MATH. SCAND. 66 (1990), 291–300

DISTRIBUTION DIFFERENTIAL EQUATIONS. REGULARIZATION AND BENDING OF A ROD

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1. Introduction.

In this note we extend the results of Persson [6], [7], on the Cauchy problem for linear distribution differential equations. Our new existence and uniqueness theorem, Theorem 1.8 below, shows that the earlier distinction between equations of odd and even order has been artificial. Furthermore some new cases with lower regularity of the right member are included. However we only treat cases where at least the solutions themselves are pointwise defined. If the coefficients of a linear meaure differential equation are regularized one sometimes finds that the limit of the solutions of the regularized problem is a solution of the unregularized problem as in Persson [4]. This is not always the case Persson [6]. However Theorem 1.9 below shows that if there is no point mass in the coefficient before $u^{(n-1)}$ no anomaly appears. We also treat a boundary value problem for a distribution differential equation which is not a measure differential equation. This corresponds to the bending of a clamped rod with point moments acting at the rod.

Just to keep the note self-contained we repeat the definitions and propositions of [7].

DEFINITION 1.1: If g is a complex Borel measure on R then g is said to be in $\mathscr{P}^0$. If g is a complex valued Borel measurable function on R then g is said to be in $\mathscr{B}^0$. We agree that two function in $\mathscr{B}^0$ are different if they are different at at least one point. Let D be distribution differentiation and let $D^{-1}g$ denote a primitive distribution of $g \in \mathscr{D}'(\mathsf{R})$. If $D^j g \in \mathscr{P}^0$ for some integer j then g is said to be in $\mathscr{P}^j$. Outside distribution theory we agree that all functions in $\mathscr{P}^1$ are right continuous and that all functions of $\mathscr{P}^j, j > 1$, are continuous. In the same way if $D^j g \in \mathscr{B}^0$ for some integer j then g is said to be in $\mathscr{B}^j$. Outside distribution theory we agree that all functions in $\mathscr{B}^j, j > 0$, are continuous. Further if $f \in \mathscr{B}^j$ then we fix a certain $g \in \mathscr{B}^0$ such that $D^j f = g$. When we write $f \in \mathscr{B}^j$ then we mean the pair (f, g). By convention we mean (f, f) when $f \in \mathscr{B}^0$. Let $f_1 \in \mathscr{B}^j$ with $D^j f_1 = g_1$. Let $j > 0$.

Received July 25, 1988; in revised form June 22, 1989.

Then $f = f_1$ in $\mathscr{B}^j$ if $(f, g) = (f_1, g_1)$ pointwise. If $j < 0$ then $f = f_1$ in $\mathscr{B}^j$ if $f = f_1$ in $\mathscr{D}'(\mathsf{R})$ and $g - g_1$ is a polynomial of at most order $-1-j$.

We define multiplication of elements in $\mathscr{B}^j$ by elements in $\mathscr{P}^{-j}$.

DEFINITION 1.2: Let j be an integer, let $a \in \mathscr{P}^{-j}$ and let $f \in \mathscr{B}^j$. We choose $b \in \mathscr{P}^0$ and $g \in \mathscr{B}^0$ such that $D^j b = a$ and $D^j f = g$. If $j \geqq 0$ then the distribution fa at $\phi \in \mathscr{D}(\mathsf{R})$ is defined as

$$\langle fa, \phi \rangle = (-1)^j \int D^j(g\phi)\, db. \tag{1.1}$$

If $j < 0$ then

$$\langle fa, \phi \rangle = (-1)^j \int g d(D^{-j}(\phi a)). \tag{1.2}$$

Definition 1.2 gives

PROPOSITION 1.3. *Let j be an integer. If $j \geqq 0$ then $\mathscr{P}^{-j}$ is a $\mathscr{B}^j$ module. If $j > 0$ then $\mathscr{B}^{-j}$ is a $\mathscr{P}^j$ module.*

REMARK. We regard the modules of Proposition 1.3 as two-sided to make the bookkeeping easier. We also notice that one has $\mathscr{P}^{j-1} \subset \mathscr{B}^j$ if for $f \in \mathscr{P}^{j-1}$ one agrees that in $(f, g) \in \mathscr{B}^j$, $D^j f = g \in \mathscr{P}^1$, i.e. g is unique and right continuous. In the same way for $(f, g) \in \mathscr{B}^j$, $j > 0$, g is unique in in $\mathscr{P}^0$, g is unique modulo a polynomial of at most order $-j-1$ if $j < 0$. Thus we get $\mathscr{B}^j \subset \mathscr{P}^j$ for all j. Proposition 1.3 shows that $\mathscr{P}^j$ is a $\mathscr{P}^{-j+1}$ module, $j \leqq 0$, and that $\mathscr{P}^{-j+1}$ is a $\mathscr{P}^j$ module, $j > 0$. We shall not use the spaces $\mathscr{B}^j$ in the formulation of the theorems below. Still we keep them in the propositions since one can make Theorem 1.8 a little more general as is done in the corresponding theorems of [7] and [8].

PROPOSITION 1.4. *Let $j > 0$ be an integer, let $a \in \mathscr{P}^{-j}$, $f \in \mathscr{B}^j$ and let $b \in \mathscr{P}^0$* be such that $D^j b = a$. *Then*

$$D^{-1}(fa) = D^{-1}(fD^j b) = fD^{j-1}b - D^{-1}((Df)D^{j-1}b) + \text{constant}. \tag{1.3}$$

PROPOSITION 1.5. *Let $j > 0$ be an integer, let $a \in \mathscr{P}^j$, $f \in \mathscr{B}^{-j}$ and let $g \in \mathscr{B}^0$ be such that $D^j g = f$. Then*

$$D^{-1}(af) = D^{-1}(aD^j g) = aD^{j-1}g - D^{-1}((Da)D^{j-1}g) + \text{constant}. \tag{1.4}$$

These propositions are proved in [7] and the proofs are not repeated here. Induction then gives as in [7].

PROPOSITION 1.6. *Let j, b, and f be as in Proposition 1.4. Then*

$$D^{-j}(fD^j b) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} D^{-k}((D^k f)b) + p, \tag{1.5}$$

where p is a polynomial of at most degree $j - 1$.

PROPOSITION 1.7. *Let j, a and g be as in Proposition* 1.5. *Then*

$$D^{-j}(aD^j g) = \sum_{k=0}^{j} (-1)^k \binom{j}{k} D^{-k}(gD^k a) + p, \tag{1.6}$$

where p is a polynomial of at most degree $j - 1$.

We want to solve the Cauchy problem with initial values given at $x = 0$ for the equation

$$u^{(n)} + a_{n-1}u^{(n-1)} + \ldots + a_0 u = f. \tag{1.7}$$

If $g \in \mathscr{P}^k$, $k \geqq 0$, then we agree to define D^{-1} in $D^{-1}g$ as

$$\int_{0^+}^{x} = \int_{(0,x]} \text{ if } x \geqq 0 \quad \text{and} \quad -\int_{x^+}^{0} = -\int_{(x,0]} \text{ if } x < 0.$$

THEOREM 1.8. *Let m and n be integers such that* $0 \leqq m < n$. *Let* $d = \max(0, n - 2m)$. *Let* $a_j \in \mathscr{P}^{-m}$, $0 \leqq j < d$, *let* $a_j \in \mathscr{P}^{m+j-n+1}$, $d \leqq j < n$, *and let* $f \in \mathscr{P}^{-m}$. *Let*

$$a_{n-1}(\{x\}) \neq -1, x \in \mathrm{R}, \textit{ if } m = 0. \tag{1.8}$$

Choose $b_j \in \mathscr{P}^0$ *such that* $D^m b_j = a_j$, $0 \leqq j < d$ *and* $D^{n-m-j-1}b_j = a_j$, $d \leqq j < n$. *Choose* $g \in \mathscr{P}^0$ *such that* $D^m g = f$. *Then to each choice of* $c = (c_1, \ldots, c_n) \in \mathrm{C}^n$ *there is a unique* $u \in \mathscr{P}^{n-m}$ *fulfilling*

$$\begin{aligned} u &+ \sum_{j=0}^{d-1} \sum_{k=0}^{m} (-1)^k \binom{m}{k} D^{m-n-k}((D^{j+k}u)b_j) \\ &+ \sum_{j=d}^{n-m-1} \sum_{k=0}^{n-m-j-1} (-1)^k \binom{n-m-j-1}{k} D^{-m-j-k-1}((D^{j+k}u)b_j) \\ &+ \sum_{j=n-m}^{n-1} \sum_{k=0}^{m+j+1-n} (-1)^k \binom{m+j-n+1}{k} D^{m+j+1-2n-k}((D^k a_j)D^{n-m-1}u) \\ &= D^{m-n}g + \sum_{k=0}^{n-1} c_k x^k/k!. \end{aligned} \tag{1.9}$$

As to the proof one applies D^{n-m-1} to both sides of (1.9). Then one solves the resulting equation as an integral equation in $v = D^{n-m-1}u$. It is just a slight modification of the proof in Persson [7], [8]. We do not repeat it here. One notices that u in (1.9) solves (1.7) in $\mathscr{D}'$. Further the affine solution space of (1.7) is invariant under the choice of the measures b_j and g, Persson [9], since the proof in [9] also applies to the slightly more general situation of Theorem 1.8.

Let $\varepsilon > 0, \varphi \in C_0^\infty, \varphi \geqq 0, \int \varphi = 1, \varphi(x) = 0, x < -1, x > 0, \varphi(x, \varepsilon) = \varepsilon^{-1}\varphi(x/\varepsilon)$. Let

$$b(x, \varepsilon) = \int \varphi(x - y, \varepsilon)\, db(y), \qquad b \in \mathscr{P}^0. \tag{1.10}$$

THEOREM 1.9. *Let the hypothesis be that of Theorem 1.8 except that we require*

$$a_{n-1}(\{x\}) = 0, \qquad x \in \mathrm{R}. \tag{1.11}$$

If the b_j, $D^k a_j$, and g of (1.9) are regularized with $\varphi(x, \varepsilon)$ as in (1.10) then the solution $u(x, \varepsilon)$ of the regularized version of (1.9) exists and is unique. Further $D^k u(x, \varepsilon) \to D^k u(x)$, $\varepsilon \to 0$, $0 \leqq k \leqq n - m - 1$, and the convergence is uniform on compact sets for $0 \leqq k < n - m - 1$.

The proof of Theorem 1.9 is given in Section 2. One may notice that a special case of Theorem 1.9 is treated in Persson [4]. In Section 3 we construct Green's function for the boundary value problem mentioned at the beginning of the introduction. Here many problems are left open as for example eigenfunction expansions of Green's function. See also Persson [4] and [5].

After the completion of this note I got the article J. Ligęza [2] from its author. In [2] I found many valuable references which I have ignored till now above all Pfaff [10], [11] and [12]. In [10] Pfaff treats the effects of regularization of the Sturm-Liouville problem for measure differential equations which later on is done in greater detail in Persson [4]. In [11] and [12] one uses L^p theory to define multiplication of distributions. See also Fisher [1]. In [12] one also finds the definition of multiplication expressed in Prop. 1.6 and Prop. 1.7 when $\mathscr{B}^j$ is replaced by $\mathscr{P}^{j+1}$, i.e. the multiplication used in Theorem 1.8. The results of [11] and [12] cover a part of Theorem 1.8 and vice versa. See also Ligęza [2, Chap. 1 and Chap. 5]. As to further information on distribution differential equations see Persson [7], [8], [9].

2. Proof of Theorem 1.9.

We notice that the case $m = 0$ is already proved in Persson [6]. So we assume that $m > 0$. In this case (1.11) is automatically fulfilled. We start by letting

$$w = u - \sum_{k=0}^{n-1} c_k x^k/k!. \tag{2.1}$$

Let $p(x) = \sum_{k=0}^{n-1} c_k x^k/k!$. Let $v = D^{n-m-1}w$. Since $u = w + p = D^{m+1-n}v + p$,

$$D^{j+k}u = D^{j+k-n+m+1}v + D^{j+k}p. \tag{2.2}$$

We differentiate (1.9) $n - m - 1$ times and get

$$(2.3)\quad v + \sum_{j=0}^{d-1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}D^{-k-1}((D^{j+k+1+m-n}v)b_j)$$
$$+\sum_{j=d}^{n-m-1}\sum_{k=0}^{n-m-j-1}(-1)^k\binom{n-m-j-1}{k}D^{n-2m-j-2-k}((D^{j+k+1+m-n}v)b_j)$$
$$+\sum_{j=n-m}^{n-1}\sum_{k=0}^{m+j+1-n}(-1)^k\binom{m+j+1-n}{k}D^{j-n-k}((D^k a_j)v)$$
$$= D^{-1}g - \sum_{j=0}^{d-1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}D^{-k-1}((D^{j+k}p)b_j)$$
$$-\sum_{j=d}^{n-m-1}\sum_{k=0}^{n-m-j-1}(-1)^k\binom{n-m-j-1}{k}D^{n-2m-j-k-2}((D^{j+k}p)b_j)$$
$$-\sum_{j=n-m}^{n-1}\sum_{k=0}^{m+j-n+1}(-1)^k\binom{m+j-n+1}{k}D^{j-n-k}((D^k a_j)D^{n-m-1}p).$$

We regularize the measures b_j, $0 \leqq j \leqq n - m - 1$, the coefficients a_j, $n - m \leqq j \leqq n - 1$ and the measure g chosen to represent $D^{n-m}f$ as is done in (1.10) with b. We rewrite (2.3) as the equation

$$(2.4)\qquad v + L_1 v + L_2 v = D^{-1}g - L(D^{n-m-1}p)$$

where

$$(2.5)\quad L_1 v = \sum_{j=0}^{d-1}\sum_{k=1}^{m}(-1)^k\binom{m}{k}D^{-k-1}((D^{j+k+1-n+m}v)b_j)$$
$$+\sum_{m=d}^{n-m-1}\sum_{k=0}^{n-m-1-j}(-1)^k\binom{n-m-j-1}{k}D^{n-2m-j-2-k}((D^{m+j+1-n}v)b_j)$$
$$+\sum_{j=n-m}^{n-2}\sum_{k=0}^{j+m+1-n}(-1)^k\binom{j+1+m-n}{k}D^{j-n-k}((D^k a_j)v)$$
$$+\sum_{k=1}^{m}(-1)^k\binom{m}{k}D^{-k-1}((D^k a_{n-1})v),$$

and

$$(2.6)\qquad L_2 v = \sum_{j=0}^{d-1}D^{-1}((D^{j+1+m-n}v)b_j) + D^{-1}(a_{n-1}v).$$

The regularized version of (2.4) is written as

$$(2.7)\qquad v(x,\varepsilon) + L_1(\varepsilon)v(x,\varepsilon) + L_2(\varepsilon)v(x,\varepsilon) =$$
$$D^{-1}g(x,\varepsilon) - L_1(\varepsilon)D^{n-m-1}p - L_2(\varepsilon)D^{n-m-1}p.$$

We shall prove that $v(x, \varepsilon), 0 < \varepsilon \leqq 1$, is equibounded on compact sets. We solve (2.7) by successive approximations. Let $L = L_1 + L_2$ and let $L(\varepsilon) = L_1(\varepsilon) + L_2(\varepsilon)$. We define

$$v_0(x, \varepsilon) = (D^{-1}g)(x, \varepsilon) + L(\varepsilon)D^{n-m-1}p, \tag{2.8}$$

and let

$$v_{j+1}(x, \varepsilon) = -L(\varepsilon)v_j(x, \varepsilon), \qquad j = 0, 1, 2, \ldots \tag{2.9}$$

We shall use exponential majorization adjusted to the operator $L(\varepsilon)$. We let

$$\begin{aligned} \psi(x, \varepsilon) = &\sum_{j=0}^{d-1} \sum_{k=0}^{m} \binom{m}{k} \left(\left| \int_0^x |b_j|(s, \varepsilon)\, ds \right| + |x| \right) \\ &+ \sum_{j=d}^{n-m-1} \sum_{k=0}^{n-m-j-1} \binom{n-m-j-1}{k} \left(\left| \int_0^x |b_j|(s, \varepsilon)\, ds \right| + |x| \right) \\ &+ \sum_{j=n-m}^{n-1} \sum_{k=0}^{j+1+m-n} \binom{j+1+m-n}{k} \left(\left| \int_0^x |D^k a_j|(s, \varepsilon)\, ds \right| + |x| \right). \end{aligned} \tag{2.10}$$

We choose an arbitrary fixed number a and assert that

$$|v_j(x, \varepsilon)| \leqq C2^{-j}e^{2\psi(x, \varepsilon)}, \; |x| \leqq a, \qquad j = 0, 1, \ldots, \tag{2.11}$$

for a certain constant C. Let $C = \sup |v_0(x, \varepsilon)|, |x| \leqq a$. Then (2.11) is true for $j = 0$ and each $\varepsilon > 0$. We now assume that (2.11) is true for such a C and a certain j, Then we notice that (2.10) and (2.11) show that

$$|D^{-1}| \, v_j(x, \varepsilon)|| \leqq C2^{-j} |D^{-1}(\psi' e^{2\psi})| \leqq C2^{-1-j}(e^{2\psi} - 1) \leqq C2^{-j}e^{2\psi},$$

and that

$$|D^{-k}|v_j(x, \varepsilon)|| \leqq C2^{-j}e^{2\psi(x, \varepsilon)}, \qquad k = 1, 2, \ldots. \tag{2.12}$$

It also follows from (2.10) that

$$|D^{-1}((|b_j|(x, \varepsilon) + 1)e^{2\psi(x, \varepsilon)})| \leqq 2^{-1}e^{2\psi} \leqq (|b_j|(x, e) + 1)e^{2\psi},$$

and

$$|D^{-1}(|D^k a_j|(x, \varepsilon) + 1)e^{2\psi}| \leqq 2^{-1}e^{2\psi} \leqq (|D^k a_j|(x, \varepsilon) + 1)e^{2\psi}.$$

Together with (2.9) and (2.11) these observations give

(2.13)
$$|v_{j+1}(x,\varepsilon)| \leqq C2^{-j}\left(\sum_{j=0}^{d-1}\sum_{k=0}^{m}\binom{m}{k}|D^{-k-1}((|b_j|(x,\varepsilon)+1)e^{2\psi})|\right.$$
$$+\sum_{d}^{n-m-1}\sum_{k=0}^{n-m-1-j}\binom{n-m-1-j}{k}|D^{n-2m-2-j-k}((|b_j|(x,e)+1)e^{2\psi})|$$
$$\left.+\sum_{j=n-m-1}^{n-1}\sum_{k=0}^{j+1+m-n}\binom{j+1+m-n}{k}|D^{j-n-k}((|D^k a_j|(x,\varepsilon)+1)e^{2\psi})|\right)$$
$$\leqq C2^{-j}|D^{-1}(\psi' e^{2\psi})| \leqq C2^{-j-1}e^{2\psi}.$$

So (2.11) is true for all j. It follows that $v(x,\varepsilon)=\sum_{j=0}^{\infty} v_j(x,\varepsilon)$ converges uniformly on $|x|\leqq a$ to a solution of (2.7). Let $w(x,\varepsilon)=D^{m-n-1}v(x,\varepsilon)$ and let $u(x,\varepsilon)=w(x,\varepsilon)+\sum_{k=0}^{n-1}c_k x^k/k!$ Then $u(x,\varepsilon)$ solves the regularized version of (1.2). It is obvious from the definition of $\psi(x,\varepsilon)$ in (2.10) that we can replace the last member of (2.13) with a new bigger function ψ independent of ε. The same applies to the constant C. With the new C and ψ in the last member of (2.13) we let

$$b=\max(\psi(a),\psi(-a)).$$

If follows that

$$|v(x,\varepsilon)|\leqq 2Ce^{2b}, \qquad |x|\leqq a,\ 0<\varepsilon\leqq 1.$$

By that we have proved

LEMMA 2.1. *Let $u(x,\varepsilon)$ solve the regularized problem* (1.9). *Then the functions $D^k u(x,\varepsilon)$, $0\leqq k<n-m$, are equibounded on compect sets, $0<\varepsilon\leqq 1$.*

We start by assuming that there are no point masses in the measures defining $L=L_1+L_2$. Let

(2.14)
$$v(x)+Lv=D^{-1}g+L(D^{n-m-1}p).$$

Then

(2.15)
$$v(x,\varepsilon)-v(x)+L(v(x,\varepsilon)-v(x))$$
$$=D^{-1}(g(x,\varepsilon)-g)+(L-L(\varepsilon))v(x,\varepsilon)+(L-L(\varepsilon))(D^{n-m-1}p).$$

Since there are no point masses it follows from Lemma 2.1 that the right hand side of (2.15) goes uniformly to zero on compact sets. It then follows from (2.15) and the proof of Theorem 1.8, see the corresponding proof in [7], that $v(x,\varepsilon)-v(x)$ tends to zero uniformly on compact sets. By that we have proved Theorem 1.9 in this special case.

Let there be point masses at $x=c$, $c>0$, and nowhere else. For $-a\leqq x<c$, $v(x,\varepsilon)\to v(x)$, pointwise when $\varepsilon\to 0$. That means that

(2.16)
$$D^{-r}v(x,\varepsilon)\to D^{-r}v(x),\ r>0, \text{ uniformly in } -a\leqq x\leqq c.$$

We also notice that $D^{-r}v(x)$, $r > 0$, is continuous. Then we notice from (2.5) that

$$L_1(\varepsilon)v(c, \varepsilon) \to L_1 v(c),\ \varepsilon \to 0,$$

since there is no point mass in $x < c$, and since there are at least two integrations in each term of L_1 and $L_1(\varepsilon)$. It also follows that $L_1(\varepsilon)v(x, \varepsilon) \to L_1 v(x)$, $\varepsilon \to 0$, uniformly in $-a \leqq x \leqq c$.

We then see that

$$\begin{aligned} v(x, \varepsilon) - v(x) + L_2(\varepsilon)(v(x, \varepsilon) - v(x)) \\ = (L_2(\varepsilon) - L_2)v(x) + L_1 v(x) - L_1(\varepsilon)v(x, \varepsilon) \\ + D^{-1}(g(x, \varepsilon) - g) + (L - L(\varepsilon))(D^{n-m-1}p). \end{aligned} \tag{2.17}$$

In (2.17) we have already noticed that

$$L_1 v(x) - L_1(\varepsilon)v(x, \varepsilon) \to 0, \text{ and } (L - L(\varepsilon))(D^{n-m-1}p) \to 0,\ \varepsilon \to 0.$$

We notice that $a_{n-1} \in \mathscr{P}^m$, $m > 0$. Then (2.6), (2.16) and the remark after (2.16) show that

$$(L_2(\varepsilon) - L_2)v(x) \to 0, \text{ and } L_2(\varepsilon)(v(x, \varepsilon) - v(x)) \to 0.$$

Since $D^{-1}(g(x, \varepsilon) - g) \to 0$ pointwise in $-a \leqq x \leqq c$, $\varepsilon \to 0$, it follows from (2.17) that $v(x, \varepsilon) - v(x) \to 0$, $\varepsilon \to 0$, pointwise in $-a \leqq x \leqq c$. Then we redefine D^{-1} as integration from c. We get new Cauchy data for $x = c$. If we use the Cauchy data for the unregularized problem as Cauchy data for the regularized problems we commit an error tending to zero for $\varepsilon \to 0$. The first part of the proof translated to $x = c$ then gives that $v(x, \varepsilon) \to v(x)$, $\varepsilon \to 0$, uniformly on compact sets in $x \geqq c$.

The case with point masses at a finite number of points can be proved by repeating the procedure above a finite number of times. The case with $c \leqq 0$ adds no extra difficulty. The general case is proved in such a way that one removes all but a finite number of point masses from the measures in (2.3). One applies the result above for these modified equations. Then one lets the total of the absolute values of the removed masses tend to zero. Comparison between the modified equations and the unmodified ones completes the proof of Theorem 1.9.

3. Bending of a rod.

Look at a rod clamped at $x = 0$ and $x = 1$. Let u be the deviation of its elastic line when moments and forces are acting on the rod. We get the problem

$$u^{(4)} - ku'' + (\eta u)' + \mu u = f + g' \tag{3.1}$$

$$u(0) = 0,\ u'(0) = 0,\ u(1) = 0,\ u'(1) = 0. \tag{3.2}$$

Here f and μ are densities of forces and g and η are densities of moments. If we allow point moments and point forces then (3.1) is turned into a distribution differential equation. The case $\eta = 0$ and $g = 0$ turns (3.1)–(3.2) into a problem for a measure differential equation partly treated already in Persson [5]. One cannot use [5] to get eigenfunction expansions of Green's function here since (3.1)–(3.2) is not selfadjoint in general. Here we simply assume that (3.1)–(3.2) always has a unique solution.

In Theorem 1.8 we choose $n = 4$ and $m = 2$. Then $a_0 \in \mathscr{P}^{-1}, a_1 \in \mathscr{P}^0$. Since the right member of (3.1) is in $\mathscr{P}^{-1}$ we can apply Theorem 1.8 to (3.1). We choose $c^{(j)} = (c_1^{(j)}, \ldots, c_4^{(j)})$ such that $c_{j-1}^{(j)} = 1$ and $c_k^{(j)} = 0$, $k \neq j-1$. Then we let $f = g = 0$. The solution of (3.1) with these Cauchy data is called u_j. These solutions form a fundamental set of solutions of (3.1) when $f = g = 0$. The Wronskian W of these solutions is is in $\mathscr{P}^0$ only and not apriori pointwise defined. Let $u_j(x, \varepsilon)$ be the solution of the regularized version of (3.1) with the Cauchy data $c^{(j)}$ at $x = 0$ and with $f = g = 0$. Let $W(\varepsilon)$ be the corresponding Wronskian. Now $W(\varepsilon) = 1$ for all x if $\varepsilon > 0$. Therefore we construct the formal Green's function for (3.1)–(3.2) from the fundamental set chosen above. See [3, pp. 34–37]. Then we construct the real Green's function for the regularized problem, $G(x, t, \varepsilon)$, with the corresponding fundamental set for small ε.

We know from Theorem 1.8 that $u_j \in \mathscr{P}^3$ and from Theorem 1.9 that $u_j^{(k)}(x, \varepsilon) \to u_j^{(k)}(x)$ pointwise and boundedly, $0 \leqq x \leqq 1$, $0 \leqq k \leqq 2$, $1 \leqq j \leqq 4$. That means that $G(x, t, \varepsilon) \to G(x, t)$ pointwise and boundedly when $\varepsilon \to 0$.

Let $v(x)$ be the solution of (3.1) with zero initial data at $x = 0$. Let $v(x, \varepsilon)$ be the solution of the regularized problem with zero initial data at $x = 0$. It follows from Theorem 1.9 that $v(x, \varepsilon) \to v(x)$, $\varepsilon \to 0$, and so do the corresponding Cauchy data at $x = 1$. It follows that the boundary values $d(\varepsilon)$ of $v(x, \varepsilon)$ tend to the boundary values d of $v(x)$. Let $w(x) = u(x) - v(x)$ and let $w(x, \varepsilon) = u(x, \varepsilon) - v(x, \varepsilon)$. It follows from the hypothesis that for small ε there is an invertible matrix $U(\varepsilon)$ giving the connection between the Cauchy data at $x = 0$ and the boundary values of any solution of the homogeneous regularized version of (3.1). Let U be the corresponding matrix for the original (3.1). From $d^{(j)}(\varepsilon) = U(\varepsilon)c^{(j)}$ it follows that $U(\varepsilon) \to U$, $\varepsilon \to 0$. Let $c(\varepsilon)$ be the Cauchy data of $w(x, \varepsilon)$ and let c be the Cauchy data of $w(x)$ at $x = 0$. Then $c(\varepsilon) = -U^{-1}(\varepsilon)d(\varepsilon) \to -U^{-1}d = c$, $\varepsilon \to 0$. Then a very slight modification of the proof of Theorem 1.9 shows that $w(x, \varepsilon) \to w(x)$, $\varepsilon \to 0$. That means that $u(x, \varepsilon) \to u(x)$, $\varepsilon \to 0$. If we now take the limit of $u(x, \varepsilon) = \int_0^1 G(x, t, \varepsilon)(f(t) + g'(t))\, dt$ one realizes that $G(x, t)$ is the Green's function of (3.1)–(3.2) as long as $f + g'$ is a function. A closer look at the adjoint problem shows that $G(x, \cdot) \in \mathscr{P}^4$ locally. But letting it zero outside $0 \leqq t \leqq 1$ it is in $\mathscr{P}^3$ and can be used as a testfunction on $\mathscr{P}^{-2}$. This is the maximum irregularity admitted by Theorem 1.8 for the right member of (3.1).

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