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SMOOTHNESS OF SUMS OF CONVEX SETS WITH REAL ANALYTIC BOUNDARIES

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1. Introduction.

Given two compact, convex sets in R^n, A and B, with smooth boundaries, how smooth must the boundary of the vector sum $A + B$ be? For plane sets this problem was studied in [K2]. If ∂A and ∂B are C^∞, $\partial(A + B)$ must be C^4; if ∂A and ∂B are real analytic, $\partial(A + B)$ must be $C^{20/3}$. Counterexamples show that those statements are sharp (see also [B]).

If the dimension n is arbitrary and the boundaries of A and B are C^∞, it is known that $\partial(A + B)$ must be $C^{1,1}$ (see [KP]) and that $\partial(A + B)$ is not always C^2. The latter statement follows from an example given by C. O. Kiselman in a different context: a convex set in R^3 with C^∞ boundary is constructed, whose plane shadow does not have C^2 boundary ([K2], [K1] p. 243). For dimensions $n > 2$ and the boundaries of A and B real analytic nothing seems to be known about our problem apart from the fact that $\partial(A + B)$ must be $C^{1,1}$ and the two-dimensional result that $\partial(A + B)$ may be as bad as $C^{20/3}$. Here we give the solution to this problem for $n \geqq 4$.

THEOREM 1. *There exist two compact, convex sets A, B in R^4 with real analytic boundaries, such that the boundary of $A + B$ is not C^2.*

The analogous statement is true in R^n for any $n \geqq 4$; see Remark 2 at the end of the paper. However, we do not know if there exists a similar example in R^3.

2. Preliminaries.

As was explained in [K2] our problem is equivalent to the study of the infimal convolution

$$f \,\square\, g(x) = \inf_y (f(y) + g(x - y))$$

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of two convex germs at the origin. Here f and g are (germs of) functions of $n-1=d \geqq 2$ variables, whose epigraphs are equal to A and B locally near $x=0$. We may assume $f(0)=g(0)$, and $f'(0)=g'(0)=(0,\ldots,0)$.

Set $f \,\square\, g = h$. If f and g are C^1 we have (see [K2] and [B])

$$h'(x) = f'(y) = g'(x-y), \tag{1}$$

where y is a solution to the equation

$$f'(y) = g'(x-y). \tag{2}$$

If f or g is strictly convex, the solution to (2) is unique and depends continuously on x. If f and g are C^2 and $f''(0)+g''(0)$ is non-singular, one can prove $h \in C^2$ and obtain an expression for $h''(x)$ generalizing (5) in [B] as follows. Let $\partial y/\partial x$ be the matrix $(a_{jk}) = (\partial y_j/\partial x_k)$ and I the identity matrix. Differentiating (1) and (2) we obtain

$$h''(x) = f''(y)\,\partial y/\partial x = g''(x-y)(I - \partial y/\partial x).$$

Hence $\partial y/\partial x$ can be solved from the equation

$$(f''(y) + g''(x-y))\,\partial y/\partial x = g''(x-y),$$

so that

$$h''(x) = f''(y)(f''(y)+g''(x-y))^{-1}g''(x-y). \tag{3}$$

3. Definition of the convex germs.

It is natural to ask if the expression (3) must have a limit as $|x| \to 0$, if $f''(0)+g''(0)$ is a singular matrix. Looking at this as a question of pure matrix algebra we may ask if the matrix $F(F+G)^{-1}G$ must tend to a limit, if the positive definite, symmetric matrices F and G tend to a singular matrix. The answer is no; a counterexample is given by

$$F = \begin{pmatrix} 1 & b \\ b & d^2 \end{pmatrix} \qquad G = \begin{pmatrix} 1 & 0 \\ 0 & d^2 \end{pmatrix},$$

where $|b| < d$ and $d \to 0$. In fact $F(F+G)^{-1}G = H = (h_{ij})$, where $h_{11} = (2d^2 - b^2)/(4d^2 - b^2)$. We therefore look for functions f and g whose second derivatives resemble F and G, respectively. For reasons that will be explained later this idea does not work for functions of two variables, so we have to consider functions of three variables. Choose

$$g_0(x) = g_0(x_1,x_2,x_3) = x_1^2/2 + x_2^2/2 + (x_3^2/2)(x_1^2 + x_2^2 + x_3^2/6)$$

and

$$f_0(x) = g_0(x) + c x_1 x_2 x_3,$$

where $|c| \leqq 1$. The matrix $g_0''(x) = (\partial_j \partial_k g_0(x))$ can be written

$$g_0''(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & |x|^2 \end{pmatrix} + R(x),$$

where $R(x) = (r_{ij}(x)) = O(|x|^2)$ as $|x| \to 0$, and $r_{33}(x) = 0$. Similarly

$$f_0''(x) = g_0''(x) + \begin{pmatrix} 0 & cx_3 & cx_2 \\ cx_3 & 0 & cx_1 \\ cx_2 & cx_1 & 0 \end{pmatrix}.$$

It is clear that g_0 is strictly convex. To see that f_0 is strictly convex we compute

$$\det f_0''(x) = |x^2| - c^2(x_1^2 + x_2^2) + O(|x|^3), \quad \text{as} \quad |x| \to 0,$$

which is positive if $|c| < 1$ and $|x|$ is sufficiently small, $|x| \neq 0$. The fact that f_0'' is positive definite for small x outside the origin now follows from Jacobi's criterion [S, p. 127]: a sequence of ascending minors have positive determinant.

PROPOSITION 1. *Let f_0 and g_0 be defined as above, $|c| < 1$. Then $f_0 \square g_0 \notin C^2$. More generally, if f and g are arbitrary real analytic germs, whose Taylor expansions at the origin to fourth order coincide with those of f_0 and g_0, then $f \square g \notin C^2$.*

The second statement of the proposition will be needed in the proof of Theorem 1.

It is obvious that $f_0''(y) + g_0''(x-y)$ is nonsingular for sufficiently small $|x| \neq 0$, hence formula (3) is valid for such x. Our goal is to show that this expression does not have a limit as $|x| \to 0$.

4. Study of h''.

LEMMA 1. *Let f and g be defined as in Proposition 1, and $h = f \square g$. Then $\partial_1^2 h(x) = -c^2 y_2^2 / Q(x, y) + w(x)$, where Q is a positive definite quadratic form in x and y, $y = y(x)$ is the solution to (2), and $w(x)$ is a continuous function. More precisely,*

$$Q(x, y) = 4|y|^2 + 4|x-y|^2 - 2c^2(y_1^2 + y_2^2).$$

PROOF. Set $f''(y) + g''(x-y) = T$ and $f''(y) - g''(x-y) = S$. Then according to (3)

$$h''(x) = (1/4)(T+S)T^{-1}(T-S) = -(1/4)ST^{-1}S + \dots, \tag{4}$$

where the omitted terms are continuous. Write $S = (s_{ij})$ and denote the cofactors of T by T_{jk}. By (3) and (4)

$$4\partial_1^2 h(x) + (\det T)^{-1} \sum_{j,k} s_{1j} T_{jk} s_{k1}$$

is continuous. Now $s_{ij} = O(|x| + |y|)$ for all i, j, and $\det T = Q(x, y) + O(|x|^3 + |y|^3)$, as $|x| \to 0$. Therefore only zero order terms in T_{jk} can give non-continuous contributions to $\partial_1^2 h$. The only cofactor containing a constant term is

$$T_{33} = 4 + O(|x|^2 + |y|^2), \quad \text{as} \quad |x| \to 0.$$

Hence

$$-4\partial_1^2 h(x) = (1/Q)s_{13}T_{33}s_{31} + \dots = 4c^2 y_2^2/Q + \dots,$$

where the omitted terms are continuous. The proof is complete.

To be able to prove that the function $x \mapsto y_2^2/Q(x, y)$ is discontinuous we must study the solution $y = y(x)$ to the equation (2).

LEMMA 2. *Let $y = y(x)$ be determined by the equation* (2), *where f and g are defined as in Proposition* 1. *Then there exists a number $\delta > 0$ such that*

$$(5) \qquad |x| < \delta \quad \text{and} \quad x_1 x_2 = 0$$

implies

$$(6) \qquad |y| \leqq C|x| \quad \text{and} \quad y_j = x_j/2 + O(|x|^2) \quad \text{as} \quad |x| \to 0 \quad \text{for} \quad j = 1, 2.$$

PROOF. The first two equations of the system (2) are

$$y_1 a(y_3) + c y_3 y_2 = (x_1 - y_1)a(x_3 - y_3) + \dots$$

$$y_2 a(y_3) + c y_3 y_1 = (x_2 - y_2)a(x_3 - y_3) + \dots,$$

where $a(t) = 1 + t^2$ and the higher order terms are omitted. These equations can be written

$$(7) \qquad \begin{cases} 2y_1 = x_1 + w_1 \\ 2y_2 = x_2 + w_2, \end{cases}$$

where $w_j = w_j(x, y) = O(|x|^2 + |y|^2)$. Applying the Implicit Function Theorem we can solve y_1 and y_2 in terms of x and y_3 from this system; we then obtain (7) with new functions w_j satisfying the same estimates and depending only on x and y_3. We next want to use the third equation of the system (2) for estimating y_3. This equation can be written

$$(8) \qquad y_3(b(y) + b(x - y)) + c y_1 y_2 = x_3 b(x - y) + \dots,$$

where $b(y) = y_1^2 + y_2^2 + y_3^2/3$, and terms of order $\geqq 4$ are omitted. Now

$$b(y) + b(x - y) \geqq (1/3)(|y|^2 + |x - y|^2) \geqq |x|^2/6.$$

We need an estimate for $y_1 y_2$. From (7) we obtain, if $x_1 x_2 = 0$,

$$(9) \quad 4|y_1 y_2| \leqq |x_1 w_2 + x_2 w_1 + w_1 w_2| \leqq C(|x|(|x|^2 + |y_3|^2) + |x|^4 + |y_3|^4).$$

The higher order terms in (8) are initially known to be $\leqq C(|x|^4 + |y|^4)$; applying (7) we see that those terms are in fact majorized by $C(|x|^4 + y_3^4)$. Using alternatively $|y|^2$ and $|x|^2/2$ as lower bound for $|y|^2 + |x - y|^2$ we obtain from (8) and (9)

$$|y_3| \leqq C(|x| + |y_3|^2),$$

which implies

$$|y_3| \leqq C|x|,$$

if $|x| < \delta$ and δ is small enough. Combining this with (7) we obtain (6). The proof is complete.

PROOF OF PROPOSITION 1. By Lemma 1 it is enough to prove that the function $k(x) = y_2^2/Q(x, y)$ is discontinuous at the origin. But Lemma 2 shows that $k(0, t, 0)$ for $t \neq 0$ is bounded away from zero, whereas $k(t, 0, 0) = 0$.

REMARK. The conclusion of Lemma 2 is not true if the assumption $x_1 x_2 = 0$ in (5) is omitted. In fact, if $x = \omega t, \omega \in S^2, t \in \mathsf{R}$, the solution to (2) is $y_j = \omega_j t/2 + o(t)$ for $j = 1, 2$, and

$$y_3 = -(3c\omega_1\omega_2/8)^{1/3}|t|^{2/3} + o(|t|^{2/3}), \text{ as } t \to 0.$$

This implies that the limit of $y_2^2/Q(x, y)$ is zero on all rays through the origin except $t \mapsto (0, t, 0)$.

5. Construction of the convex sets.

PROOF OF THEOREM 1. Denoting points in R^4 by (x, z), $x \in \mathsf{R}^3$, set $u(z) = z^4 - z$ and

$$F(x, z) = u(z) + \varepsilon f_0(x) + |x|^6, \quad G(x, z) = u(z) + \varepsilon g_0(x) + |x|^6,$$

for some small $\varepsilon > 0$ to be determined later, and let A and B be the sets determined by $F \leqq 0$ and $G \leqq 0$, respectively. Then A and B are compact, and since F and G are convex, A and B must be convex. To see that the boundary of A is real analytic we must check that the gradient of F does not vanish when $F = 0$. Now $\partial F/\partial z = u'(z)$ vanishes only for $z = z_0 = 4^{-1/3}$, the minimum point of $u(z)$. It is clear that we may choose ε so small that the gradient of $|x|^6 + \varepsilon f_0(x)$ is different from zero whenever $F(x, z_0) = 0$, i.e. $|x|^6 + \varepsilon f_0(x) = -u(z_0) = 3 \cdot 4^{-4/3}$.

We finally need to check that ∂A near $x = z = 0$ is given by an equation

$$z = \varepsilon f_0(x) + v(x),$$

where v is real analytic and $v(x) = O(|x|^5)$ as $|x| \to 0$. In fact, since $u'(0) = -1$, $u''(0) = 0$, and $f_0(x) = O(|x|^2)$, the equation $F(x, z) = 0$ gives $z = \varepsilon f_0(x) + O(|x|^6)$. The corresponding statements for the set B are of course verified similarly. The fact that $\partial(A + B)$ is not C^2 at the origin is now a consequence of Proposition 1.

REMARKS. 1. It is rather easy to see that the function $x_1 x_2 x_3$ occurring in the definition of $f_0(x)$ can be replaced by any function of the form $x_3 q(x_1, x_2)$, where q is a second order homogeneous polynomial with some non-degenerate zero outside the origin. For instance we could take q equal to $x_1^2 - x_2^2$. The fact that there exists no polynomial in one variable with those properties is the reason why we could not construct functions f and g on R^2 with the properties in Proposition 1.

2. It is easy to see that the statement of Theorem 1 is true in R^n for any $n \geqq 4$. Denote points in R^n, $n \geqq 5$, by (x, z, t), $x \in \mathsf{R}^3$, $z \in \mathsf{R}$, $t \in \mathsf{R}^{n-4}$, let A be the set $F \leqq 0$, where

$$F(x, z, t) = \varepsilon f_0(x) + |x|^6 + u(z) + |t|^2,$$

and construct B similarly. Then the boundaries of A and B near the origin will be given by $z = f(x, t)$ and $z = g(x, t)$, where $f(x, t) = \varepsilon f_0(x) + |t|^2 + O(|x|^6 + |t|^6)$, $g(x, t) = \varepsilon g_0(x) + |t|^2 + O(|x|^6 + |t|^6)$, as $(x, t) \to (0, 0)$. The arguments in Lemma 1 and Lemma 2 are valid for this pair of functions and therefore the boundary of $A + B$ will not be C^2.

3. The fact that $f, g \in C^2$ implies $f \,\Box\, g \in C^{1,1}$, the class of C^1-functions with Lipschitz continuous first derivatives, can be proved as follows. For $\varepsilon > 0$, set $f_\varepsilon(x) = f(x) + \varepsilon |x|^2$ and set $h_\varepsilon = f_\varepsilon \,\Box\, g$. Then $f_\varepsilon'' \geqq 2\varepsilon I$, so that $h_\varepsilon \in C^2$ and (3) holds. If F and G are symmetric, positive definite matrices, the norm of $F(F + G)^{-1} G$ can be estimated by $\|F\| (\|F\| + \|G\|)^{-1} \|G\|$ ([AD], Theorem 25). Hence the second derivative h_ε'' must be uniformly bounded as $\varepsilon \to 0$. By Arzela's theorem there is a uniformly convergent sequence h'_{ε_k}, $k = 1, 2, \ldots$, converging to some function u, which is Lipschitz continuous. By a theorem of elementary calculus u must be equal to h'.

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