0.54248356	5.2060654
12	5617059/16777216	0.33480280	5.5408682
13	32932169/67108864	0.49072755	6.0315957
14	74146540/268435456	0.27621738	6.3078131
15	459815105/1073741824	0.42823618	6.7360493
16	1101432174/4294967296	0.25644716	6.9924964
17	6466239450/17179869184	0.37638467	7.3688811
18	14383614901/68719476736	0.20930914	7.5781903
19	88513143507/274877906944	0.32200893	7.9001992
20	205912251644/1099511627776	0.18727610	8.0874754
21	1202802586025/4398046511104	0.27348564	8.3609610

II. The main lemma.

II.1. STATEMENT. Recall that if A is a set then $|A|$ = number of elements in A; if A and B are sets of integers and c is an integer, then $A + B = \{a + b : a \in A \,\&\, b \in B\}$, $A + c = \{a + c : a \in A\}$, and $2A = A + A$; and if x is a real number, then $[x]$ denotes the integer part of x.

MAIN LEMMA. *Let ε be any given positive real number. Then there is a constant C such that for any positive integers n and q we have*

$$K(n,q) := |\{X \subseteq \{1,2,\ldots,n\} : |2X| \leq q\}| < C \cdot 2^{\varepsilon n + \frac{1}{2}q}. \tag{33}$$

II.2. REMARKS.

1. We could as well have formulated the lemma for subsets of *any* fixed arithmetic sequence $\alpha + \beta, \alpha + 2\beta, \ldots, \alpha + n\beta$ of length n, by means of obvious

mappings; and, in fact, in this paper the lemma is indeed applied with this apparent generalization tacitly understood.

2. In some respects this lemma is close to optimal. Loosely spoken, if we fix n and let q vary (over $1,\dots,2n-1$), then for any q we have that there are at least $2^{\frac{1}{2}q}$ subsets X of $\{1,\dots,[(q+1)/2]\}$ (and of any other arithmetic subsequence of this length), and $|2X| \leq q$ for each such X. (Thus in particular $\sup\limits_{n,q}(K(n,q)/2^{\varepsilon n + cq}) = \infty$ for $c < c + 2\varepsilon < \frac{1}{2}$.) On the other hand, for any given degree d we may fix a $q \geq \binom{d+2}{2}$ and let n grow; then, since if $|X| \leq d+1$ then $|2X| \leq \binom{d+2}{2}$, we get

$$K(n,q) > \binom{n}{d+1},$$

whence $K(n,q)$ grows faster than polynomially in n.

3. It would be very interesting to find explicit formulas or good approximations for the quantities

$$K(n,k,q) := |\{X \subseteq \{1,\dots,n\} : |X| = k \,\&\, |2X| = q\}|.$$

(E.g. for $q < 3k-3$ fairly good results can be obtained by means of [1, thm. 1.9]; perhaps the weaker but quite general "fundamental theorem" [1, thm. 2.8] may yield good results in general.)

II.3. Proof. The lemma will be proved in the following, apparently weaker, formulation:

For any given positive ε there is a polynomial $p(x)$ (with real coefficients) such that for any positive integers n and q we have

$$K(n,q) \leq p(n) \cdot 2^{\varepsilon n + \frac{1}{2}q}. \tag{34}$$

Actually (33) for $\varepsilon = \varepsilon_1 > 0$ follows from (34), applied on some $\varepsilon = \varepsilon_2$, where $0 < \varepsilon_2 < \varepsilon_1$. Choose an $\varepsilon > 0$. For a while, fix an integer $m \geq 3/\varepsilon$. For any set A of integers, and for $i = 1,2,\dots,m$, let $A_i := \{j \in A : j \equiv i (\operatorname{mod} m)\}$. We shall find upper bounds for the quantities

$$K^m(n,q) := |\{X \subseteq \{1,\dots,n\} : |2X| \leq q \,\&\, X_i \neq \emptyset \quad \text{for} \quad i = 1,\dots,m\}|.$$

$K^m(n,q) = 0$ if $m > n$, whence we subsequently assume $m \leq n$. Let α vary over the set of $2m$-tuples $(a_1, a_2, \dots, a_m, b_1, \dots, b_m)$ such that $a_i \equiv b_i \equiv i$ and $1 \leq a_i \leq b_i \leq n$ for $i = 1,\dots,m$. There are less than n^{2m} such α; and clearly

$$K^m(n,q) = \left|\bigcup_{\alpha} \mathscr{V}(n,q,m,\alpha)\right|, \text{ where} \tag{35}$$

$\mathscr{V}(n,q,m,\alpha) := \{X \subseteq \{1,\dots,n\} : |2X| \leq q \,\&\, (\min X_i = a_i \,\&\, \max X_i = b_i \text{ for } i = 1, \dots, m)\}$. Fix such an $\alpha = (a_1,\dots,b_m)$. Since $|2X| = \sum_{r=1}^{m} |(2X)_r|$, we have

$$\mathscr{V} := \mathscr{V}(n,q,m,\alpha) = \bigcup_{r=1}^{m} \mathscr{V}^r, \tag{36}$$

where $\mathscr{V}^r := \{X \in \mathscr{V} : |(2X)_r| \leq q/m\}$. Fix an $r \in \{1,\dots,m\}$. Note that for any $i,j \in \{1,\dots,m\}$ such that $i + j \equiv r (\operatorname{mod} m)$ and for any $X \in \mathscr{V}^r$ we have

$$A_i(X) := (a_i + X_j) \cup (X_i + b_j) \subseteq X_i + X_j \subseteq (2X)_r.$$

Conversely, given such an $A_i(X) \subseteq (2X)_r$ we may reclaim X_j and X_i, since then $X_j = \{x - a_i : a_i + b_j \geq x \in A_i(X)\}$ and $X_i = \{x - b_j : a_i + b_j \leq x \in A_i(X)\}$. The set $\{1,\dots,m\}$ may be partitioned into c pairs $p_\mu := (i_\mu, j_\mu)$ with $i_\mu < j_\mu$ and $i_\mu + j_\mu \equiv r$, and d "singletons" s_ν with $2s_\nu \equiv r$. By construction, $2c + d = m$. Furthermore, the family

$$\mathscr{F}(X) := (A_{i_1}(X), A_{i_2}(X), \dots, A_{i_c}(X), A_{s_1}(X), \dots, A_{s_d}(X))$$

is completely determined by and completely determines X.

Let $B(X) := \bigcup_{A \in \mathscr{F}(X)} A$; then $B(X) \subseteq (2X)_r \subseteq \{2,\dots,2n-1\}_r$; and $|B(X)| \leq q/m$. Thus there are less than $2^{(2n/m)+1} \leq 2^{3n/m} \leq 2^{\varepsilon n}$ possible different sets $B = B(X)$. Fix such a B. For each $\mu = 1,\dots,c$ there are less than $2^{q/m}$ possible different $A_{i_\mu}(X)$. Furthermore, if $k \in \{1,\dots,m\}$ is a singleton, then let

$$D_k := \begin{cases} \{x \in B : x > a_k + b_k\} & \text{if this set has less than } q/2m \text{ elements} \\ \{x \in B : x < a_k + b_k\} & \text{else} \end{cases}.$$

Clearly $A_k(X) \cap D_k$ completely determines $X_k(X)$. Thus, for each $\nu = 1,\dots,d$ there are at most $2^{|D_j|_\nu} < 2^{q/2m}$ possible $A_{j^\nu}(X)$ (with $B(X) = B$). Thus # possible $\mathscr{F}(X)$ where $B(X) = B$ is less than $2^{c \cdot q/m + d \cdot q/2m} = 2^{\frac{1}{2}q}$. Summing over the different possible $B(X)$ we get

$$|\mathscr{V}^r| = \#\,\text{possible } \mathscr{F}(X) < 2^{\varepsilon n + \frac{1}{2}q}. \tag{37}$$

Thus and by (35) and (36) we get

$$K^m(n,q) < n^{2m} \cdot m \cdot 2^{\varepsilon n + \frac{1}{2}q} \leqq n^{2m+1} \cdot 2^{\varepsilon n + \frac{1}{2}q}. \tag{38}$$

Next, as is well known, we may choose a finite number of pairwise relatively prime integers (e.g. primes) $m_1,\dots,m_t$, say, such that $m_i \geqq 3/\varepsilon$ for $i = 1,\dots,t$, and that

$$c := \prod_{i=1}^{t} \frac{m_i - 1}{m_i} \leqq \varepsilon. \tag{39}$$

We may choose them in such a way that $m_1 = \max_i m_i$. Furthermore, let

$$D := \prod_{i=1}^{t} m_i.$$

Fix n and q. For any $X \subseteq \{1,\ldots,n\}$ such that $|2X| \leq q$ one of the following statements must hold:

(A) *There is an m_i such that there are elements of X in all residue classes modulo m_i*;

or

(B) *There is a family $\mathscr{g} = (g_i)_{i=1}^{t}$ of integers, such that for $i = 1,\ldots,t$ we have $1 \leq g_i \leq m_i$ and $x \not\equiv g_i (\operatorname{mod} m_i)$ for all $x \in X$.*

Clearly, the number of sets X fulfilling (A) does not exceed $\sum_{i}^{t} K^{m_i}(n,q) < t \cdot n^{2m_1+1} \cdot 2^{\varepsilon n + \frac{1}{2}q}$. Thus it is sufficient to get a good upper bound for the number of sets fulfilling (B). We shall do this for *all* subsets X of $\{1,\ldots,n\}$, relaxing the condition on $|2X|$.

For any family $\mathscr{g}$ as above, let

$$\mathscr{E}(\mathscr{g}) := \{X \subseteq \{1,\ldots,n\} : (\forall x \in X)(\forall i \in \{1,\ldots,t\})(x \not= g_i (\operatorname{mod} m_i))\}.$$

Then the number of X fulfilling (B) does not exceed $\sum_{\mathscr{g}} |\mathscr{E}(\mathscr{g})|$. There are D different families $\mathscr{g}$ of this kind. For any fixed $\mathscr{g}$ and any sequence $S = \{u+1, u+2,\ldots,u+D\}$ of D successive integers there are exactly cD elements $s \in S$ such that $s \not\equiv g_i (\operatorname{mod} m_i)$ for all i. Hence

$$|\{s \in \{1,\ldots,n\} : (\forall i)(s \not\equiv g_i (\operatorname{mod} m_i))\}| < cn + D \leq \varepsilon n + D,$$

whence $\sum_{\mathscr{g}} |\mathscr{E}(\mathscr{g})| < D \cdot 2^D \cdot 2^{\varepsilon n}$.

To sum up: For all n and q

$$I(n,q) < (t \cdot n^{2m_1+1} + D \cdot 2^D) \cdot 2^{\varepsilon n + \frac{1}{2}q}, \tag{40}$$

where t, m_1 and D are independent of n and q. (34) and thus the main lemma follow.

REFERENCES

1. G. A. Freĭman, *Foundations of a Structural Theory of Set Addition*, Transl. Math. Monographs 37, AMS, 1973.

2. R. Fröberg, C. Gottlieb, R. Häggkvist, *On numerical semigroups*, Semigroup Forum 35 (1987), 63–83.

3. H. S. Wilf, *Circle-of-lights algorithm for money-changing problem*, Amer. Math. Monthly 85 (1978), 562–565.

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