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THE SUM OF TWO PLANE CONVEX C^∞ SETS IS NOT ALWAYS C^5

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1. Introduction.

In a recent article [K] in this journal Christer O. Kiselman investigated the smoothness of the boundary of the vector sum $A + B$ of two plane convex sets A and B with smooth boundaries. In the special case when the boundaries of A and B are C^∞ and have no infinitely flat points Kiselman shows that the boundary of $A + B$ must be of class $C^{20/3}$ and that this statement is sharp in the sense that 20/3 can not be replaced by any smaller number. In the case where infinitely flat points are allowed Kiselman states that the boundary of $A + B$ must be C^4, but on the negative side he gives no information on this problem apart from the already cited result. This means that the minimal smoothness for the sum of two bounded convex sets with C^∞ boundary is shown to be somewhere between C^4 and $C^{20/3}$ but is not exactly determined. Here we will show that the minimal smoothness is in fact C^4 (Theorem 2). Since Kiselman's statement that the boundary must be C^4 seems to be non-trivial, we have included here a proof for this fact, or more precisely

THEOREM 1. *Assume A and B are bounded, convex sets in the plane, whose boundaries are of class C^r, $r = 1, 2, 3, 4$. Then the boundary of $A + B$ is of class C^r.*

THEOREM 2. *There exist two bounded, strictly convex plane sets A and B with C^∞ boundary such that the boundary of $A + B$ is not of class C^5.*

It is easily seen from the proof that one can construct A and B so that the boundary of $A + B$ is not of class $C^{4+\varepsilon}$ for any positive ε. (See Remark 1 at the end of the paper).

The analogous problem in higher dimensions is considered in an other article in this issue [B].

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2. Preliminaries.

We use the notation and terminology of [K]. The problem is local, so it suffices to study the smoothness of the infimal convolution $h = f \,\square\, g$ i.e.

$$h(x) = \inf_y (f(y) + g(x - y)), \tag{1}$$

where f and g are (germs of) smooth convex functions which vanish and have vanishing derivative at the origin. Theorems 1 and 2 are immediate consequences of the corresponding statements for infimal convolution of germs.

PROPOSITION 1. *Let r be 1, 2, 3, or 4, and assume f and g are convex germs of class C^r, such that $f'(0) = g'(0)$. Then $f \,\square\, g$ is of class C^r.*

PROPOSITION 2. *There exist two strictly convex C^∞ germs f and g with $f'(0) = g'(0) = 0$, such that $f \,\square\, g$ is not in C^5.*

In what follows we assume throughout that f and g are convex germs of class at least C^1 defined in some neighbourhood J of the origin, and that $f(0) = g(0) = f'(0) = g'(0) = 0$.

If f or g is strictly convex, the infimum in (1) is attained at the unique point y for which

$$f'(y) = g'(x - y). \tag{2}$$

If f and g are only assumed to be convex, the set of solutions to (2) may be an interval, and the infimum is attained at any point of that interval. Denote the set of solutions to (2) by $Y(x)$. The function $x \mapsto Y(x)$ is Lipschitz continuous in the following sense: given x_1, x_2, and $y_1 \in Y(x_1)$, there exists $y_2 \in Y(x_2)$ such that $|y_1 - y_2| \leqq |x_1 - x_2|$. To prove this one uses the fact that f' and g' are increasing together with the intermediate value theorem. It is easy to see that h must be differentiable and

$$h'(x) = f'(y) = g'(x - y), \tag{3}$$

for any $y \in Y(x)$. By the continuity of $x \mapsto Y(x)$, h must be C^1.

Assume now, f, $g \in C^k$, $k \geqq 2$. Denote by E the set of all $x \in J$ for which $f''(y) = g''(x - y) = 0$ for some (equivalently all) y satisfying (2). Applying the Implicit Function Theorem to (2) we find that the function $x \mapsto y$ is in C^{k-1} in $J \setminus E$, and

$$dy/dx = g''(x - y)/(f''(y) + g''(x - y)), \quad x \in J \setminus E. \tag{4}$$

Hence by (3)

$$h''(x) = f''(y)g''(x - y)/(f''(y) + g''(x - y)), \quad x \in J \setminus E, \tag{5}$$

and if $f, g \in C^3$

$$h^{(3)}(x) = ((g'')^3 f^{(3)} + (f'')^3 g^{(3)})/(f'' + g'')^3, \tag{6}$$

$$= (dy/dx)^3 f^{(3)} + (1 - dy/dx)^3 g^{(3)}, \quad x \in J \setminus E.$$

In the last formula we have abbreviated $f''(y)$ to f'', $g''(x - y)$ to g'' etc. Due to convexity $f^{(3)}(x - y)$ must vanish whenever $x \in E$. It is clear that the expressions (5) and (6) must tend to zero if x tends to some point of E. It is easy to see that h'' and $h^{(3)}$ exist and are equal to zero at points of E. Thus we have proved Proposition 1 for $r = 1, 2, 3$.

3. $f \,\Box\, g \in \mathbf{C}^4$.

We now consider the case $r = 4$, the only non-trivial case. First note that it is sufficient to prove that $h^{(4)}$ exists and is continuous at $x = 0$. This is obvious for geometric reasons and can also be seen as follows. Let $y_0 \in Y(x_0)$ and set $f_1(t) = f(y_0 + t) - f(y_0) - tf'(y_0)$, $g_1(t) = g(x_0 - y_0 + t) - g(x_0 - y_0) - tg'(x_0 - y_0)$; then $(f \,\Box\, g)(x_0 + t) = (f_1 \,\Box\, g_1)(t)$ plus linear terms for t near 0. Assume $0 \in E$, i.e. $f''(0) = g''(0) = 0$. If $f^{(4)}(0) = g^{(4)}(0) = 0$ we can deduce from (6) that $h^{(3)}(x)/x \to 0$, i.e. that $h^{(4)}(0)$ exist and is equal to zero. If e.g. $f^{(4)}(0) > 0$, then f'' must be positive in a punctured neighbourhood of the origin; the fact that $h^{(4)}(0)$ exists and is equal to $\lim_{x \to 0} h^{(4)}(x)$ therefore follows from the existence of the latter limit. To prove that $h^{(4)}$ is continuous we shall prove that the expression (9) below tends to a limit as x tends to zero through points of E. For this we need to study dy/dx. If $f^{(4)}(0) = g^{(4)}(0) = 0$, it will be sufficient to know that dy/dx is bounded (which is obvious by (4)), but if $f^{(4)}(0)$ or $g^{(4)}(0)$ is positive, we shall need to know that dy/dx tends to a limit as $x \to 0$.

LEMMA 1. *Assume* $0 \in E$, i.e. $f''(0) = g''(0) = 0$, *and* $f^{(4)}(0)$ *or* $g^{(4)}(0)$ *is* > 0. *Then the solution* $y(x)$ *to* (2) *is* C^1 *near* $x = 0$.

PROOF. By assumption $f'(x) = x^3 a(x)$ and $g'(x) = x^3 b(x)$, where $a(x)$ and $b(x)$ are continuous, $6a(0) = f^{(4)}(0)$, $6b(0) = g^{(4)}(0)$, $a(0) \geqq 0$, $b(0) \geqq 0$, and $a(0) + b(0) > 0$. Equation (2) means

$$y^3 a(y) = (x - y)^3 b(x - y).$$

Dividing by x^3 and letting x tend to zero we obtain

$$y/x = b(x - y)^{1/3}/(a(y)^{1/3} + b(x - y)^{1/3}) \to \tag{7}$$

$$b(0)^{1/3}/(a(0)^{1/3} + b(0)^{1/3}) = g^{(4)}(0)^{1/3}/(f^{(4)}(0)^{1/3} + g^{(4)}(0)^{1/3}) \text{ as } x \to 0.$$

Similarly, $f''(x) = x^2 c(x)$, $g''(x) = x^2 d(x)$, where $c(x)$ and $d(x)$ are non-negative, continuous, and $c(0) + d(0) > 0$. By (4)

(8)
$$\frac{dy}{dx} = \frac{g''(x-y)}{f''(y)+g''(x-y)} = \frac{(x-y)^2 d(x-y)}{y^2 c(y) + (x-y)^2 d(x-y)}$$
$$= \frac{(1-y/x)^2\, d(x-y)}{(y/x)^2 c(y) + (1-y/x)^2 d(x-y)}.$$

We have to prove that this expression tends to a limit as $x \to 0$. If both $f^{(4)}(0) = 2c(0)$ and $g^{(4)}(0) = 2d(0)$ are positive, this is immediate, since y/x tends to a limit between 0 and 1. Assume now $c(0)$ or $d(0)$ is zero; we may assume $c(0) = 0$, since the opposite case is analogous. Then by (7) $\lim(y/x) = 1$, so the expression (8) has the form 0/0 at $x = 0$. We shall prove $f''(y)/g''(x-y) \to 0$, which implies $dy/dx \to 1$. Since $g^{(4)}(0) > 0$ we have $g''(x-y) \approx k(x-y)^2$ for some $k > 0$. From (7) we get

$$(x-y)/x = 1 - (y/x) = (a(y)/b(0))^{1/3}(1 + o(1)), \text{ as } x \to 0.$$

Hence, for a new $k > 0$,

$$f''(y)/g''(x-y) = kf''(y)\, x^{-2}\, a(y)^{-2/3}(1 + o(1)), \text{ as } x \to 0.$$

Since $a(y) = f'(y)/y^3$, this gives

$$(f''(y)/g''(x-y))^{3/2} = k(y/|x|)^3 f''(y)^{3/2} f'(y)^{-1}(1 + o(1)), \text{ as } x \to 0.$$

It remains only to show that $f''(y)^{3/2}/f'(y)$ tends to zero as y tends to zero through points where $f'(y) \neq 0$. By l'Hospital's rule it suffices to prove that $f^{(3)}(y)/f''(y)^{1/2} \to 0$ as y tends to zero through points where $f''(y) > 0$. Squaring, applying l'Hospital's rule once more and remembering that $f^{(4)}(0) = 0$ we verify this fact. The proof of the Lemma is complete.

The conclusion of the lemma that $y(x)$ is C^1 can not be improved. For example, if we take $f(x) = x^4$, $g(x) = x^4/\log(1/|x|)$, we get $y(x) \approx x/\log(1/|x|))^{1/3}$. Moreover, we shall see later that dy/dx not necessarily exists, if both f and g have all derivatives of order $\leqq 4$ equal to zero.

PROOF OF PROPOSITION 1. To compute $h^{(4)}$ we differentiate (6) and get

$$h^{(4)}(x) = (dy/dx)^4 f^{(4)} + (1 - dy/dx)^4 g^{(4)} +$$
$$+ 3d^2y/dx^2[(dy/dx)^2 f^{(3)} - (1 - dy/dx)^2 g^{(3)}], \quad x \in J \setminus E.$$

From (4) we compute

$$d^2y/dx^2 = -[(dy/dx)^2 f^{(3)} - (1 - dy/dx)^2 g^{(3)}]/(f'' + g'').$$

Combining the last two equations we obtain

$$h^{(4)}(x) = (dy/dx)^4 f^{(4)} + (1 - dy/dx)^4 g^{(4)}$$
(9)
$$- 3[(dy/dx)^2 f^{(3)} - (1 - dy/dx)^2 g^{(3)}]^2/(f'' + g'').$$

Assume now $0 \in E$. It is sufficient to prove that the expression (9) tends to a limit as $x \to 0$. Consider first the term $(dy/dx)^4 f^{(4)}$. If $f^{(4)}(0) = 0$, this term tends to zero, since dy/dx is bounded. If $f^{(4)}(0) > 0$, dy/dx must tend to a limit according to Lemma 1. Hence $(dy/dx)^4 f^{(4)}$ must always tend to a limit. The second term is treated similarly. The third term in (9) is a little more difficult. Expanding the square in the numerator we get three terms, one of which is

$$(dy/dx)^2(1 - dy/dx)^2 \frac{f^{(3)}g^{(3)}}{f'' + g''} = (dy/dx)^2(1 - dy/dx)^2 \frac{f^{(3)}}{\sqrt{f''}} \frac{g^{(3)}}{\sqrt{g''}} \frac{\sqrt{f'' g''}}{f'' + g''}$$

up to a constant factor. We need only discuss this term, since the other two terms are easier. We may assume $f''(y) > 0$ and $g''(x - y) > 0$, since if not, $f^{(3)}(y) = 0$ or $g^{(3)}(x - y) = 0$, which makes the expression on the left hand side equal to zero. Arguing as above we find that it suffices to prove that the limit of $|f^{(3)}(y)|/f''(y)^{1/2}$ exists as y tends to zero through points where $f''(y) > 0$, and that this limit is zero, if $f^{(4)}(0) = 0$. As we saw above, this is proved by squaring and applying l'Hospital's rule. The proof of Proposition 1 is complete.

4. An example where $f \,\Box\, g \notin C^5$.

Our first step towards a proof of Proposition 2 will be to construct f and g so that the fifth derivative of h is very large. Let $f(x) = f_a(x) = a^2x^2/2$ for $a > 0$ and $g(x) = x^4/4$. To compute $h_a = f_a \,\Box\, g$ we consider the equation (2), i.e. $a^2 y = (x - y)^3$. This equation has a real analytic solution

$$y = a\varphi(x/a), \tag{10}$$

where $y = \varphi(x)$ is the real root of the equation $y = (x - y)^3$. By (3)

$$h_a'(x) = f_a'(a\varphi(x/a)) = a^3\varphi(x/a),$$

and hence

$$h_a^{(5)}(x) = (1/a)\varphi^{(4)}(x/a).$$

We want to make this expression large by taking a small. Now

$$\varphi(x) = x^3 - 3x^5 + 0(x^7), \text{ as } x \to 0,$$

so that $\varphi^{(4)}(0) = 0$. But $\varphi^{(4)}$ is not identically zero, in fact $\varphi^{(4)}(x) = -360x + 0(x^3)$, so obviously

$$\sup_{|x| < a} |h_a^{(5)}(x)| \geqq C/a \tag{11}$$

with $C \neq 0$.

To complete the proof of Proposition 2 we need to piece together infinite sequences of germs of the form f_a and g, or more exactly, multiples of such germs plus linear terms. The following lemma is useful for this purpose.

LEMMA 2. *Assume f_k is a strictly convex C^∞ function defined on $[-4^{-k}, 4^{-k}]$, $k = 1, 2, \ldots$, such that $f_k(0) = f_k'(0)$ and*

$$\sup_{x,k} |f_k^{(r)}(x)| \leqq M_r < \infty, \ r = 1, 2, \ldots . \tag{12}$$

Let b_k be a sequence of positive numbers such that $2^k b_k$ is decreasing and

$$\lim_{k \to \infty} 2^{kN} b_k = 0 \ \text{ for all } N. \tag{13}$$

Then there exists a convex C^∞ germ f at 0, such that $f(0) = f'(0) = 0$ and for all sufficiently large k

$$f(x) - b_k f_k(x - 4^{-k}) - b_k x$$

is constant for $|x - 4^{-k}| < 4^{-k-1}$.

PROOF. Take $\Psi \in C^\infty(R)$ so that $\Psi \geqq 0$, $\Psi = 1$ in $[3/4, 5/4]$, $\operatorname{supp} \Psi \subset [2/3, 3/2]$, and $\sum_{k=1} \Psi(2^k) = 1$ for $0 < x < 1$, and set $\Psi_k(x) = \Psi(2^k x)$. Choose $f(x) = \exp(1/x)$ for $x < 0$, and define $f(x)$ for $x \geqq 0$ so that $f(0) = f'(0) = 0$ and

$$f''(x) = \sum_{k=k} b_k \Psi_{2k}(x) f_k''(x - 4^{-k}) + \sum_{k=K} \alpha_k \Psi_{2k+1}(x),$$

where K and α_k are still to be determined. The requirement that $f'(4^{-k}) = b_k$ leads to a relation for the determination of α_k

$$b_{k-1} - b_k = b_{k-1} A_k + b_k B_k + \alpha_k 2^{-2k-1} d; \tag{14}$$

here $d = \int \Psi \, dx$ and A_k, B_k are positive numbers not greater than $dM_2/4^{k-1}$. For sufficiently large k the values of α_k that are determined from (14) will be positive, hence $f''(x)$ is positive for $x \neq 0$, if K is large enough. Since the sequence b_k is rapidly decreasing according to (13), the sequence α_k must have the same property. This fact together with (12) and (13) implies that $f \in C^\infty$.

PROOF OF PROPOSITION 2. Let a_k denote a sequence of numbers between 0 and 1, which will be specified in a moment, and set $f_k(x) = f_{a_k}(x) = a_k^2 x^2/2$, $g_k(x) = x^4/4$. The estimate (12) is satisfied, and we can construct f by means of Lemma 2. Replacing f_k by g_k we construct g similarly. Set $h = f \,\square\, g$ and $h_k = f_k \,\square\, g_k$. Then

$$f'(x) = b_k + b_k f_k'(x - 4^{-k})$$

for $|x - 4^{-k}| < 4^{-k-1}$, and $g'(x)$ is given by a similar formula. For $|x - 2 \cdot 4^{-k}| < 4^{-k-1}$ the equation (2) therefore means

$$f_k'(y - 4^{-k}) = g_k'(x - y - 4^{-k}) = g_k'(x - 2 \cdot 4^{-k} - (y - 4^{-k})).$$

By (10) the solution $y(x)$ is

$$y - 4^{-k} = a_k \varphi((x - 2\cdot 4^{-k})/a_k), \tag{15}$$

so that

$$h'(x) = f'(y) = b_k + b_k f_k'(a_k \varphi((x - 2\cdot 4^{-k})/a_k)) = b_k + b_k h_k'(x - 2\cdot 4^{-k})$$

for $|x - 2\cdot 4^{-k}| < 4^{-k-1}$. But $h_k = h_{a_k}$, so by (11) the fifth derivative of h must be unbounded, if a_k tends to zero sufficiently fast. The proof is complete.

REMARK. Let $\varepsilon > 0$. If a_k decays so fast that a_k^ε/b_k tends to zero as $k \to \infty$, the function h will not be in $C^{4+\varepsilon}$, i.e. $h^{(4)}$ will not be in Lip (ε). More generally, let ρ be an arbitrary modulus of continuity, i.e. a continuous, increasing, subadditive function from R_+ into R_+ such that $\lim_{t\to 0} \rho(t) = 0$, and choose a_k such that $\rho(a_k)/b_k \to 0$ as $k \to \infty$. Then the modulus of continuity of $h^{(4)}$ can not be estimated by $C\rho(t)$. To see this one can argue as follows. We have seen that $h^{(5)}(2\cdot 4^{-k} + t) = b_k h_k^{(5)}(t) = b_k a_k^{-1} \varphi^{(4)}(t/a_k)$ for $|t| < 4^{-k-1}$. Choose $\delta, 0 < \delta < 1$, so that $\varphi^{(4)}(t) > \delta$ for $\delta < t < 2\delta$. Assume a_k is chosen so that $2\delta a_k < 4^{-k-1}$ for all k. Then $h^{(5)} > b_k\delta/a_k$ on an interval whose length is $\geqq \delta a_k$. This implies that

$$\sup_{x_1,x_2} \frac{|h^{(4)}(x_1) - h^{(4)}(x_2)|}{\rho(|x_1 - x_2|)} > \frac{(b_k\delta/a_k)\delta a_k}{\rho(\delta a_k)} \geqq \frac{\delta^2 b_k}{\rho(a_k)} \to \infty, \text{ as } k \to \infty,$$

which proves the claim.

5. The continuity of the mapping $(f, g) \to f \,\Box\, g$.

It is rather easy to see from (6) that the third derivative

$$h^{(3)}(x) = \Phi_3(x, f, g)$$

is a continuous function on $R \times C^3(J) \times C^3(J)$. The corresponding statement for the fourth derivative is not true, though. Consider for simplicity

$$\Phi_4(f, g) = h^{(4)}(0).$$

This functional is not even continuous with respect to the C^∞-topology. To see this just note that the function

$$h_a(x) = (a^2x^2/2) \,\Box\, (x^4/4)$$

considered in Section 4 has fourth derivative $h_a^{(4)}(0) = \varphi^{(3)}(0) = 6$ for all $a > 0$, whereas $h_a = 0$ for $a = 0$. Incidentally, this shows that $(f + \varepsilon x^2) \,\Box\, g$ in general does not converge in C^4 to $f \,\Box\, g$ as $\varepsilon \to 0$. More generally, using (6) and the arguments in the proof of Lemma 1 one can prove that

$$h^{(4)}(0) = f^{(4)}(0)\, g^{(4)}(0)/(f^{(4)}(0)^{1/3} + g^{(4}(0)^{1/3})^3,$$

if $f''(0) = g''(0) = 0$; the expression on the right hand side should be interpreted as zero if $f^{(4)}(0) = g^{(4)}(0) = 0$. But if $g''(0) = 0$ and $f''(0) > 0$, we have $g^{(3)}(0) = 0$ and $dy/dx = 0$ at $x = 0$, so that (9) gives $h^{(4)}(0) = g^{(4)}(0)$.

These facts together with the remark at the end of Section 4 show that the continuity of $h^{(4)}(x)$ asserted in Proposition 1 is quite delicate.

6. Remarks.

1. Here is an attempt to explain what property of the pair of functions $f_a(x) = a^2x^2/2$ and $g(x) = x^4/4$ is responsible for the fifth derivative of $f_a \,\square\, g$ to be large. Looking at (4) we see that $dy/dx = 0$ if $g''(x - y) = 0$ and $f''(y) > 0$, and $dy/dx = 1$ in the opposite case. Thus, if f'' and g'' vanish at two different, closely situated points, dy/dx must change its value from 0 to 1 over a short interval, hence d^2y/dx^2 must be large; the hope is that this will make $h^{(5)}$ large. This situation occurs for the pair $f_a(x) = (x - a)^4 + 4a^3x - a^4$ and $g_a(x) = f_a(-x)$, if a is small. In an earlier version of this article that pair of functions was used to obtain (11). Later C. O. Kiselman pointed out that one can use the pair considered above and that this gives simpler computations. This pair has actually similar properties although f_a'' is constant; indeed $g''(x)/f_a''(x) = 3x^2/a^2$ varies from 0 to 3 as x runs from 0 to a.

2. In the example constructed above the limit of dy/dx as $x \to 0$ does not exist. This follows from the fact that the quantity $g''(x - y)/f''(y)$ does not have a limit; compare the previous remark. Moreover, we claim that dy/dx does not exist at the origin. In fact, for $x = (2 + t)/4^k$, $|t| < 1/4$ we obtain from (15) $y - 4^{-k} = a_k\varphi(t4^{-k}/a_k)$ and hence

$$y/x = (1 + 4^k a_k \varphi(t/4^k a_k))/(2 + t).$$

But $4^k a_k \to 0$ and $\lim_{\varepsilon \to 0} \varepsilon\varphi(t/\varepsilon) = t$, hence for $|t| < 1/4$

$$\frac{y}{x} = \frac{1 + t}{2 + t}(1 + o(1)), \text{ as } k \to \infty.$$

Since the function $(1 + t)/(2 + t)$ is non-constant, this proves that y/x does not have a limit.

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