MATH. SCAND. 66 (1990), 301–306

A NOTE ON HEREDITARY RINGS OR NON-SINGULAR RINGS WITH CHAIN CONDITION

NGUYEN V. DUNG

1. Introduction.

A well-known result of Osofsky [10, 11] states that a ring R is semisimple artinian iff every cyclic R-module is injective. From this it follows that a right self-injective right hereditary ring is semisimple artinian. In this paper we shall extend these results of Osofsky in some ways.

Following Chatters-Hajarnavis [1] a ring R is called right CS if every closed right ideal of R is a direct summand of R_R. Clearly, the class of right CS rings includes all right self-injective rings. In Theorem 3.1 we prove that a right CS right hereditary ring is right noetherian. By the way, this result closely relates to a result of Goodearl [8] (see also Colby-Rutter [3]) characterizing rings over which non-singular right modules are projective (Corollary 3.2).

Further we study right hereditary rings R whose injective hulls $E(R_R)$ are finitely generated. In some cases we are able to show that the rings in question are right artinian (Theorem 3.4, Corollary 3.7). Finally we give a new characterization of semisimple artinian rings as right self-injective rings whose non-zero singular right modules contain non-zero injective submodules (Theorem 3.8). This result extends a result of Smith [15, Theorem 2.10].

2. Definitions and notations.

Throughout this paper we consider associative rings with identity and unitary modules. We write M_R to indicate that M is a right module over a ring R. To say that M_R has finite Goldie dimension means that M_R does not contain infinite direct sums of non-zero submodules. A submodule K of M is called essential in M if $K \cap L \neq 0$ for each non-zero submodule L of M. In this case M is called an essential extension of K. A submodule C of M is called a closed submodule of M if C is the only essential extension of C in M. A right ideal of R is closed (essential) if it is closed (essential) as submodule of R_R.

Received August 16, 1989.

For a right R-module M, $E(M)$ denotes the injective hull of M, and the submodule

$$Z(M) = \{x \in M \mid xK = 0 \text{ for some essential right ideal } K \text{ of } R\}$$

is called the singular submodule of M. If $Z(M) = M$, M is called a singular module. If $Z(M) = 0$, M is said to be non-singular. A ring R is right non-singular if R_R is non-singular.

3. Results.

We start with proving the following theorem.

THEOREM 3.1. *A right* CS *right hereditary ring is right noetherian.*

PROOF. Let R be a right CS, right hereditary ring. First we show that R does not contain an infinite set of non-zero orthogonal idempotents. We do this by applying the technique of Osofsky in [11]. Assume on the contrary that there is an infinite set $\{e_i, i \in I\}$ of orthogonal idempotents $e_i \neq 0$ in R. Let P be a subset of I and put $A = \bigoplus_{i \in P} e_i R$. Since R is right CS, there exists an idempotent f in R such that A_R is essential in fR. Clearly we have $fe_i = e_i$ for all $i \in P$. Let $e_j \in \{e_i\}$ with $j \notin P$. Since R is right non-singular, R has a maximal right quotient ring Q which is von Neumann regular (see e.g. [9] or [17]). Hence there is an idempotent g of Q such that $Qe_j f = Qg$. Suppose that $fgQ \cap A \neq 0$. Then there is an x in A such that $0 \neq x = fgq$ with $q \in Q$. Hence $e_j fgq = e_j fq = 0$. Since $g \in Qe_j f$ we have $gq = 0$, implying $x = 0$, a contradiction. Thus $fgQ \cap A = 0$. Since fR is essential in fQ_R (see e.g. [9, Chapter 2]), it follows that A_R is essential in fQ_R, so we must have $fgQ = 0$, in particular $fg = 0$. Therefore $e_j f = e_j fg = 0$ for all $j \notin P$. Thus we can apply [11, Theorem] to arrive at a contradiction: Let $B = \sum_{i \in I} (e_i R + \ker \varphi)$ where φ: $R \to \prod_{i \in I} e_i R$, $\varphi(x) = \langle e_i x \rangle$, $x \in R$. Then B is a right ideal of R. Now by [11, Theorem] M_R/B_R is not injective for all $M_R \supset R_R$ (clearly $B \neq R$). But since R is right hereditary, $E(R_R)/B_R$ has to be injective. This contradiction shows that R does not contain an infinite set of non-zero orthogonal idempotents. Hence there are orthogonal idempotents $e_1, \ldots, e_n$ in R such that

$$R_R = e_1 R \oplus \ldots \oplus e_n R,$$

where each $e_i R$ is an indecomposable right R-module. From this we could use [1, Proposition 2.3], but we prefer giving a direct proof here for completeness. Let C be a non-zero closed submodule of $e_i R$. Then C_R is closed in R_R too. Hence $R_R = C \oplus C'$ for some submodule C' of R_R. From this we see that C_R is also a direct summand of $e_i R_R$. It follows that $C = e_i R$ are uniform right ideals of R.

Hence R_R has finite Goldie dimension. Since R is right hereditary, by [14, Corollary 2] we know that R is right noetherian.

The proof of Theorem 3.1 is complete.

Using Theorem 3.1 we can prove the equivalence (a)$\Leftrightarrow$(b) in the following result given in [3, Theorem 3.2] and [8, Theorem 2.15].

COROLLARY 3.2. *For a ring R the following conditions are equipvalent:*
(a) *R is right hereditary and $E(R_R)$ is projective.*
(b) *$Z(R_R) = 0$ and every non-singular right R-module is projective.*
(c) *R is artinian, serial and hereditary on both right and left sides.*

PROOF. (b)$\Leftrightarrow$(c) is proved in [8, Theorem 2.15]. (b)$\Rightarrow$(a) is clear. (a)$\Rightarrow$(b): Assume (a). Then every submodule of $E(R_R)$ is projective. Let A be any right ideal of R and B be a maximal essential extension of A_R in R_R. Then R/B_R is a non-singular cyclic right R-module. Using [6, Lemma 4] we see that R/B_R is embedded in $E(R_R)$. Therefore R/B_R is projective, so B_R is a direct summand of R_R, proving that R is right CS. Hence R is right noetherian by Theorem 3.1. Now let M be a non-singular right R-module, and let N denote the injective hull of M. Then N is also non-singular, and $N = \bigoplus_{i\in I} N_i$ where each N_i is indecomposable and injective. Let X_i be a non-zero cyclic submodule of N_i. Since X_i is non-singular we see by using [6, Lemma 4] that X_i is isomorphic to a submodule of $E(R_R)$, and since N_i is uniform, N_i can be embedded in $E(R_R)$, too. It follows that each N_i is projective, therefore N is also projective. Since R is right hereditary, M is projective, proving (b).

We note that if R is a (right non-singular) ring with $E(R_R)$ projective, then $E(R_R)$ is finitely generated. From this and Corollary 3.2 it arises a question: Does a right hereditary ring R with finitely generated $E(R_R)$ satisfy some kind of chain conditions on right ideals?

The purpose of the remainder of this paper is to answer this question in some special cases and to prove a related result. Recall that a module M is called completely injective if every homomorphic image of M is injective. By the same proof as that of [4, Proposition] without any change we obtain.

LEMMA 3.3. *Let M_R be a cyclic finitely presented completely injective module such that* $\mathrm{End}_R(M_R)$ *is a von Neumann regular right self-injective ring. Then M_R has finite Goldie dimension.*

Now we can give a sufficient condition for a right hereditary ring to be right artinian.

THEOREM 3.4. *Let R be a right hereditary ring such that every closed right ideal* *of R is finitely generated and $E(R_R)$ is a direct sum of cyclic submodules. Then R is right artinian.*

PROOF. Suppose that R is as in Theorem 3.4. Then $E(R_R) = E_1 \oplus \ldots \oplus E_n$, where each E_i is cyclic. Since R is right hereditary, in particular right non-singular, each E_i is non-singular. Hence for each i there is a closed right ideal A_i of R such that $E_i \approx R/A_i$. Moreover by hypothesis each A_i is finitely generated. From this and since R is right hereditary, each E_i is a cyclic finitely presented completely injective module. Then by [7, Corollary 19.29], each $\mathrm{End}_R E_i$ is von Neumann regular and right self-injective. Then by Lemma 3.3, each E_i has finite Goldie dimension. It follows that R has finite right Goldie dimension. Hence R is right noetherian by [14, Corollary 2]. Since $E(R_R)$ is moreover finitely generated, it follows that R is right artinian by [18, Theorem A].

The proof of Theorem 3.4 is complete.

Next we consider right non-singular rings with cyclic injective hulls. The following lemma is similar to [12, Lemma 1.7].

LEMMA 3.5. *Let R be a ring such that $E(R_R)$ is cyclic. If R_R does not contain an infinite direct sum of isomorphic submodules, then R is right self-injective.*

PROOF. Let R be a ring such that $E(R_R) \approx R/A$ for some right ideal A of R. Then there is a short exact sequence

$$0 \to A \to R \xrightarrow{\varphi} E(R_R) \to 0.$$

Take $c \in R$ such that $\varphi(c) = 1$ where 1 is the identity of R. Suppose that $cx \in A$ for some $x \in R$. Then $0 = \varphi(cx) = \varphi(c)x = x$. It follows that $cx \in A$ iff $x = 0$ and hence $cR \cap A = 0$. Now assume that R_R does not contain an infinite direct sum of (non-zero) isomorphic submodules. Consider the infinite sum of submodules $c^i A$:

$$(*) \qquad A + cA + c^2A + \ldots + c^nA + \ldots$$

Clearly $c^nA \approx A$ for all n. Suppose that there are $x_0, \ldots, x_n$ in A such that $x_0 + cx_1 + \ldots + c^n x_n = 0$. Then $-x_0 = c(x_1 + \ldots + c^{n-1}x_n)$. It follows that $x_0 = 0$ and $x_1 + cx_2 + \ldots + c^{n-1}x_n = 0$. From this we can easily verify that $x_0 = x_1 = \ldots = x_n = 0$, proving that the sum (*) is direct, a contradiction to the assumption in case $A \neq 0$. Hence $A = 0$, showing that R is right self-injective.

PROPOSITION 3.6. *Let R be a right non-singular ring such that $E(R_R)$ is a cyclic finitely presented completely injective right R-module. Then R is semisimple artinian.*

PROOF. Clearly $E(R_R)$ is non-singular, hence $S = \mathrm{End}_R E(R_R)$ is von Neumann regular and right self-injective. From Lemma 3.3 we obtain that S is semisimple artinian. On the other hand, Lemma 3.5 shows that R is right self-injective. Hence R is semisimple artinian.

COROLLARY 3.7. *If R is a right hereditary ring such that $E(R_R)$ is cyclic and finitely presented, then R is semisimple artinian.*

QUESTION. Let R be a right hereditary ring such that $E(R_R)$ is cyclic. Is R necessarily semisimple artinian?

Following Goodearl [8], a ring R is defined to be a right SI ring if every singular right R-module is injective. If every singular cyclic right R-module is injective, then R is called right RIC by Smith [15, 16]. Recently Osofsky and Smith [13] have proved that every right RIC ring is right SI. Now a ring R is called right weakly SI (briefly, right WSI) if every singular right R-module $\neq 0$ contains a non-zero injective submodule. Clearly right SI rings are right WSI, but we do not know if the converse holds.

Let R be a right SI ring. Then for each essential right ideal B of R, R/B is semisimple (see [8, Theorem 3.11]). From this and [5, Theorem 5] we see that a right or left self-injective right SI ring is quasi-Frobenius. Then using [15, Theorem 2.10] we obtain that R is semisimple artinian. For right WSI rings we have the following result.

THEOREM 3.8. *A right self-injective, right* WSI *ring is semisimple artinian.*

PROOF. Let R be a right self-injective, right WSI ring, and let Z denote the singular submodule of R_R. Suppose that $Z \neq 0$. Then Z contains a non-zero injective right ideal of R which is clearly projective, a contradiction. Hence $Z = 0$, or with other words, R is right non-singular. Hence R is a von Neumann regular ring (see [7, Corollary 19.28]).

Let A be a countably generated right ideal of R which is not finitely generated. Since R_R is injective, there exists an indempotent e of R such that $E(A_R) = eR$. Let $C = A \oplus (1-e)R$. Then C_R is countably generated and essential in R_R with $C \neq R$. Now, since R/C_R is a non-zero singular right R-module, by hypothesis we have $R/C_R = I \oplus N$ for some non-zero injective module I and a module N. Then there is a right ideal D of R such that $C \subset D \subset R$, $D/C \approx N$ and $R/D \approx I$. Since N_R is cyclic and C_R is countably generated, it follows that D_R is also countably generated. Clearly, $D \neq R$ and D_R is not finitely generated, for otherwise D_R would be a direct summand of R_R, a contradiction to the fact that D is essential in R_R. Now by using [7, Proposition 19.25] and by an easy induction proof we see that there is an infinite set of orthogonal idempotents $\{e_i\}_{i=1}^{\infty}$ in R such that $D = \bigoplus_{i=1}^{\infty} e_iR$. Then by [10, Lemma 5] we see that R/D_R is not injective, a contradiction to $R/D_R \approx I$ injective. This shows that every countably generated right ideal of R is finitely generated. Then, since R is von Neumann regular, R is semisimple artinian.

The proof of Theorem 3.8 is complete.

The author wishes to thank Professor Dinh van Huynh for many useful discussions during the preparation of this paper.

REFERENCES

1. A. W. Chatters and C. R. Hajarnavis, *Rings in which every complement right ideal is a direct summand*, Quart. J. Math. Oxford (2) 28 (1977), 61–80.
2. A. W. Chatters and S. M. Khuri, *Endomorphism rings of modules over non-singular* CS *rings*, J. London Math. Soc. (2) 21 (1980) 434–444.
3. R. R. Colby and E. A. Rutter, *Generalizations of* QF-3 *algebras*, Trans. Amer. Math. Soc. 153 (1971), 371–386.
4. R. F. Damiano, *A right* PCI *ring is right noetherian*, Proc. Amer. Math. Soc. 77 (1979), 11–14.
5. Dinh van Huynh, Nguyen V. Dung and Robert Wisbauer, *Quasi-injective modules with* ACC *or* DCC *on essential submodules*, Arch. Math. (Basel) 53 (1989), 252–255.
6. C. Faith, *When are proper cyclics injective*? Pacific J. Math. 45 (1973), 97–112.
7. C. Faith, *Algebra II, Ring Theory*, Springer-Verlag, 1976.
8. K. R. Goodearl, *Singular torsion and splitting properties*. Mem. Amer. Math. Soc. 124(1972).
9. K. R. Goodearl, *Ring Theory*, Marcel Dekker Inc., 1976.
10. B. L. Osofsky, *Rings all of whose finitely generated modules are injective*, Pacific J. Math. 14 (1964), 645–650.
11. B. L. Osofsky, *Non-injective cyclic modules*, Proc. Amer. Math. Soc. 19 (1968), 1383-1384.
12. B. L. Osofsky, *Noncommutative rings whose cyclic modules have cyclic injective hulls*, Pacific J. Math. 25 (1968), 331–340.
13. B. L. Osofsky and P. F. Smith, *Cyclic modules whose quotients have complements direct summands*, Preprint (1989).
14. F. L. Sandomierski, *Nonsingular rings*, Proc. Amer. Math. Soc. 19 (1968), 225–230.
15. P. F. Smith, *Decomposing modules into projectives and injectives*, Pacific J. Math. 76 (1978), 247–266.
16. P. F. Smith, *Rings characterized by their cyclic modules*, Canad. J. Math. 24 (1979), 93–111.
17. B. Stenström, *Rings of Quotients*, Springer-Verlag, 1975.
18. C. I. Vinsonhaler,*Orders in QF-*3 *rings. Supplement*, J. Algebra 17 (1971), 149–151.

INSTITUTE OF MATHEMATICS
P.O. BOX 631 BO HO
HANOI
VIETNAM