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ON COMPLEX INTERPOLATION WITH AN ANALYTIC FUNCTIONAL

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Abstract.

We consider the variant of the Calderón complex interpolation method associated to an analytic functional.

Some examples related to the domain of positive operators and L^p spaces are given.

§1. Introduction.

Given an interpolation pair (A_0, A_1) of complex Banach spaces, the Calderón method (see [1]) $[A_0, A_1]_\theta$ has been extended (see [5] and [6]) to define the interpolation space $[A_0, A_1]_R$ associated to an analytic functional, $R \in \mathscr{H}'(\Omega)$, on the strip $\Omega = \{0 < \operatorname{Re} z < 1\}$. As in the case of distributions, R acts on vector valued functions $f \in \mathscr{H}(\Omega; A_0 + A_1)$, and $R(f)$ can be considered instead of the evaluation $f(\theta)$. That is,

$$[A_0, A_1]_R = \{R(f); f \in \mathscr{F}(A_0, A_1)\}, \text{ with } \|b\|_R = \inf\{\|f\|_{\mathscr{F}}; f \in \mathscr{F}, R(f) = b\}.$$

We say that an analytic functional T on an open set $G \subset C$ is of finite support, if T admits a representation of the type

$$T = \sum_{j=0}^{n} \sum_{l=0}^{m(j)} a_{jl} \delta^{(l)}(z_j), \tag{1}$$

and $\{z_0, z_1, \ldots, z_n\}$ is said to be the support of T.

We have studied the complex interpolation when R is of this type in [2] and [3]. Now, the aim is to identify some interpolation spaces when the analytic functional is not of finite support and to get some consequences of this identification.

We include the extension of the above method to complex interpolation families (c.i.f.) in the sense of [4]:

Let D denote the disc $\{|z| < 1\}$ and Γ its boundary, and let $\{B(\gamma), \gamma \in \Gamma\}$ be a c.i.f.

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on Γ with the containing complex Banach space $\mathscr{V}$ and with $\mathscr{B}$ as log-intersection space. The Banach space $B[z] = \{f(z); f \in \mathscr{F}\}$ with the norm $\|b\|_z = \inf\{\|f\|_{\mathscr{F}}; f(z) = b\}$, for every $z \in D$, is defined in [4]. Here $\mathscr{F} = \mathscr{F}(B(\cdot), \Gamma)$ is a Banach space of $\mathscr{V}$-valued analytic functions f on D with a.e. non-tangential boundary values $f(\gamma) = \mathscr{V}\text{-}\lim_{\xi \to \gamma} f(\xi)$, which can be described as the completion of the space

$$\mathscr{G} = \{g = \sum_{j=1}^{N} \varphi_j(\cdot) b_j; \mathrm{b}_j \in \mathscr{B}, \varphi_j \in N^+(D), \operatorname*{ess\,sup}_{\gamma \in \Gamma} \|g(\gamma)\|_\gamma < +\infty\},$$

with the norm $\|f\|_{\mathscr{F}} = \operatorname*{ess\,sup}_{\gamma \in \Gamma} \|f(\gamma)\|_\gamma$.

If T is an analytic functional on D, we define, for the c.i.f. $\{B(\gamma)\}$, the space

$$B[T] = \{T(f); f \in \mathscr{F}\}, \text{ with } \|b\|_{[T]} = \inf\{\|f\|_{\mathscr{F}}; f \in \mathscr{F}, T(f) = b\}.$$

$B[T]$ is a Banach space and the following theorem, which is an extension of a similar result derived in [4] for $B[z]$ $(T = \delta(z))$, is easily proved.

THEOREM 1. *Let $\{A(\gamma)\}$ and $\{B(\gamma)\}$ be two* c.i.f. *on Γ, with the containing spaces $\mathscr{U}$ and $\mathscr{V}$, and the log-intersection spaces $\mathscr{A}$ and $\mathscr{B}$, respectively.*

Let $L: \mathscr{U} \to \mathscr{V}$ be a linear bounded operator and suppose that $L(\mathscr{A}) \subset \cap_{\gamma \in \Gamma} B(\gamma)$ with

$$\|La\|_\gamma \leqq M(\gamma)\|a\|_\gamma \quad \text{a.e. } \gamma \in \Gamma,$$

where $\log M(\gamma)$ *is integrable. Then $L: A[GT] \to B[T]$ continuously with norm $\leqq 1$, for*

$$G(z) = \exp\left(\int_\Gamma -\log M(\gamma)\, dH_z(\gamma)\right)$$

and $H_z(\gamma) = (1/2\pi)(e^{i\gamma} + z/e^{i\gamma} - z)$ the Herglotz kernel.

Let $\alpha: \Gamma \to [0, 1]$ be a measurable function and consider the c.i.f.

$$\{A(\gamma) = [A_0, A_1]_{\alpha(\gamma)}; \gamma \in \Gamma\}.$$

Define the analytic funtional S on Ω by $S(\varphi) = T(\varphi \circ w)$, where $w(z) = \alpha(z) + i\tilde{\alpha}(z)$, and $\alpha(z)$ is the Poisson integral of α. It is known (see [4]) that, if $T = \delta(z_0)$,

$$[A_0, A_1]_S = A[T]. \tag{2}$$

For any $F \in \mathscr{F}(A_0, A_1)$, $F \circ w \in \mathscr{F}(A(\cdot), \Gamma)$ and $\|F \circ w\|_{\mathscr{F}} \leqq \|F\|_{\mathscr{F}}$ imply the following.

PROPOSITION 2. *$[A_0, A_1]_S$ is continuously embedded in $A[T]$.*

The converse is not always true. In fact, it may happen that $S = 0$ and $T \neq 0$. However we have proved in [2] that, if T is of the type (1) and under some natural conditions, the equivalence (2) is true.

In the general case, we have not found conditions for this equivalence to hold.

Let E be a complex Banach space. In the sequel $A(\Omega; E)$ will denote the space of the E-valued analytic functions $F \in \mathscr{H}(\Omega; E)$ which are continuous and bounded on the closed strip.

§2. Complex Powers of Linear Operators.

Let $L: D(L) \to A$ be a positive linear operator densely defined in a Banach space A and such that $\|L^{it}\| \leqq M$ for each $t \in R$, where L^{α} is the complex power of L. This condition can be relaxed to $\|L^{it}\| \leqq M$ for all $-\varepsilon \leqq t \leqq \varepsilon$ (see [7]).

Let S be an analytic functional on the band Ω and $0 < \xi_0 < 1$. Throughout this paper $C: D \to \Omega$ will be a fixed conformal map with $C(0) = \xi_0$.

Define $T(\varphi) = S(\varphi \circ C^{-1})$, an analytic functional on D.

For any $F \in \mathscr{H}(\Omega)$ we have

$$S(F) = T(F \circ C) = \sum_{n=0}^{\infty} \frac{(F \circ C)^{(n)}(0)}{n!} T(z^n)$$

and, if α_j^n is the coefficient of $F^{(j)}(\xi_0)$ in $(F \circ C)^{(n)}(0)$,

$$S(F) = \sum_{n=0}^{\infty} \frac{T(z^n)}{n!} \sum_{j=0}^{n} \alpha_j^n F^{(j)}(\xi_0)$$

with absolute convergence. Thus, if we call

$$A(S, C; j) = \sum_{m=0}^{\infty} S((C^{-1}(\xi))^{m+j}) \frac{\alpha_j^{m+j}}{(m+j)!},$$

we have

$$\begin{aligned} S(F) &= \sum_{j=0}^{\infty} \sum_{n=j}^{\infty} \frac{T(z^n)}{n!} \alpha_j^n F^{(j)}(\xi_0) = \\ &= \sum_{j=0}^{\infty} \left(\sum_{m=0}^{\infty} \frac{T(z^{m+j})}{(m+j)!} \alpha_j^{m+j} \right) F^{(j)}(\xi_0) = \\ &= \sum_{j=0}^{\infty} A(S, C; j) F^{(j)}(\xi_0). \end{aligned}$$

Let us consider $a \in D(L^3)$. The function $F(\xi) = L^\xi a : \bar{\Omega} \to A$ is analytic in Ω and

$$F'(0) = 2\int_0^\infty \left(\log t + \frac{C'(0)}{2}\right) L^2(L + tI)^{-3} a\, dt,$$

where $C(\alpha) = 2/\Gamma(1 + \alpha)\Gamma(2 - \alpha)$ (*see* [7]).

It is natural to denote $(\log L)a = F'(0)$, and $(\log L)^p a = F^{(p)}(0)$. $\log L$ and $(\log L)^p$ will be the closure of these operators. Then, a straightforward calculation proves the following

Lemma 3. *If* $\phi_m(\xi) = (\xi - \xi_0)^m/m!$, *then*

$$S(L^{-\xi+\xi_0}\phi_m) = \sum_{p=0}^{\infty} \binom{m+p}{p} A(S, C; m + p)(-\log L)^p.$$

Proof From the above remarks we have

$$\begin{aligned} S(L^{-\xi+\xi_0}\phi_m) &= \sum_{j=0}^{\infty} A(S, C; j)(L^{-\xi+\xi_0}\phi_m)^{(j)}(\xi_0) = \\ &= \sum_{j=0}^{\infty} A(S, C; j)\left(\sum_{p=0}^{j} \binom{j}{p} (L^{-\xi+\xi_0})^{(p)}_{|\xi_0} \phi_m^{(j-p)}(\xi_0)\right) = \\ &= \sum_{j=0}^{\infty} A(S, C; j)\binom{j}{j-m}(\log L)^{j-m} = \\ &= \sum_{p=0}^{\infty} \binom{m+p}{p} A(S, C; m + p)(-\log L)^p. \end{aligned}$$

Let

$$E = \{(A_m)_m \subset D(L^{\xi_0}); \exists \varphi \in A(\Omega; D(L^{\xi_0})), \varphi^{(m)}(\xi_0) = A_m, \forall m \in \mathbb{N}\}$$

with the norm $\|(A_m)_m\|_E = \|\varphi\|_\infty$.

Consider the operator $\varphi_L^S : E \to A$ defined by

$$\varphi_L^S((A_m)_m) = \sum_{m=0}^{\infty} S(L^{-\xi+\xi_0}\phi_m)A_m,$$

and its range $R[\varphi_L^S]$.

Proposition 4. *The space* $[A, D(L)]_S$ *is equivalent to the space* $R[\varphi_L^S]$ *with the norm*

$$\|x\|_R = \inf\{\|(A_m)_m\|_E; \varphi_L^S((A_m)_m) = x\}.$$

Proof Let $x \in R[\varphi_L^S]$ and $(A_m)_m \in E$ with $x = \varphi_L^S((A_m)_m)$. Let $\varphi \in A(\Omega; D(L^{\xi_0}))$ such that $\varphi^{(m)}(\xi_0) = A_m$ for each $m \in \mathbb{N}$. A straightward calculation proves that

$\varphi_L^S((A_m)_m) = S(L^{-\xi+\xi_0}\varphi)$. But $L^{-\xi+\xi_0}\varphi(\xi)$ is in $\mathscr{F}(A, D(L))$ and

$$\|L^{-\xi+\xi_0}\varphi(\xi)\|_{\mathscr{F}} \ll \|\varphi\|_\infty$$

(where $X \ll Y$ means that $X \leqq CY$ for a certain constant C).

Thus, $x = \varphi_L^S((A_m)_m) \in [A, D(L)]_S$ and $\|x\|_S \ll \|\varphi\|_\infty = \|(A_m)_m\|_E$.

Conversely, let $x \in [A, D(L)]_S$ and $F \in \mathscr{F}(A, D(L))$ with $S(F) = x$. By expresing $F(\xi) = L^{-\xi+\xi_0}L^{-\xi_0}L^{\xi}F(\xi)$ and, using the previous lemma, one can easily obtain $x = S(F) = \varphi_L^S((A_m)_m)$ with $(A_m)_m = ((L^{-\xi_0}L^{\xi}F(\xi))^{(m)}(\xi_0))_m$. Hence, $x \in R[\varphi_L^S]$. Moreover,

$$\|x\|_R \leqq \|(A_m)_m\|_E = \|L^{-\xi_0}L^{\xi}F(\xi)\|_\infty = \sup_{\xi\in\Omega} \|L^{-\xi_0}L^{\xi}F(\xi)\|_{D(L^{\xi_0})} \ll \|F\|_{\mathscr{F}(A,D(L))}.$$

EXAMPLE I. Define $L: D(L) \to L^2$ by $L(f) = \mathscr{F}^{-1}(1 + |x|^2)\mathscr{F}(f)$ for each $f \in \mathscr{S}(\mathrm{R}^n)$ and $\mathscr{F}$ the Fourier transform. It is known that $D(L) = H_2^2(\mathrm{R}^n)$ and $\|L^{it}\| \leqq 1$ for all $t \in \mathrm{R}$. Thus,

$$[L^2(\mathrm{R}^n), H_2^2(\mathrm{R}^n)]_S \equiv R[\varphi_L^S],$$

where "≡" means equality of spaces with equivalence of norms.

Let $P \in A(\Omega; C)$ such that $\mu[S, P](x) = S(P(1 + |x|^2)^{-\xi})^{-1}$ belongs to $\mathscr{S}'(\mathrm{R}^n)$, and

$$H_2^{\mu[S,P](x)} = \{f \in L^2(\mathrm{R}^n); \mathscr{F}^{-1}\mu[S, P](x)\mathscr{F}(f) \in L^2(\mathrm{R}^n)\},$$

with the norm $\|f\|_\mu = \|\mathscr{F}^{-1}\mu[S, P](x)\mathscr{F}f\|_2$, the corresponding Sobolev space.

PROPOSITION 5. *$H_2^{\mu[S,P](x)}$ is continuously embedded in $[L^2(\mathrm{R}^n), H_2^2(\mathrm{R}^n)]_S$.*

PROOF Let $f \in H_2^{\mu[S,P](x)}$ and $(\varphi_n)_n \subset \mathscr{S}(\mathrm{R}^n)$ such that $f = H_2^{\mu[S,P](x)} - \lim_n \varphi_n$. Consider $\Psi_n = P(\xi)\mathscr{F}^{-1}\mu[S, P](x)(1 + |x|^2)^{-\xi_0}\mathscr{F}(\varphi_n)$. It is easy to prove that

$$S(L^{-\xi+\xi_0}\Psi_n) = \varphi_n \qquad \forall n \in \mathrm{N},$$

and that $L^{-\xi+\xi_0}\Psi_n \in \mathscr{F}(L^2, H_2^2)$ with

$$\|L^{-\xi+\xi_0}\Psi_n\|_{\mathscr{F}(L^2,H_2^2)} \ll \|\varphi_n\|_\mu.$$

Therefore

$$\|\varphi_n\|_{[L^2(\mathrm{R}^n), H_2^2(\mathrm{R}^n)]_S} \ll \|\varphi_n\|_\mu$$

and $(\varphi_n)_n$ is a Cauchy sequence in $[L^2(\mathrm{R}^n), H_2^2(\mathrm{R}^n)]_S$.

Clearly $f = \lim_n \varphi_n$ in $[L^2(\mathrm{R}^n), \mathrm{H}_2^2(\mathrm{R}^n)]_S$, and the proof is ended.

PROPOSITION 6. *Let $\xi^* = \max(1/\xi_0, 1/(1 - \xi_0))$. Assume that there exists $P \in A(\Omega; C)$ and $(a_q)_q \subset \mathrm{R}^+$ such that*

(a) $|S((\xi - \xi_0)^q(1 + |x|^2)^{-\xi})| \ll a_q|S(P(1 + |x|^2)^{-\xi})|$ *for each $q \in \mathrm{N}$, and*

(b) $k = \sum_{q=0}^{\infty} a_q(\xi^*)^q < +\infty$, *then*

$$[L^2(\mathsf{R}^n), H_2^2(\mathsf{R}^n)]_S \equiv H_2^{\mu[S,P](x)}.$$

PROOF If $f \in [L^2(\mathsf{R}^n), H_2^2(\mathsf{R}^n)]_S$, there exists $\varphi \in A(\Omega; H_2^{2\xi_0})$ such that

$$f = \sum_{q=0}^{\infty} S(L^{-\xi+\xi_0}(\xi-\xi_o)^q)\frac{\varphi^{(q)}(\xi_0)}{q!}$$

We know that φ can be taken as $L^{-\xi_0}L^{\xi}F(\xi)$, with F in $\mathscr{F}(L^2, H_2^2)$ such that $S(F) = f$. Every $F \in \mathscr{F}(L^2, H_2^2)$ can be approximated by functions $H = \sum' \varphi_j b_j$ ($\varphi_j \in A(\Omega; C)$, $b_j \in H_2^2$) and H, $\mathscr{S}(\mathsf{R}^n)$ being dense in H_2^2, by functions $G = \sum' \varphi_j a_j$ ($\varphi_j \in A(\Omega; C)$ and $a_j \in \mathscr{S}(\mathsf{R}^n)$). Thus, let $G_n = \sum_{j=1}^{N_n} \varphi_j^n b_j^n$ ($\varphi_j^n \in A(\Omega; C)$ and $b_j^n \in \mathscr{S}(\mathsf{R}^n)$) a sequence which approximates F.

Note that $\varphi_n(\xi) = \mathrm{L}^{-\xi_0 L^{\xi}} G_n(\xi) = \mathscr{F}^{-1}(1+|x|^2)^{-\xi+\xi_0} F\, G_n(\xi)$ takes its values in $\mathscr{S}(\mathsf{R}_n)$, and $\|\varphi_n\|_{A(\Omega, H_2^{2\xi_0})} \leqslant \|\mathbf{G}_n\|_{\mathscr{F}}$.

Consider, for each $n \in \mathsf{N}$,

$$\begin{aligned} f_n &= \sum_{q=0}^{\infty} S(L^{-\xi+\xi_0}(\xi-\xi_0)^q)\frac{\varphi_n^{(q)}(\xi_0)}{q!} = \\ &= \sum_{q=0}^{\infty} S(\mathscr{F}^{-1}(1+|x|^2)^{-\xi+\xi_0}\mathscr{F}(\xi-\xi_0)^q)\frac{\varphi_n^{(q)}(\xi_0)}{q!} = \\ &= \sum_{q=0}^{\infty} \mathscr{F}^{-1}(S((1+|x|^2)^{-\xi+\xi_0}(\xi-\xi_0)^q))\mathscr{F}\frac{\varphi_n^{(q)}(\xi_0)}{q!}. \end{aligned}$$

Then,

$$\mathscr{F}^{-1}\mu[S,P](x)\mathscr{F}f_n = \sum_{q=0}^{\infty} \mathscr{F}^{-1}\mu[S,P](x)S((1+|x|^2)^{-\xi+\xi_0}(\xi-\xi_0)^q)\mathscr{F}\frac{\varphi_n^{(q)}(\xi_0)}{q!}.$$

and

$$\begin{aligned} &\left\|\mathscr{F}^{-1}\mu[S,P](x)S((1+|x|^2)^{-\xi}(\xi-\xi_0)^q)(1+|x|^2)^{\xi_0}\mathscr{F}\frac{\varphi_n^{(q)}(\xi_0)}{q!}\right\|_2 \leqslant \\ &\leqslant a_q\left\|\mathscr{F}^{-1}(1+|x|^2)^{\xi_0}\mathscr{F}\frac{\varphi_n^{(q)}(\xi_0)}{q!}\right\|_2 = a_q\left\|\frac{\varphi_n^{(q)}(\xi_0)}{q!}\right\|_{H_2^{2\xi_0}} \leqslant \\ &\leqslant a_q(\xi^*)^q\,\|\varphi_n\|_\infty. \end{aligned}$$

From (b), we have $\|\mathscr{F}^{-1}\mu[S,P](x)\mathscr{F}f_n\|_2 \leqslant \|\varphi_n\|_\infty$. So, $(f_n)_n$ is a Cauchy sequence in $H_2^{\mu[S,P](x)}$. Finally, it is clear that $f \in H_2^{\mu[S,P](x)}$ and $\|f\|_\mu \leqslant \|F\|_{\mathscr{F}}$.

REMARK (1) If $S = \delta(\xi_0)$, the previous hypotheses are satisfied for $P = 1$. Thus,

$$[L^2(\mathsf{R}^n), H_2^2(\mathsf{R}^n)]_S \equiv H_2^{\mu[S,P](x)},$$

with $\mu[S,P](x) = (1+|x|^2)^{\xi_0}$. That is,

$$[L^2(\mathsf{R}^n), H_2^2(\mathsf{R}^n)]_S \equiv H_2^{\mu[S,P](x)} = H_2^{2\xi_0}(\mathsf{R}^n),$$

as proved in [7] (2.4.2., Rem. 2).

(2) If $S = \delta'(\xi_0)$, we must take $P = \exp(\xi_0 - \xi)$ and thus,

$$\mu[S,P](x) = \frac{(1+|x|^2)^{\xi_0}}{1+\log(1+|x|^2)},$$

as proved in [3].

(3) Let S be the analytic functional on Ω defined by

$$S(\varphi) = \frac{1}{2\pi i}\int_\Sigma \varphi(\xi)(\xi-\xi_0)^{-1}\exp(-(\xi-\xi_0)^{-1})d\xi,$$

with Σ a simple rectifiable close curve in Ω with ξ_0 in its interior. One can prove that the hypotheses are satisfied for $P = 1$. Thus H_2^μ is equivalent to $[L^2(\mathsf{R}^n), H_2^2(\mathsf{R}^n)]_S$ where

$$\mu(x) = (S((1+|x|^2)^{-\xi}))^{-1} = \left(\sum_{n=0}^{\infty}\frac{\log(1+|x|^2)^n}{n!\,n!}\right)^{-1}(1+|x|^2)^{\xi_0}.$$

(4) For each $0 < \varepsilon < \frac{1}{2}$, the analytic functional

$$S(\varphi) = \frac{1}{2\varepsilon}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon}\varphi(x)\,dx$$

satisfies the hypotheses for $P = 1$, (we can choose $\xi_0 = 1/2$). So, $[L^2(\mathsf{R}^n), H_2^2(\mathsf{R}^n)]_S$ is equivalent to H_2^μ where

$$\mu = \frac{1}{2\varepsilon}(1+|x|^2)^{\frac{1}{2}}((1+|x|^2)^{\varepsilon} - (1+|x|^2)^{-\varepsilon})^{-1}\log(1+|x)^2).$$

EXAMPLE II. The following proposition, in the case $S = \delta(\theta)$ and $P = 1$, gives the well-known interpolation result with change of measures

$$[L^p, L^p(w^p)]_S \equiv L^p(w^p)$$

and, in the case $S = \delta'(\theta)$ and $P = \exp(\theta - \xi)$,

$$[L^p, L^p(w^p)]_S \equiv L^p\left(\left(\frac{w^\theta}{1+|\log w|}\right)^p\right)$$

PROPOSITION 7. *Let w be a positive measurable function on a space X such that $w(x) \geq m > 0$ for every $x \in X$, and $P \in A(\Omega; C)$ with $S(Pw^{-\xi}) \neq 0$. Under the* *hypotheses* (a) *and* (b) *of proposition* 6,

$$[L^p, L^p(w^p)]_S \equiv L^p(|S(Pw^{-\xi})|^{-p}).$$

PROOF Define $L: D(L) \to L^p$ by $L(f) = wf$. It is known that $D(L) = L^p(w^p)$ and $L^\alpha f = w^\alpha f$.

Let $f \in L^p(|S(Pw^{-\xi})|^{-p})$ and consider $(A_m)_m = (P^{(m)}(\xi_0)f(S(Pw^{-\xi+\xi_0}))^{-1})_m$. It is clear that $(A_m)_m \in E$ and $\varphi_L^S((A_m)_m) = f$. So, $f \in [L^p, L^p(w^p)]_S$ and

$$\|f\|_{[L^p,L^p(w^p)]_S} \ll \|(A_m)_m\|_E = \|Pf(S(Pw^{-\xi+\xi_0}))^{-1}\|_\infty \leqq \|P\|_\infty \|f\|_{L^p(|S(Pw^{-\xi})|^{-p})}.$$

Conversely, assume that $f \in R[\varphi_L^S]$ and let $\varphi \in A(\Omega; L^p(w^{\xi_0 p}))$ such that

$$f = \sum_{q=0}^{\infty} S(w^{-\xi+\xi_0}(\xi - \xi_0)^q)\frac{\varphi^{(q)}(\xi_0)}{q!}.$$

Then,

$$\left(\int_{\mathbb{R}^n} |f(x)|^p \frac{1}{|S(Pw^{-\xi})|^p}\,dx\right)^{\frac{1}{p}} \leqq \sum_{q=0}^{\infty}\left(\int_{\mathbb{R}^n} \frac{|S(w^{-\xi}(\xi - \xi_0)^q|^p}{|S(Pw^{-\xi})|^p} w^{p\xi_0}\left|\frac{\varphi^{(q)}(\xi_0)}{q!}\right|^p dx\right)^{\frac{1}{p}}$$

and, from the hypotheses, it follows that $\|f\|_{L^p(|S(Pw^{-\xi})|^{-p})} \ll \|\varphi\|_\infty$.

When w is a Muckenhoupt weight, we can apply this result to the cases (3) and (4) of the above remark.

COROLLARY 8. *If* $w \geqq m > 0$ *is a weight in the Muckenhoupt class* A_p, *then the following functions are also in* A_p:

(a) $(\sum_n((\log w)^n)/(n!\,n!)))^{-1} w^{\xi_0}$ *with* $0 < \xi_0 < 1$.

(b) $(1/2\varepsilon)w^{\frac{1}{2}} \log w(w^\varepsilon - w^{-\varepsilon})^{-1}$, *for each* $0 < \varepsilon < \frac{1}{2}$.

§3. On L^p spaces.

Let $\varphi(x, t)$ be a function such that $\varphi(x_0, t)$ is an increasing function of $t \in \mathbb{R}^+$ and $\varphi(x_0, 0) = 0$, for each $x_0 \in M$. Denote by $\varphi(X)$ the Frechet lattice of the measurable functions g on M such that there exist $\lambda > 0$ and $f \in X$ with $\|f\|_X \leqq 1$ and

$$(3) \qquad |g(x)| \leqq \lambda\varphi(x, \lambda|f(x)|) \qquad \text{a.e. } x \in M,$$

with the F-norm

$$\|g\|_{\varphi(X)} = \inf\{\lambda > 0;\ \lambda \text{ satisfies } (3)\}.$$

We say that f is equivalent to g when

$$af(t) \leqq g(t) \leqq bf(t) \qquad (a, b > 0).$$

Let $p_0 > p_1 \geqq 1$ and w_0 and w_1 two positive measurable functions on a measure space X. If S is an analytic functional on Ω, let us consider

$$\phi_S(x,t) = |S(w_0(x)^{(\xi-1)/p_0} w_1(x)^{(-\xi/p_1)} t^{(1-\xi)/p_0+\xi/p_1})|, \tag{4}$$

and assume that it is equivalent to an increasing function.

For instance, in the case

$$S(\varphi) = \frac{1}{2\varepsilon} \int_{(1/2)-\varepsilon}^{(1/2)+\varepsilon} \varphi(x)\, dx,$$

and $w_0 = w_1 = 1$, it is easily checked that ϕ_S is equivalent to

$$\phi_S(t) = \begin{cases} t^{1/p_0} t^{(1/p_1 - 1/p_0)((1/2)-\varepsilon)} (1 + |\log t|)^{-1} & t \leqq 1 \\ t^{1/p_0} t^{(1/p_1 - 1/p_0)((1/2)+\varepsilon)} (1 + |\log t|)^{-1} & t > 1 \end{cases}$$

and, if $\varphi(t) = \phi_S^{-1}(t)$, then $\phi_S(L^1)$ is the Orlicz space

$$L(\varphi) = \{f \text{ meas}; f(1 + |\log f|) 1_{\{|f| \leqq 1\}} \in L^{1/\alpha_1}, f(1 + |\log f|) 1_{\{|f|>1\}} \in L^{1/\alpha_2}\} \tag{5}$$

with

$$\alpha_1 = 1/p_0 + (1/p_1 - 1/p_0)((1/2) - \varepsilon) \text{ and} \tag{6}$$
$$\alpha_2 = 1/p_0 + (1/p_1 - 1/p_0)((1/2) + \varepsilon)$$

Proposition 9. *$\phi_S(L^1)$ is continuously embedded in $[L^{p_0}(w_0), L^{p_1}(w_1)]_S$.*

Proof. Let $f \in \phi_S(L^1)$ and let $h \in L^1$ and $\lambda > 0$ such that

$$|f(x)| \leqq \lambda \phi_S(x, \lambda |h(x)|) \qquad \text{a.e. } x \in X.$$

We have $\phi_S(x, \lambda |h(x)|) = |S(F)|$ with

$$F(\xi, \cdot) = w_0^{(\xi-1)/p_0} w_1^{-\xi/p_1} |\lambda h|^{(1-\xi)/p_0 + \xi/p_1}(\cdot) \in \mathscr{F}(L^{p_0}(w_0), L^{p_1}(w_1)).$$

Thus, $f \in [L^{p_0}(w_0), L^{p_1}(w_1)]_S$ and

$$\|f\|_{[L^{p_0}(w_0), L^{p_1}(w_1)]_S} \leqq \lambda \|F\|_{\mathscr{F}} \leqq \begin{cases} \lambda & \lambda < 1 \\ \lambda^2 & \lambda \geqq 1 \end{cases}$$

so the proof is ended.

We don't know whether the reverse is true in general. However we have proved in [2] that, if S is of the type (1), then the equivalence holds.

The next proposition shows this equivalence for a class of functionals S, that includes example (4) of the above remark.

Proposition 10. *If*

$$S(\varphi) = \int_0^1 \mu(\theta) \varphi(\theta)\, d\theta \qquad \forall \varphi \in H(\Omega),$$

where μ is a positive measurable function with compact suport in $[0, 1]$, *then*

$$[L^{p_0}(w_0), L^{p_1}(w_1)]_S \equiv \phi_S(L^1).$$

PROOF. From proposition 9, it only remains to prove that $[L^{p_0}(w_0), L^{p_1}(w_1)]_S$ is embedded in $\phi_S(L^1)$. Let $f \in [L^{p_0}(w_0), L^{p_1}(w_1)]_S$ and $F \in \mathscr{F}$ with $S(F) = f$. It is known (see [1]) that if $0 < \theta < 1$,

$$|F(\theta, x)| \leqq g_0(x)^{1-\theta} g_1(x)^{\theta},$$

with

$$g_j(x) = \frac{(-1)^{j+1}}{j-\theta} \int_{\mathbb{R}} |F(j + it, x)| \, \mu_j(\theta, t) \, dt \qquad (j = 0, 1).$$

As $\mu_j(\theta, t) \leqq \exp(-\pi t)/\sin \pi\theta$ for $j = 0, 1$, and μ has compact support, we have

$$|S(F)(x)| \ll \int_0^1 \mu(\theta) \, |F(\theta, x)| \, d\theta \ll \int_0^1 \mu(\theta) G_0(x)^{1-\theta} G_1(x)^{\theta} \, d\theta,$$

where

$$G_j(x) = \int_{\mathbb{R}} |F(j + it, x)| \exp(-\pi |t|) \, dt \in L^{p_j}(w_j) \qquad (j = 0,1).$$

Consider $K(x) = \sup(G_j(x)^{p_j} w_j) \in L^1$. Then

$$|S(F)(x)| \ll \int_0^1 K(x)^{1-\theta/p_0+\theta/p_1} w_0(x)^{\theta-1/p_0} w_1(x)^{-\theta/p_1} \mu(\theta) \, d\theta = \phi_S(x, K(x)).$$

Therefore the proof is ended.

If ϕ_S is the function defined in (4) for $w_0 = w_1 = 1$, we have, as a trivial consequence of the interpolation, that for each $K \in \phi_S(L^1)$, the convolution operator

$$K* : L^1 \to \phi_S(L^1)$$

is bounded.

Thus, with the notation of (5) and (6), it is clear that, for every $\alpha_1 < \alpha < \alpha_2$, the convolution operator

$$|x|^{-\alpha} * : L^1 \to L(\varphi)$$

is bounded.

REMARK ON COMPLEX INTERPOLATION FAMILIES. As a trivial consequence of propositions 2 and 9, we have that if $p(\gamma) \geqq 1$ is a measurable function on Γ, and S and T are as in proposition 2, then

COROLLARY 11. *If $\phi(t) = |S(t^\xi)|$, $\phi(L^1)$ is continuously embedded in $[L^{p(\cdot)}][T]$.*

Finally, let us consider a family of measurable functions $\mu(\gamma, x)$ on $\Gamma \times X$ such that

$$\int_\Gamma \frac{1}{p(\gamma)} \log \mu(\gamma, x)\, d\gamma < +\infty \qquad \text{a.e. } x \in X.$$

Assume that the family $\{L^{p(\gamma)}(\mu(\gamma)); \gamma \in \Gamma\}$ is a c.i.f. It is known (see [7]) that $L^{p(\gamma)}_{\mu(\gamma)} = \varphi_\gamma(L^1)$, where

$$\varphi_\gamma(x, t) = \mu(\gamma, x)^{-1/p(\gamma)} t^{1/p(\gamma)}$$

Consider the function $\varphi_z(x, t) = \mu(z, x)^{-1/p(z)} t^{w(z)}$ with

$$\mu(z, x) = \exp\left(p(z) \int_\Gamma 1/p(\gamma) \log \mu(\gamma, x)\, dH_z(\gamma)\right).$$

Define $\varphi_T(x, t) = |T(\mu(z, x)^{-\alpha(z)} t^{w(z)})|$ and assume that it is equivalent to an increasing function, that we shall continue denoting by φ_T. Under these conditions and with an argument completely similar to the one of proposition 9, we have

PROPOSITION 12. *The space $\varphi_T(L^1)$ is continuously embedded in $[L^{p(\cdot)}(\mu(\cdot))][T]$.*

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