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LEADING TERMS IN THE HEAT INVARIANTS FOR THE LAPLACIANS OF THE DE RHAM, SIGNATURE, AND SPIN COMPLEXES

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Abstract.

Let D be a vector bundle-valued differential operator with positive definite leading symbol on a compact, Riemannian manifold. Asymptotic expansions of the kernel function and L^2 trace of the *heat operator* e^{-tD}, $t > 0$, naturally lead to sequences of homogeneous local and global scalar invariants $a_n(x, D), a_n(D) \equiv \int a_n(x, D), n \in \mathbf{N}$. Within each homogeneity class of local invariants, there is a filtration by *degree*; the lowest-degree terms in $a_n(x, D)$ are linear, those in $a_n(D)$ quadratic. Information about such *leading terms* has been crucial in the work of Osgood, Phillips, and Sarnak, of Brooks, Chang, Perry, and Yang, and of Melrose on compactness problems for isospectral sets of metrics and domains, modulo gauge equivalence, in dimensions two and three. We specialize our earlier general results to give the leading terms in the heat invariants produced by the Laplacians of the de Rham, signature, and spin complexes. A main technical point is a calculation of fiber traces of quadratic expressions built from iterated covariant derivatives of the Weitzenböck operator.

1. Introduction.

Let (M, g) be a compact, m-dimensional Riemannian manifold without boundary. Let R, ρ, and τ be the Riemann, Ricci, and scalar curvatures, normalized so that on the standard sphere, $\rho = (m-1)g$ and $\tau = m(m-1)$. Let (V, ∇) be a vector bundle with connection, of fiber dimension μ, and let $B_{(V,\nabla)} = -\nabla^i \nabla_i$ be the Bochner Laplacian. Here and below, all indices are raised and lowered using the metric tensor and its inverse $g^{-1} = (g^{ij})$, and we sum over repeated indices. We shall sometimes suppress the dependence of the Bochner Laplacian on ∇, or on (V, ∇), and just write B_V or B. We denote by $\Omega = (\Omega_{ij})$ the curvature two-form of ∇.

Let D be a differential operator of the form $B - \mathscr{E}$, where $\mathscr{E}$ is a C^∞ section of End V. The *heat operator* e^{-tD}, $t > 0$ is of trace class and has smooth kernel $H(t, x, y)$. The diagonal values of H admit an asymptotic expansion

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(1.1) $$H(t,x,x) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} e_n(x,D)t^n, \quad t \downarrow 0,$$

where the $e_n(x, D)$ are local endomorphism-valued invariants of D. The $e_n(x, D)$ are built universally and polynomially from g and its inverse $g^{-1} = (g^{ij})$, and from covariant derivatives of R, Ω, and $\mathscr{E}$ using only tensor product and contraction; see [9, Sec. 1] and [10, § 1] for details. Covariant differentiation of tensor quantities is effected using the Levi-Civita connection. The factor $(4\pi)^{-m/2}$ is chosen so that $e_0(x, D)$ will be the identity endomorphism on V.

As a consequence of (1.1), the fiber and L^2 traces of the heat kernel admit asymptotic expansions

$$\operatorname{trace}_{V_x} H(t,x,x) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} a_n(x,D)t^n,$$

$$\operatorname{Tr}_{L^2} e^{-tD} = \int_M H(t,x,x) \sim (4\pi t)^{-m/2} \sum_{n=0}^{\infty} a_n(D)t^n, \quad t \downarrow 0,$$

where

$$a_n(x,D) = \operatorname{trace}_{V_x} e_n(x,D), \quad a_n(D) = \int_M a_n(x,D).$$

In particular, the $a_n(D)$ are spectral invariants of D.

Of course, similar things can be said of a much larger class of elliptic differential or pseudo-differential operators, but there is a particularly nice invariant theory for operators of the form $B - \mathscr{E}$. By [9, Sec. 1], there exist polynomial expressions for the $e_n(x, D)$ and $a_n(x, D)$ in the ingredients listed above which are independent of all data: $M, g, \nabla, \mathscr{E}$, and in fact of m and μ. (In the case of $e_n(x, D)$, these are noncommutative polynomials.) These expressions are homogeneous: let Q be a section of a vector bundle W which is an invariant monomial expression in g, g^{-1}, and covariant derivatives of $(R, \Omega, \mathscr{E})$, of degree $(k_R, k_\Omega, k_\mathscr{E})$ in $(R, \Omega, \mathscr{E})$. Suppose that k_∇ explicit covariant derivatives appear in Q. For $n \in \mathbf{N}$, we say that $Q \in \mathscr{P}_N(W)$ if

$$2(k_R + k_\Omega + k_\mathscr{E}) + k_\nabla = N.$$

Here of course, an occurrence of ρ or τ is counted as an occurrence of R. We then close $\mathscr{P}_N(W)$ under addition to make it a space of polynomials. The homogeneity properties enjoyed by the heat invariants are

$$e_n(x,D) \in \mathscr{P}_{2n}(\operatorname{End} V), \quad a_n(x,D) \in \mathscr{P}_{2n}(\Lambda^0),$$

where Λ^0 is the trivial scalar bundele. In general, we shall denote the p-form bundle by Λ^p.

There is a filtration of each $\mathscr{P}_N(W)$ by *degree*. We say that a polynomial is in $\mathscr{P}_{N,\ell}$ if it can be written as a sum of monomials with $k_R + k_\Omega + k_{\mathscr{E}} \geqq \ell$, or equivalently, $k_\nabla \leqq N - 2\ell$. $\mathscr{P}_{N,2}(W)$ consists of quadratic and higher-degree polynomials, $\mathscr{P}_{N,3}(W)$ of cubic and higher, and so on. We have

$$\mathscr{P}_N = \mathscr{P}_{N,1} \supseteq \mathscr{P}_{N,2} \supseteq \ldots \supseteq \mathscr{P}_{N,[N/2]}; \quad \mathscr{P}_{N,\ell} = 0, \ell > [N/2].$$

An expression which *a priori* appears only to be in, say, $\mathscr{P}_{6,2}$, may actually be in a more elite space like $\mathscr{P}_{6,3}$; for example,

$$\nabla^i\nabla^j\mathscr{E}\cdot\Omega_{ij} = \tfrac{1}{2}[\nabla^i, \nabla^j]\mathscr{E}\cdot\Omega_{ij} = \tfrac{1}{2}[\Omega^{ij}, \mathscr{E}]\cdot\Omega_{ij} \in \mathscr{P}_{6,3}(\operatorname{End} V).$$

When m is even, characteristic classes like the Pfaffian, the Hirzebruch polynomial (for oriented M), and the $\hat{A}$-polynomial (for oriented M with spin structure) live in the most elite class of m-homogeneous scalar polynomials, viz. $\mathscr{P}_{m,m/2}(\Lambda^0)$. In the case of the Hirzebruch and $\hat{A}$ polynomials, the orientation enters the local invariant theory through Ω. In general, all of the $\mathscr{P}_N(W)$ depend implicitly on the original bundle V through Ω and $\mathscr{E}$.

Within $\mathscr{P}_N(\Lambda^0)$, one can form the class $\mathscr{D}_N = \delta\mathscr{P}_{N-1}(\Lambda^1)$ *of exact divergences*; these, of course, are the terms that universally integrate to zero. Here and below, δ is the formal adjoint of the exterior derivative d. Note that δ: $\mathscr{P}_{N-1,\ell}(\Lambda^1) \to \mathscr{P}_{N,\ell}(\Lambda^0)$. $\mathscr{D}_N$ is filtered by the $\mathscr{D}_{N,\ell} = \mathscr{D}_N \cap \mathscr{P}_{N,\ell}(\Lambda^0)$, and if N is even, $\mathscr{D}_{N,N/2} = 0$. Our interest here is in the linear terms of the local endomorphism-valued and scalar heat invariants, and in the quadratic terms of the integrated scalar heat invariants; that is, the cosets

$$e_n(x,D) + \mathscr{P}_{2n,2}(\operatorname{End} V), \quad a_n(x,D) + \mathscr{P}_{2n,2}(\Lambda^0), \quad a_n(x,D) + \mathscr{D}_{2n} + \mathscr{P}_{2n,3}(\Lambda^0).$$

By the Bianchi identities for R and Ω, $\mathscr{P}_{2n}(\operatorname{End} V)/\mathscr{P}_{2n,2}(\operatorname{End} V)$ is two-dimensional, and is spanned by the classes of $(\Delta^{n-1}\tau)I$ and $B^{n-1}_{\operatorname{End} V}\mathscr{E}$. In particular, $\mathscr{P}_{2n,1}(\Lambda^0) = \mathscr{D}_{2n,1} + \mathscr{P}_{2n,2}(\Lambda^0)$. It was proved in [9, Theorem 4.1] that

$$(1.2) \quad c_n^{-1}e_n(x,D) \cong_{\operatorname{mod}\mathscr{P}_{2n,2}(\operatorname{End} V)} -2n(\Delta^{n-1}\tau)I - 4(2n+1)B^{n-1}\mathscr{E} \quad n \geqq 1,$$

where here and below,

$$c_n = \frac{(-1)^n n!}{2(2n+1)!}.$$

It was proved in [11] that

$$\begin{aligned}(1.3)\; c_n^{-1}a_n(x,D) \cong_{\operatorname{mod}(\mathscr{D}_{2n}+\mathscr{P}_{2n,3}(\Lambda^0))} & \operatorname{trace}_{V_x}\{(n^2-n-1)|\nabla^{n-2}\tau|^2 I + 2|\nabla^{n-2}\rho|^2 I \\ & + 4(2n+1)(n-1)\nabla^{n-2}\tau\cdot\nabla^{n-2}\mathscr{E} + 2(2n+1)\nabla^{n-2}\Omega^{ij}\cdot\nabla^{n-2}\Omega_{ij} \\ & + 4(2n+1)(2n-1)\nabla^{n-2}\mathscr{E}\cdot\nabla^{n-2}\mathscr{E}\}, \quad n \geqq 3;\end{aligned}$$

see also [3] for a more functional proof. In particular, $c_n^{-1} a_n(D)$ is given, up to cubic and higher-degree terms, by the integral of the expression on the right. If we add the fiber traces of the terms on the right in (1.2) to the expression on the right in (1.3), we can replace "mod $\mathscr{D}_{2n} + \mathscr{P}_{2n,3}(\Lambda^0)$" with "mod $\mathscr{D}_{2n,2} + \mathscr{P}_{2n,3}(\Lambda^0)$". The quadratic terms in $a_2(x, D)$ are qualitatively different, involving the full Riemann tensor R: by, e.g., [8, p. 610],

$$(1.4)\quad 180 e_2(x, D) = -30(\tfrac{1}{5}\Delta\tau I + B\mathscr{E}) + 90(\tfrac{1}{6}\tau I + \mathscr{E})^2 + (|R|^2 - |\rho|^2)I + 15\Omega^{ij}\Omega_{ij}.$$

The main purpose of this paper is to survey the implications of formulas (1.2) and (1.3) in the case of the differential form Laplacians, including the Laplacians of the signature complex in the even-dimensional, oriented case. We also give the corresponding leading terms for the Laplacians of the spin complex. Leading term analysis differs from more traditional work on the heat invariants in that it attempts to provide partial information on all of the heat invariants (calculate "some of the terms all of the time") rather than total information on the first few invariants ("all of the terms some of the time").

At one time, it was hoped that one could "hear" the geometry of a compact, Riemannian manifold (M, g); that is, that the spectrum of the ordinary Laplacian Δ on functions would completely determine the isometry type of (M, g). Numerous counterexamples to this and weaker conjectures are now known. But recently, attention has turned to conjectures asserting that Δ-isospectral sets of metrics, modulo gauge equivalence (diffeomorphism) should be small; more precisely, that they should be compact in the C^∞ topology. Osgood, Phillips, and Sarnak [14] proved that this is the case in dimension $m = 2$; their proof uses the leading terms in the heat invariants in a crucial way. The idea is to set up an inductive scheme to get higher and higher Sobolev estimates on the metric. The variation across conformal classes is controlled using the nonlocal *functional determinant*; this also provides an initial Sobolev estimate. The higher estimates are obtained using the fact that the integrated heat invariants are isospectral invariants. At the nth stage, the leading (quadratic) term in $a_n(\Delta)$ controls the highest-order derivative; the remaining terms involve lower derivatives which have been controlled at a previous stage.

In higher dimensions, the moduli space of conformal structures is not nearly so well understood. However, working in dimension $m = 3$, Brooks, Chang, Perry, and Yang [4–7] were able to show that an isospectral family of metrics contained within a conformal class is compact modulo gauge equivalence. Their methods also give a similar two-dimensional result without using the functional determinant. Melrose [13] has extended the Osgood-Phillips-Sarnak methods to boundary value problems.

There are, of course, many more natural operators D besides the scalar Laplacian with the right ellipticity properties to produce heat invariants, and to make elementary sense of the isospectrality question. We hope that our calculation of the leading terms for the Laplacians of the classical (real) complexes will eventually be of use in the study of isospectrality questions, and more broadly, in the study of the topology and geometry of moduli spaces of metrics.

2. The Levi-Civita curvature and the Weitzenböck operator on differential forms.

Let $\Delta_p = \delta d + d\delta$ be the p-form Laplacian, and let ι_i and ε^i be interior and exterior multiplication by elements of some local frame $\{X_i\}$ and dual coframe $\{\eta^i\}$. Let $u \in C^\infty(\Lambda^p)$. By the classical formulas

$$(du)_{i_0 \ldots i_p} = \sum_{s=0}^{p} (-1)^s \nabla_{i_s} u_{i_0 \ldots \widehat{i_s} \ldots i_p}, \quad (\delta u)_{i_2 \ldots i_p} = -\nabla^j u_{j i_2 \ldots i_p}$$

and the Ricci identity

$$[\nabla_i, \nabla_j] u = -R^k{}_{lij} \varepsilon^l \iota_k u, \tag{2.1}$$

the *Weitzenböck operator* is

$$-\mathscr{E}_p = \Delta_p - B_{\Lambda^p} = \Delta_p - \nabla^* \nabla = -R^i{}_j{}^k{}_l \varepsilon^l \iota_k \varepsilon^j \iota_i. \tag{2.2}$$

Note that (2.1) is a formula for the curvature of the Levi-Civita connection on Λ^p. When applied in Λ^1, (2.1) also serves to fix our convention on the placement of indices in R. With this convention, $\rho_{ij} = R^k{}_{ikj}$. It is clear from these formulas that we shall have to compute with expressions $\varepsilon^i \iota_j \ldots \varepsilon^k \iota_l$ of various lengths. By the identity

$$\iota_i \varepsilon^j + \varepsilon^j \iota_i = \delta_i{}^j, \tag{2.3}$$

all such expressions can be written in terms of the

$$\varepsilon^{j_1} \ldots \varepsilon^{j_r} \iota_{i_r} \ldots \iota_{i_1}. \tag{2.4}$$

Because of the identities

$$\varepsilon^i \varepsilon^j = -\varepsilon^j \varepsilon^i, \quad \iota_i \iota_j = -\iota_j \iota_i, \tag{2.5}$$

it is easy to take the fiber trace of (2.4):

$$\operatorname{trace}_{\Lambda^p_x} \varepsilon^{j_1} \ldots \varepsilon^{j_r} \iota_{i_r} \ldots \iota_{i_1} = \binom{m-r}{p-r} \theta_{i_1 \ldots i_r}{}^{j_1 \ldots j_r}, \tag{2.6}$$

where

$$\theta_{i_1 \ldots i_r}{}^{j_1 \ldots j_r} = \begin{cases} 1 \text{ if } (j_1, \ldots, j_r) \text{ is an even permutation of } (i_1, \ldots, i_r), \\ -1 \text{ if } (j_1, \ldots, j_r) \text{ is an odd permutation of } (i_1, \ldots, i_r), \\ 0 \text{ otherwise} \end{cases}$$
$$= \langle X_{i_1} \wedge \ldots \wedge X_{i_r}, \eta^{j_1} \wedge \ldots \wedge \eta^{j_r} \rangle.$$

The binomial coefficients are set equal to zero if the entries are outside the usual range:

$$\binom{N}{P} = 0 \text{ if } P \notin \{0, \dots, N\}.$$

If T is any $2r$-tensor, we shall denote the operator

$$T^{i_1}{}_{j_1 \dots}{}^{\dots i_r}{}_{j_r} \varepsilon^{j_1} \dots \varepsilon^{j_r} \iota_{i_r} \dots \iota_{i_1}$$

by $\tilde{T}$. Note that $\tilde{T}$ is itself a tensor quantity; specifically, a C^∞ section of $\Lambda^p \otimes (\Lambda^p)^*$, and is independent of any choice of frame.

3. Leading terms in the heat invariants for the Laplacians of the de Rham complex.

To calculate the linear terms in $e_n(x, \Delta_p)$, we just need to calculate $B^{n-1}\mathscr{E}_p$ modulo quadratic terms. For $n = 1$ there is nothing to do but put things in the normal form (2.4):

$$-\mathscr{E}_p = \rho^i{}_l \varepsilon^l \iota_i + R^i{}_j{}^k{}_l \varepsilon^l \varepsilon^j \iota_k \iota_i = \tilde{\rho} - \tilde{R}.$$

Note that this gives

$$(3.1) \qquad -\operatorname{trace}_{\Lambda^p_x} \mathscr{E}_p = \binom{m-2}{p-1} \tau.$$

Now exterior and interior multiplication are natural operations of tensors on tensors; that is, ε and ι are parallel sections of $\Lambda^{p+1} \otimes (\Lambda^p)^* \otimes TM$ and $\Lambda^{p-1} \otimes (\Lambda^p)^* \otimes T^*$M respectively. Thus

$$-B^{n-1}\mathscr{E}_p = (-1)^n (\nabla^{a_1}\nabla_{a_1} \dots \nabla^{a_{n-1}}\nabla_{a_{n-1}} R^i{}_j{}^k{}_l) \varepsilon^l \iota_k \varepsilon^j \iota_i, \quad n \geqq 1.$$

By the Bianchi identity $R_{ijkl|a} + R_{ijla|k} + R_{ijak|l} = 0$ and the curvature symmetry $R_{ijkl} = R_{klij}$,

$$\begin{aligned} R^i{}_j{}^k{}_{l|a}{}^a &= -R^i{}_{jla|}{}^{ka} - R^i{}_{ja}{}^k{}_{|l}{}^a \\ &\cong_{\operatorname{mod} \mathscr{P}_{4,2}} -R^i{}_{jla|}{}^{ak} + R^i{}_j{}^k{}_{a|}{}^a{}_l \\ &= \rho_{lj|}{}^{ik} - \rho_l{}^i{}_{|j}{}^k - \rho^k{}_{j|}{}^i{}_l + \rho^{ki}{}_{|jl}, \end{aligned}$$

where indices after the bar indicate covariant differentiations. By the contracted Bianchi identity $\rho_{ij|}{}^i = \frac{1}{2}\tau_{|j}$, (2.3), and (2.5), this gives

$$R^i{}_j{}^k{}_{l|a}{}^a \varepsilon^l \iota_k \varepsilon^j \iota_i \cong_{\operatorname{mod} \mathscr{P}_{4,2}} 2\rho^k{}_{j|l}{}^i \varepsilon^l \varepsilon^j \iota_k \iota_i - \rho^i{}_{l|a}{}^a \varepsilon^l \iota_i.$$

Thus

$$-B^{n-1}\mathscr{E}_p \cong_{\operatorname{mod} \mathscr{P}_{2n,2}(\operatorname{End} \Lambda^p)} 2(B^{n-2}\nabla\nabla\rho)^\sim + (B^{n-1}\rho)^\sim, \quad n \geqq 2.$$

In view of (1.2), we have proved:

PROPOSITION 3.1.

$$c_n^{-1}e_n(x, \Delta_p) \cong_{\operatorname{mod}\mathscr{P}_{2n,2}(\operatorname{End}\Lambda^p)}$$

$$-2n(\Delta^{n-1}\tau)I + 4(2n+1)\{2(B^{n-2}\nabla\nabla\rho)^{\sim} + (B^{n-1}\rho)^{\sim}\}, \quad n \geqq 2,$$

$$e_1(x, \Delta_p) = \tfrac{1}{6}\tau I - \tilde{\rho} + \tilde{R}.$$

In particular,

$$(3.2)\quad c_n^{-1}a_n(x, \Delta_p) \cong_{\operatorname{mod}\mathscr{P}_{2n,2}(\Lambda^0)} \left\{-2n\binom{m}{p} + 4(2n+1)\binom{m-2}{p-1}\right\}\Delta^{n-1}\tau, \quad n \geqq 1.$$

Note that our formula for $a_n(x, \Delta_p)$ is invariant under $p \to m - p$, as it should be because of the Hodge $*$ operator. Orientability is not an issue; the $a_n(x, \Delta_p)$ are locally determined, and one can always pick a local orientation to define $*$. In the cases $n = 1, 2$, (3.2) agrees with Patodi's formula in [15, Proposition 2.1]. (Note that Patodi uses the opposite sign convention for Δ).

We now turn to the calculation of the quadratic terms. Since covariant differentiation commutes with the fiber trace, (3.1) gives

$$-\operatorname{trace}_{\Lambda_x^p}\nabla^{n-2}\tau\cdot\nabla^{n-2}\mathscr{E}_p = \binom{m-2}{p-1}|\nabla^{n-2}\tau|^2, \quad n \geqq 2.$$

Thus it remains only to compute the pure bundle curvature and Weitzenböck terms in (1.3). We shall suppress the dependence of the bundle curvature and Weitzenböck operator on p and denote them just by Ω and $-\mathscr{E}$. By the above identities and the naturality of ε and ι,

$$\operatorname{trace}_{\Lambda_x^p}\nabla^{n-2}\Omega^{ij}\cdot\nabla^{n-2}\Omega_{ij} = (\nabla^{n-2}R^k{}_l{}^{ij})(\nabla^{n-2}R^a{}_{bij})\operatorname{trace}_{\Lambda_x^p}\varepsilon^l\iota_k\varepsilon^b\iota_a$$

$$= -\binom{m-2}{p-1}|\nabla^{n-2}R|^2, \quad n \geqq 2.$$

This is also invariant under $p \to m - p$. Modulo higher-degree terms and divergences, the Bianchi identity simplifies this expression when $n \geqq 3$:

$$(3.3)\qquad |\nabla^{n-2}R|^2 \cong_{\operatorname{mod}(\mathscr{D}_{2n}+\mathscr{P}_{2n,3}(\Lambda^0))} 4|\nabla^{n-2}\rho|^2 - |\nabla^{n-2}\tau|^2, \quad n \geqq 3.$$

(See [3, Remark 1.4].)

Finally, consider the pure Weitzenböck term. We shall first do the computation in the case $n = 2$, then indicate how things change for $n \geqq 3$. By (2.2), the curvature symmetries $R_{ijkl} = R_{klij} = -R_{ijlk}$, (2.3), the possibility of commuting operators under the trace, and (2.6), we get

$$(3.4)\quad \operatorname{trace}_{\Lambda_x^p}\mathscr{E}\cdot\mathscr{E} = R^a{}_b{}^c{}_d R^i{}_j{}^k{}_l \operatorname{trace}_{\Lambda_x^p}\varepsilon^d\iota_c\varepsilon^b\iota_a\varepsilon^l\iota_k\varepsilon^j\iota_i$$

$$= -R^a{}_b{}^c{}_d R^i{}_j{}^k{}_l \operatorname{trace}_{\Lambda_x^p}\iota_c\varepsilon^d\varepsilon^b\iota_a\varepsilon^l\iota_k\varepsilon^j\iota_i$$

$$= R^a{}_b{}^c{}_d R^i{}_j{}^k{}_l \operatorname{trace}_{\Lambda_x^{p+1}} \varepsilon^d \iota_a \varepsilon^b \varepsilon^l \iota_k \varepsilon^j \iota_i \iota_c$$

$$= R^a{}_b{}^c{}_d R^i{}_j{}^k{}_l \operatorname{trace}_{\Lambda_x^{p+1}} (\delta_a{}^d - \iota_a \varepsilon^d) \varepsilon^b \varepsilon^l (\delta_k{}^j - \varepsilon^j \iota_k) \iota_i \iota_c$$

$$= \rho^c{}_b \rho^i{}_l \operatorname{trace}_{\Lambda_x^{p+1}} \varepsilon^b \varepsilon^l \iota_i \iota_c + \rho^c{}_b R^i{}_j{}^k{}_l \operatorname{trace}_{\Lambda_x^{p+1}} \varepsilon^b \varepsilon^l \varepsilon^j \iota_k \iota_i \iota_c$$

$$+ R^a{}_b{}^c{}_d \rho^i{}_l \operatorname{trace}_{\Lambda_x^{p+2}} \varepsilon^d \varepsilon^b \varepsilon^l \iota_i \iota_c \iota_a$$

$$+ R^a{}_b{}^c{}_d R^i{}_j{}^k{}_l \operatorname{trace}_{\Lambda_x^{p+2}} \varepsilon^d \varepsilon^b \varepsilon^l \varepsilon^j \iota_k \iota_i \iota_c \iota_a$$

$$= \rho^c{}_b \rho^i{}_l \binom{m-2}{p-1} \theta_{ci}{}^{bl} + \rho^c{}_b R^i{}_j{}^k{}_l \binom{m-3}{p-2} \theta_{cik}{}^{blj}$$

$$+ R^a{}_b{}^c{}_d \rho^i{}_l \binom{m-3}{p-1} \theta_{aci}{}^{dbl} + R^a{}_b{}^c{}_d R^i{}_j{}^k{}_l \binom{m-4}{p-2} \theta_{acik}{}^{dblj}.$$

The first term on the right in (3.4) reduces to $\binom{m-2}{p-1}(-|\rho|^2 + \tau^2)$, and the second and third combine to make $\binom{m-2}{p-1}(2|\rho|^2 - \tau^2)$. By the curvature symmetries above and the Bianchi identity $R_{ijkl} + R_{iklj} + R_{iljk} = 0$, the fourth term reduces to $\binom{m-4}{p-2}(-4|\rho|^2 + \tau^2 + |R|^2)$. The total is

$$\text{(3.5)} \qquad \operatorname{trace}_{\Lambda_x^p} \mathscr{E} \cdot \mathscr{E} = \binom{m-2}{p-1} |\rho|^2 + \binom{m-4}{p-2}(-4|\rho|^2 + \tau^2 + |R|^2).$$

To calculate $\operatorname{trace}_{\Lambda_x^p} \nabla^{n-2}\mathscr{E} \cdot \nabla^{n-2}\mathscr{E}$ for $n \geqq 3$, we just need to place a ∇^{n-2} in front of each curvature quantity in the above calculation; this gives us an analogue of (3.5). Weakening the equality to congruence modulo $\mathscr{D}_{2n} + \mathscr{P}_{2n,3}(\Lambda^0)$ results in a considerable simplification via (3.3):

$$\operatorname{trace}_{\Lambda_x^p} \nabla^{n-2}\mathscr{E} \cdot \nabla^{n-2}\mathscr{E} \cong_{\operatorname{mod}(\mathscr{D}_{2n} + \mathscr{P}_{2n,3}(\Lambda^0))} \binom{m-2}{p-1} |\nabla^{n-2}\rho|^2, \quad n \geqq 3.$$

Substituting into (1.3) and (1.4), we have:

Proposition 3.2.

$$c_n^{-1} a_n(x, \Delta_p) \cong_{\operatorname{mod}(\mathscr{D}_{2n} + \mathscr{P}_{2n,3}(\Lambda^0))}$$

$$\left\{(n^2 - n - 1)\binom{m}{p} - 2(2n+1)(2n-3)\binom{m-2}{p-1}\right\} |\nabla^{n-2}\tau|^2$$

$$+ \left\{2\binom{m}{p} + 4(2n+1)(2n-3)\binom{m-2}{p-1}\right\} |\nabla^{n-2}\rho|^2, \quad n \geqq 3,$$

$180a_2(x, \Delta_p) \cong_{\mathrm{mod}\, \mathscr{D}_4}$

$$\left\{\frac{5}{2}\binom{m}{p} - 30\binom{m-2}{p-1} + 90\binom{m-4}{p-2}\right\}\tau^2 + \left\{-\binom{m}{p} + 90\binom{m-2}{p-1} - 360\binom{m-4}{p-2}\right\}|\rho|^2 + \left\{\binom{m}{p} - 15\binom{m-2}{p-1} + 90\binom{m-4}{p-2}\right\}|R|^2. \tag{3.6}$$

In particular, the $a_n(\Delta_p)$ for $n \geqq 2$ are given by the integrals of the expressions on the right (times c_n and 180 respectively), modulo cubic and higher-degree terms.

REMARK 3.3. In addition to checking invariance under $p \to m - p$, which we have immediately because all our coefficients are linear combinations of the $\binom{m-2r}{p-r}$, we can check Propositions 3.1 and 3.2 against the local Chern-Gauss-Bonnet, or "Fantastic Cancellation" Theorem [16], [10, Theorem 2.4.8]. This asserts that

$$\sum_{p=0}^{m} (-1)^p a_n(x, \Delta_p) \quad \begin{array}{l} = 0, \ m > 2n, \\ = (4\pi)^{m/2}\,\mathrm{Pff},\ m = 2n, \\ \in \mathscr{D}_{2n},\ m < 2n, \end{array}$$

where in even dimensions m, $\mathrm{Pff} \in \mathscr{P}_{m,m/2}(\Lambda^0)$ is the Pfafffian. The consequence for the linear terms is that

$$1 \leqq n \leqq m/2 \Rightarrow \sum_{p=0}^{m} (-1)^p a_n(x, \Delta_p) \cong_{\mathrm{mod}\, \mathscr{P}_{2n,2}(\Lambda^0)} \begin{cases} \tau, & m = 2,\ n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For the quadratic terms, we must have

$$n \geqq 2 \Rightarrow \sum_{p=0}^{m} (-1)^p a_n(x, \Delta_p) \cong_{\mathrm{mod}(\mathscr{D}_{2n} + \mathscr{P}_{2n,3}(\Lambda^0))} \begin{cases} \frac{1}{2}\tau^2 - 2|\rho|^2 + \frac{1}{2}|R|^2, & m = 4,\ n = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where we have substituted the formulas for the Pfaffians in dimensions two and four. But since

$$\sum_{p=0}^{m} (-1)^p \binom{m-2r}{p-r} = \begin{cases} 1, & m = 2r, \\ 0 & \text{otherwise,} \end{cases}$$

these are exactly the results we get from (3.2) and Proposition 3.2. We can also check (3.6) against [15, Proposition 2.1]. Note that $\mathscr{D}_4$ is just $\mathrm{R}\Delta\tau$.

4. Leading terms in the heat invariants for the Laplacians of the signature complex.

Consider now the more specialized category of compact, oriented Riemannian manifolds. The Hodge $*$-operator on p-forms has $** = (-1)^{p(m-p)}$, so one can split the bundles $\Lambda^p \oplus \Lambda^{m-p}(p \neq m/2)$ and $\Lambda^{m/2}$ (m even) into ± 1 eigenbundles of $\nu \equiv (\sqrt{-1})^{p(m-p)}*$, called $(\Lambda^p \oplus \Lambda^{m-p})_{\pm}$ and $\Lambda^{m/2}_{\pm}$. When $m \in 4\mathbf{Z}^+$, $\Lambda^{m/2}_{\pm}$ are the bundles of self-dual and anti-self-dual middle-forms. The splittings determined by ν are equivariant for the structure group $SO(m)$, and since $\Lambda^p \cong_{SO(m)} \Lambda^{m-p}$ is an irreducible $SO(m)$-*bundle for* $m \neq 2p$, we must have $(\Lambda^p \oplus \Lambda^{m-p})_{\pm} \cong_{SO(m)} \Lambda^p$ in this case. Since the bundle curvatures and Weitzenböck operators can be written purely in terms of R and the representations to which the Λ^p are associated, the heat invariants of the form Laplacians depend only on the $SO(m)$-isometry types of the bundles involves. Thus the operator $(\Delta_p \oplus \Delta_{m-p})_{\pm}$ on $(\Lambda^p \oplus \Lambda^{m-p})_{\pm}$ produces the same heat invariants as the operator Δ_p on Λ^p when $m \neq 2p$.

The situation is potentially more interesting in the middle order for even m, where the local form of the Hirzebruch Signature Theorem [10, Theorem 3.1.1] says that

$$a_n(x,(\Delta_{m/2})_+) - a_n(x,(\Delta_{m/2})_-) \begin{cases} = 0, \ m > 2n \text{ or } m \in 2(2\mathbf{N}+1), \\ = (4\pi)^{m/2} L_m, \ m = 2n \in 4\mathbf{Z}^+, \\ \in \mathcal{D}_{2n} \text{ otherwise.} \end{cases}$$

Here $L_m \in \mathcal{P}_{m,m/2}(\Lambda^0)$ is the Hirzebruch polynomial. This implies that the quadratic, non-divergence terms in $a_n(x, D_{m/2})$ will be evenly split between $a_n(x, \Delta_{m/2})_+)$ and $a_n(x,(\Delta_{m/2})_-)$ unless $m = 4$, where we have a formula for the Hirzebruch polynomial:

$$L_4 = \frac{1}{48\pi^2}(|C_+|^2 - |C_-|^2),$$

where $C_{\pm}$ are the self-dual and anti-self-dual parts of the Weyl conformal curvature tensor C. Combining this information with Propositions 3.1 and 3.2, we can conclude:

PROPOSITION 4.1. *If* $m \neq 2p$, *then*

$$a_n(x,(\Delta_p \oplus \Delta_{m-p})_{\pm}) = a_n(x, \Delta_p);$$

in particular, the leading terms are as given in (3.2) *and Proposition* 3.2. *If* $m \in 2(2\mathbf{N}+1)$ *or* $2n < m \in 4\mathbf{Z}^+$,

$$a_n(x,(\Delta_{m/2})_{\pm}) = \tfrac{1}{2}a_n(x, \Delta_{m/2});$$

in particular, the leading terms are half those given in (3.2) *and Proposition* 3.2. *If* $m = 4$,

$$(4.1)\ 120a_2(x,(\Delta_2)_{\pm}) = 8\Delta\tau + 15\tau^2 - 62|\rho|^2 + 22|R|^2 \pm 20(|C_+|^2 - |C_-|^2).$$

The proposition does not give the linear terms in the case $2n > m = 2p \in 4\mathbf{Z}^+$. The expression on the right in (4.1) is given in terms of six local scalar invariants, only five of which are linearly independent. (4.1) is most intelligible when written in terms of the representatives τ, $b \equiv \rho - \tau g/4$, C_+, and C_- of the four SO(4)-irreducible summands of the bundle of curvature tensors in dimension four. One easily computes that $|\rho|^2 = |b|^2 + \frac{1}{4}\tau^2$, $|R|^2 = |C|^2 + 2|b|^2 + \frac{1}{6}\tau^2$, so that

$$720a_2(x,(\Delta_2)_\pm) = 48\Delta\tau + 19\tau^2 - 108|b|^2 + 252|C_\pm|^2 + 12|C_\mp|^2, \quad m = 4.$$

5. Leading terms in the heat invariants for the Laplacians of the spin complex.

We now specialize further to the category of compact, oriented Riemannian manifolds with a spin structure. Let Σ be the full spinor bundle of fiber dimension $\mu = 2^{[m/2]}$. If m is even, there is a Spin(m)-equivalent splitting $\Sigma = \Sigma_+ \oplus \Sigma_-$ into bundles of *positive* and *negative* spinors. These are the ± 1 eigenbundles of the bundle map $\xi = (\sqrt{-1})^{m(m+1)/2}\gamma^1 \ldots \gamma^m$, where γ is the fundamental section of $TM \otimes \operatorname{End}\Sigma$ (or *spin representation*), normalized so that $\gamma^i\gamma^j + \gamma^j\gamma^i = -2g^{ij}$, and the local indices are taken in an orthonormal frame, and in an order consistent with the orientation. Σ carries a natural connection ∇, determined by the condition that $\nabla\gamma = 0$. The *Dirac operator* $P = \gamma^i\nabla_i$ carries $C^\infty(\Sigma)$ to itself, and carries $C^\infty(\Sigma_\pm)$ to $C^\infty(\Sigma_\mp)$ for even m. We denote by $P_\pm$ the restriction of P to $C^\infty(\Sigma_\pm)$.

With our normalizations, the curvature operator of Σ is

$$\Omega_{ij} = -\tfrac{1}{4}R^k{}_{lij}\gamma_k\gamma^l,$$

by, e.g., [12, I.2.7]. Tha analogue of the Weitzenböck formula in this setting is the Lichnerowicz formula, which says that $P^2 = \nabla^*\nabla + \tau/4 = B_{(\Sigma,\nabla)} + \tau/4$. By [2, pp. 98, 99] and analogous calculations with ∇^{n-2} operating on each curvature quantity,

$$\operatorname{trace}_\Sigma \nabla^{n-2}\Omega^{ij}\cdot\nabla^{n-2}\Omega_{ij} = -\frac{\mu}{8}|\nabla^{n-2}R|^2, \quad n \geqq 2.$$

These considerations, (3.3), and (1.4) immediately give us the leading terms in the heat invariants for P^2:

PROPOSITION 5.1.

$$c_n^{-1}e_n(x,P^2) \cong_{\operatorname{mod}\mathscr{P}_{2n,2}(\operatorname{End}\Sigma)}(\Delta^{n-1}\tau)I, \quad n \geqq 1,$$

$$c_n^{-1}2^{-[m/2]}a_n(x,P^2) \cong_{\operatorname{mod}\mathscr{P}_{2n,2}(\Lambda^0)}\Delta^{n-1}\tau, \ n \geqq 1,$$

$$c_n^{-1}2^{-[m/2]}a_n(x,P^2) \cong_{\operatorname{mod}(\mathscr{Q}_{2n}+\mathscr{P}_{2n,3}(\Lambda^0))}\tfrac{1}{2}n|\nabla^{n-2}\tau|^2 - (2n-1)|\nabla^{n-2}\rho|^2, \quad n \geqq 3,$$

$$180\cdot 2^{3-[m/2]}a_2(x,P^2) = 12\Delta\tau + 5\tau^2 - 8|\rho|^2 - 7|R|^2.$$

When m is even, the local form of the index theorem for the spin complex [10, Theorem 3.4.4] says that

$$a_n(x, P_-P_+) - a_n(x, P_+P_-) \quad \begin{array}{l} = 0, m > 2n, \\ = (4\pi)^{m/2}\hat{A}_m, m = 2n, \\ \in \mathscr{D}_{2n} \text{ otherwise.} \end{array}$$

Here $\hat{A}_m \in \mathscr{P}_{m,m/2}(\Lambda^0)$ is the $\hat{A}$-polynomial. This implies that all leading terms are evenly split between positive and negative spinors except when $m = 4$, where, by e.g. [1, 6.72], $\hat{A}_4 = L_4/16$:

PROPOSITION 5.2. *Suppose that m is even. Then*

$$c_n^{-1}e_n(x, P_\mp P_\pm) \cong_{\operatorname{mod} \mathscr{P}_{2n,2}(\operatorname{End}\Sigma_\pm)}(\Delta^{n-1}\tau)I_{\Sigma_\pm}, \quad n \geqq 1.$$

The leading terms in $a_n(x, P_\mp P_\pm)$ are half those given in Proposition 5.1 unless $m = 4$ and $n = 2$. In this case,

$$\begin{aligned} 1440a_2(x, P_\mp P_\pm) &= 24\Delta\tau + 10\tau^2 - 16|\rho|^2 - 14|R|^2 \pm 15(|C_+|^2 - |C_-|^2) \\ &= 24\Delta\tau + \tfrac{11}{3}\tau^2 - 44|b|^2 + |C_\pm|^2 - 29|C_\mp|^2. \end{aligned}$$

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