MATH. SCAND. 67 (1990), 5–14

CANCELLATION OF ABELIAN GROUPS OF FINITE RANK MODULO ELEMENTARY EQUIVALENCE

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Introduction.

Following R. Hirshon, we say that a group A *may be cancelled in direct products* if, for any groups G, H, $A \times G \cong A \times H$ implies $G \cong H$. Results, examples and references concerning the cancellation properties of groups can be found in [5] and other papers of R. Hirshon. Finite groups and many other familiar groups may be cancelled. On the other hand, [5] gives some examples of non abelian groups G, H which satisfy $\mathbf{Z} \times G \cong \mathbf{Z} \times H$ without being isomorphic.

According to [7], if G and H are groups such that $\mathbf{Z} \times G \cong \mathbf{Z} \times H$, then G and H are elementarily equivalent. By [9], the converse is true for finitely generated finite-by-nilpotent groups. On the other hand, [8, Proposition, p. 1042] gives an example of two polycyclic abelian-by finite groups G, H which are elementarily equivalent while $\mathbf{Z} \times G$ and $\mathbf{Z} \times H$ are not isomorphic. According to [4], the result from [7] remains true if we replace $\mathbf{Z}$ by any subgroup of $\mathbf{Q}^n$ for an integer $n \in \mathbf{N}$.

In the present paper, we give conditions on the abelian groups A, B which imply that, for any groups G, H, if $A \times G$ and $B \times H$ are isomorphic, then G and H are elementary equivalent. In particular, we prove the following result:

Let A be an abelian group such that, for each prime number p and for each subgroup S of A, S/pS is finite. If G and H are groups such that $A \times G$ and $A \times H$ are isomorphic, then G and H are elementarily equivalent.

The condition on A is satisfied by any abelian group which is an homomorphic image of a subroup of $\mathbf{Q}^n$ for an integer $n \in \mathbf{N}$. In particular, our result generalizes [7] and [4] (see Examples 2 and 3 below). On the other hand, we have $M^{(\omega)} \times \{1\} \cong M^{(\omega)} \times M$ for each group M, while $\{1\}$ and M are elementarily equivalent if and only if M is trivial.

Received June 20, 1989

Definitions and main theorem.

For each integer $n \geqq 1$, we consider $\mathbf{Z}(n) = \mathbf{Z}/n\mathbf{Z}$. For each prime number p, we denote by $\hat{\mathbf{Z}}_p$ the p-adic completion of $\mathbf{Z}$ and we write $\mathbf{Z}(p^\infty) = \{a/p^k \mid a \in \mathbf{Z}$ and $k \in \mathbf{N}\}/\mathbf{Z}$.

If S is a subset of a group M, we denote by $\langle S \rangle$ the subgroup of M which is generated by S. If G is a subgroup of a group M, we say that a subgroup A of M is a *supplementary* of G in M if we have $A \cap G = \{1\}$, $[A, G] = \{1\}$ and $\langle A, G \rangle = M$. Similarly, if R is a ring and if G is a submodule of an R-module M, we say that a submodule A of M is a *supplementary* of G in M if we have $A \cap G = \{0\}$ and $A + G = M$.

Now, let us consider an abelian group M with additive notation. We denote by $t(M)$ the torsion subgroup of M and $d(M)$ the subgroup which consists of the elements which are divisible by each integer $k \geqq 1$. For each prime number p, we write $M[p] = \{x \in M \mid px = 0\}$; we denote by $t_p(M)$ the subgroup which consists of the elements $x \in M$ which satisfy $p^k x = 0$ for an integer $k \geqq 1$ and $d_p(M)$ the subgroup which consists of the elements which are divisible by each integer k which is not divisible by p.

For each prime number p and for each integer k, we consider the following invariants, according to the notations of [2]:

$$\mathrm{Tf}(p; M) = \inf_{h \in \mathbf{N}} \dim (p^h M / p^{h+1} M) \text{ if finite, } \infty \text{ otherwise;}$$
$$D(p; M) = \inf_{h \in \mathbf{N}} \dim ((p^h M)[p]) \text{ if finite, } \infty \text{ otherwise;}$$
$$U(p, k; M) = \dim (((p^k M)[p])/((p^{k+1} M)[p])) \text{ if finite, } \infty \text{ otherwise.}$$

For each prime number p, $\dim (p^h M / p^{h+1} M)$ and $\dim ((p^h M)[p])$ are monotonically decreasing functions of h. The invariants $\mathrm{Tf}(p; M)$, $D(p; M)$ and $U(p, k; M)$ are first-order definable.

Now, we investigate the abelian groups M which satisfy the following property:

(P) For each subgroup S of M and for each prime number p, S/pS is finite.

This condition is equivalent to saying that the torsion-free rank $r_0(M)$ and all the p-torsion ranks $r_p(M)$ are finite, or equivalently, that the injective hull of M has only finitely many copies of $\mathbf{Q}$ and of $\mathbf{Z}(p^\infty)$ for each prime number p (the torsion-free rank and the p-torsion ranks are defined in [3, § 16]; the injective hull is defined in [3, § 24]). Finite direct products, subgroups and homomorphic images of abelian groups which satisfy (P) also satisfy (P).

If M satisfies (P), then, for each subgroup S of M and for each integer $n \geqq 1$, S/nS is finite. The invariants $\mathrm{Tf}(p; M)$, $D(p; M)$ and $U(p, k; M)$ are finite, as well as $\dim (M[p]) = D(p; M) + \sum_{k \in \mathbf{N}} U(p, k; M)$. The subgroup $d(M)$ is divisible: for each

$x \in d(M)$ and for each integer $n \geqq 1$, there are finitely many elements $y \in M$ such that $ny = x$; as x is divisible by $nk!$ for each integer k, one of these elements is divisible by $k!$ for infinitely many integers k and therefore belongs to $d(M)$. We can show in a similar way that $n\, d_p(M) = d_p(M)$ for each prime number p and for each integer n which is not divisible by p. The divisible torsion-free group $d(M)/(d(M) \cap t(M))$ is isomorphic to the additive structure of a vector space over $\mathbb{Q}$. We denote by $Q(M)$ the finite dimension of this vector space. The invariant Q is not first-order definable.

We consider the following relation between abelian groups which satisfy (P):

(R) A and B are elementarily equivalent and satisfy $Q(A) = Q(B)$.

It follows from [2, Theorem 2.2 and Theorem 2.6] that A and B satisfy (R) if and only if they satisfy $Q(A) = Q(B)$, $\mathrm{Tf}(p; A) = \mathrm{Tf}(p; B)$, $D(p; A) = D(p; B)$ and $U(p, k; A) = U(p, k; B)$ for each prime number p and for each integer $k \in \mathbb{N}$.

Now, we state the main theorem:

THEOREM. *Let A,B be abelian groups such that, for each prime number p and for each subgroup S of A (respectively B), S/pS is finite. Let us suppose that A and B are elementarily equivalent and that the divisible torsion-free groups $d(A)/(d(A) \cap t(A))$ and $d(B)/(d(B) \cap t(B))$ have the same dimension over $\mathbb{Q}$. If G and H are groups such that $A \times G$ and $B \times H$ are isomorphic, then G and H are elementarily equivalent.*

REMARK 1. We do not suppose that G and H are abelian.

REMARK 2. We must suppose that A and B satisfy (R) and not only that they are elementarily equivalent. For instance, $A = \mathbb{Q}$ and $B = \mathbb{Q} \times \mathbb{Q}$ are elementarily equivalent by [2, Theorem 2.6] and do not satisfy $Q(A) = Q(B)$; we have $A \times G \cong B \times H$ for $G = \mathbb{Q}$ and $H = \{1\}$, but G and H are not elementarily equivalent.

EXAMPLE 1. In [6], B. Jonsson gives an example of a subgroup A of $\mathbb{Q}$ and two nonisomorphic subgroups G, H of $\mathbb{Q}^2$ such that $A \times G \cong A \times H$.

EXAMPLE 2. If G and H are abelian groups, then $\mathbb{Z} \times G \cong \mathbb{Z} \times H$ implies $G \cong H$. On the other hand, [5] gives some examples of nonisomorphic finitely generated nilpotent groups G, H which satisfy $\mathbb{Z} \times G \cong \mathbb{Z} \times H$. By [7], if G and H are groups such that $\mathbb{Z} \times G \cong \mathbb{Z} \times H$, then G and H are elementarily equivalent. This result is generalized by the theorem above.

EXAMPLE 3. If M is a homomorphic image of a subgroup of $\mathbb{Q}^n$ for an integer $n \in \mathbb{N}$, then M satisfies (P). In order to prove this result, it suffices to show that $\mathbb{Q}$ satisfies (P). As a matter of fact, for each subgroup S of $\mathbb{Q}$ and for each prime number p, we have $|S/pS| \leqq p$ since any finitely generated subgroup of S is generated by one element.

So, the theorem above can be applied for A, B homomorphic images of subgroups of $\mathbb{Q}^n$. In particular, it generalizes [4].

EXAMPLE 4. The abelian group $M = \bigoplus_{p \text{ prime}} \mathbb{Z}(p)^p$ is not a homomorphic image of a subgroup of $\mathbb{Q}^n$ for an integer $n \in \mathbb{N}$. Anyhow, M satisfies (P) since we have $S = \bigoplus_{p \text{ prime}} (S \cap \mathbb{Z}(p)^p)$ for each subgroup S of M.

Proof of the theorem.

If $I(M)$ is any of the invariants $U(p,k;M)$, $\mathrm{Tf}(p;M)$, $D(p;M)$ and $Q(M)$ defined above, then we have $I(M \times N) = I(M) + I(N)$ for any abelian groups M,N which satisfy (P). Moreover, $I(A)$ and $I(B)$ are finite since A and B satisfy (P) and we have $I(A) = I(B)$ since A and B satisfy (R). If G and H are abelian and satisfy (P), it follows that $I(G) = I(M) - I(A) = I(M) - I(B) = I(H)$. So, G and H satisfy (R) if they satisfy (P). This particular case of the theorem will be used in the proof of lemma 1 below.

We shall prove the following result, which is clearly equivalent to our theorem: If M is a group, if A,B,G,H are subgroups of M, if A and B are abelian and satisfy (P) and (R), if $M = \langle A, G\rangle = \langle B, H\rangle$ and if $A \cap G = [A, G] = B \cap H = [B, H] = \{1\}$, then G and H are elementarily equivalent.

We write $S = \langle A, B\rangle$; in S, A is a supplementary of $S \cap G$ and B is a supplementary of $S \cap H$; however, we must not suppose that $S \cap G$ and $S \cap H$ have a common supplementary in S, or that $S \cap G \cap H$ has supplementaries in $S \cap G$ and $S \cap H$, or, even, that $S \cap G \cap H$ is a pure subgroup of S; on the other hand, we have the following result:

LEMMA 1. *$S \cap G$ and $S \cap H$ satisfy* (P) *and* (R); *moreover,* $(S \cap G)/(S \cap G \cap H)$ *and* $(S \cap H)/(S \cap G \cap H)$ *satisfy* (P) *and* (R).

PROOF. The groups $S \cap G$ and $S \cap H$ satisfy (P) since they are respectively isomorphic to $B/(A \cap B)$ and $A/(A \cap B)$. The groups $(S \cap G)/(S \cap G \cap H)$ and $(S \cap H)/(S \cap G \cap H)$ also satisfy (P) since they are images of $S \cap G$ and $S \cap H$ respectively.

The groups $S \cap G$ and $S \cap H$ satisfy (R) since their supplementaries A, B satisfy (R). Let us consider $T = \langle S \cap G, S \cap H\rangle$; $T \cap A$ and $T \cap B$ satisfy (R) since, in T, $T \cap A$ is a supplementary of $S \cap G$ and $T \cap B$ is a supplementary of $S \cap H$. The groups $(S \cap G)/(S \cap G \cap H)$ and $(S \cap H)/(S \cap G \cap H)$ satisfy (R) since they are respectively isomorphic to $T/(S \cap H) \cong T \cap B$ and $T/(S \cap G) \cong T \cap A$.

We are going to prove that, for each countably incomplete ultrafilter U, we have $G^U \cong H^U$; G and H will be elementarily equivalent according to [1, Corollary 4.1.10].

By [1, Theorem 6.1.1], if U is a countably incomplete ultrafilter and if K is a structure associated with a countable language, then K^U is ω_1-saturated. In particular, if K is an abelian group which satisfies (P), it follows from [2, Theorem 1.11] that we have the following decomposition of K^U:

$$\text{(D)}\ K^U \cong \left[\prod_{p\,\text{prime}} \left(\hat{\mathsf{Z}}_p^{\mathrm{Tf}(p;K)} \oplus \left(\bigoplus_{n\geq 1} \mathsf{Z}(p^n)^{U(p,n-1;K)}\right)\right)\right] \oplus \left(\bigoplus_{p\,\text{prime}} \mathsf{Z}(p^\infty)^{D(p;K)}\right) \oplus \mathsf{Q}^{(\delta)}$$

with $\delta = 0$ if K is finite and $\delta \geq \omega_1$ if K is infinite. As a matter of fact, if K is infinite, we have $\delta = |K^U| = |\mathsf{N}^U|$.

In order to prove the last point, we consider, for each countably incomplete ultrafilter U over a set I, a sequence $(A_n)_{n\in\mathsf{N}}$ of pairwise disjoint subsets of I which do not belong to U and such that $\bigcup_{n\in\mathsf{N}} A_n = I$. As K is a group of unbounded order, there exists, for each integer $n \geq 1$, an element $x_n \in K$ such that $kx_n \neq 0$ for each $k\in\{1,\ldots,(n!)^3\}$. We define an injection from $[0,1[\, \subset \mathsf{R}$ to $d(K^U)/(d(K^U)\cap t(K^U))$ as follows: For each $u\in[0,1[$, we consider the sequence of integers $(u_n)_{n\in\mathsf{N}}$ such that $u_n \leq n!\,u < u_n + 1$ for each integer n, and the element $y(u)\in G^U$ which admits the system of representatives $(y_i(u))_{i\in I}$ in G^I with $y_i(u) = u_n\, n!\, x_n$ for each $n\in\mathsf{N}$ and each $i\in A_n$; the element $y(u)$ is divisible in K^U since, for each integer n, the elements $y_i(u)$ for $i\in \bigcup_{m\geq n} A_m$ are divisible by $n!$. For any elements $u \neq v$ in $[0,1[$, we have $n!\ (y_i(u) - y_i(v)) \neq 0$ for each integer n such that $n!\ |u - v| \geq 1$ and for each $i\in \bigcup_{m\geq n} A_m$; it follows that $y(u) - y(v)$ does not belong to $t(K^U)$. Consequently, we have $|d(K^U)/(d(K^U)\cap t(K^U))| \geq 2^\omega$, which implies $\delta \geq 2^\omega$, and therefore $\delta = |K^U|$ since K^U satisfies (D) with

$$\left|\left[\prod_{p\,\text{prime}} \left(\hat{\mathsf{Z}}_p^{\mathrm{Tf}(p;K)} \oplus \left(\bigoplus_{n\geq 1} \mathsf{Z}(p^n)^{U(p,n-1;K)}\right)\right)\right] \oplus \left(\bigoplus_{p\,\text{prime}} \mathsf{Z}(p^\infty)^{D(p;K)}\right)\right| \leq 2^\omega.$$

It follows that, if U is a countably incomplete ultrafilter and if K, L are abelian groups which satisfy (P) and (R), then K^U and L^U are isomorphic.

It suffices to show that $G^U \cap \langle A^U, B^U\rangle$ and $H^U \cap \langle A^U, B^U\rangle$ have a common supplementary R in $\langle A^U, B^U\rangle$; then, R is also a supplementary of G^U and H^U in M^U and we have $G^U \cong M^U/R \cong H^U$.

We write $S' = S^U = \langle A^U, B^U\rangle, A' = A^U, B' = B^U, G' = (G\cap S)^U = G^U \cap \langle A^U, B^U\rangle$ and $H' = (H\cap S)^U = H^U \cap \langle A^U, B^U\rangle$. In S', A' is a supplementary of G' and B' is a supplementary of H'. According to lemma 1 and the definition of U, A' is isomorphic to B', G' is isomorphic to H' and $G'/(G'\cap H')$ is isomorphic to $H'/(G'\cap H')$. We must show that G' and H' have a common supplementary in S'.

From now on, we only have to consider subgroups of the abelian group S'. So, we use the additive notation instead of the multiplicative notation. We write $S_1 = t(d(S')) = t(S')\cap d(S')$ and $S_2 = d(S')$; we define similarly A_1, A_2 in A', B_1, B_2 in B', G_1, G_2 in G' and H_1, H_2 in H'.

We have $d(S') \cap A' = d(A')$ since A' has a supplementary in S'. Clearly, we also have $t(S') \cap A' = t(A')$. It follows that we have $S_1 \cap A' = A_1$ and $S_2 \cap A' = A_2$. Similar equalities hold for B', G' and H'.

According to [3, Theorem 21.2], S_1 has a supplementary in S_2 and S_2 has a supplementary in S'. In the three following sections, we are going to show that:

1) G_1 and H_1 have a common supplementary in S_1;
2) $\langle G_2, S_1\rangle/S_1$ and $\langle H_2, S_1\rangle/S_1$ have a common supplementary in S_2/S_1;
3) $\langle G', S_2\rangle/S_2$ and $\langle H', S_2\rangle/S_2$ have a common supplementary in S'/S_2.

This result implies that G' and H' have a common supplementary in S' according to the following lemma:

LEMMA 2. *Let S be a group, let G, H be subgroups of S and let M be a subgroup of S which has a supplementary in S. If $G \cap M$ and $H \cap M$ have a common supplementary in M and if $\langle G, M\rangle/M$ and $\langle H, M\rangle/M$ have a common supplementary in S/M, then G and H have a common supplementary in S.*

PROOF OF LEMMA 2. Let N be a supplementary of M in S, let A be a supplementary of $G \cap M$ and $H \cap M$ in M and let B be a supplementary of $\langle G, M\rangle/M$ and $\langle H, M\rangle/M$ in S/M; let us denote by C the subgroup of N which consists of the representatives of elements of B in N.

As the subgroup A is a supplementary of $G \cap M$ in M, it is also a supplementary of G in $\langle G, M\rangle$. Moreover, C is a supplementary of $\langle G, M\rangle$ in S. So, $\langle A, C\rangle$ is a supplementary of G in S. We prove in a similar way that $\langle A, C\rangle$ is a supplementary of H in S.

G_1 and H_1 have a common supplementary in S_1.

We see from the decomposition (D) that S_1 is isomorphic to $\bigoplus_{p \text{ prime}} \mathbf{Z}(p^\infty)^{D(p;S)}$. So, G_1 and H_1 have a common supplementary in S_1 according to the second of the two following lemmas:

LEMMA 3. *Let K be a field, let S be a vector space over K and let G, H be two subspaces of S. If $G/(G \cap H)$ and $H/(G \cap H)$ are isomorphic, and in particular if G and H have the same finite dimension over K, then G and H have a common supplementary in S.*

PROOF. Let M and N be supplementaries of $G \cap H$ in G and H respectively; let f be an isomorphism from M to N. Then $A = \{x + f(x) \mid x \in M\}$ is a common supplementary of G and H in $G + H$. If B is a supplementary of $G + H$ in S, then $A + B$ is a common supplementary of G and H in S.

LEMMA 4. *Let S be a torsion group such that, for each prime number p, $\{x \in S \mid px = 0\}$ is finite. Let G, H be isomorphic subgroups of S which have supplementaries in S. Then G and H have a common supplementary in S.*

PROOF. We have $S = \bigoplus_{p \text{ prime}} t_p(S)$, $G = \bigoplus_{p \text{ prime}} t_p(G)$, $H = \bigoplus_{p \text{ prime}} t_p(H)$ and, for each prime number p, $t_p(S) \cap G = t_p(G)$ and $t_p(S) \cap H = t_p(H)$. So, it suffices to show that, for each prime number p, $t_p(G)$ and $t_p(H)$, which are isomorphic and have supplementaries in $t_p(S)$, have a common supplementary in $t_p(S)$. Consequently, we can suppose for the remainder of the proof that S is a p-torsion group for a prime number p.

For each integer i, we write $S(i) = \{x \in p^i S \mid px = 0\}$, $G(i) = \{x \in p^i G \mid px = 0\}$ and $H(i) = \{x \in p^i H \mid px = 0\}$; we have $S(i) \cap G = G(i)$ and $S(i) \cap H = H(i)$ since G and H have supplementaries in S. We have $S(j) \subset S(i)$ for any integers $i < j$. As $S(0) = \{x \in S \mid px = 0\}$ is finite, there exists an integer n such that $S(n) = \bigcap_{i \in \mathbf{N}} S(i)$.

$G(n)$ and $H(n)$ are isomorphic since G and H are isomorphic. So, according to lemma 3, $G(n)$ and $H(n)$ have a common supplementary in $S(n)$. We consider a basis B of this supplementary. We define by induction on $k \geqq 1$ a set $B(k)$ which consists of elements of S which are divisible by p^i for each integer $i \geqq 1$ as follows: we write $B(1) = B$; for each integer $k \geqq 1$ and for each $x \in B(k)$, there are finitely many elements $y \in S$ such that $py = x$; one of these elements is necessarily divisible by p^i for each integer $i \geqq 1$ since x is divisible by p^i for each integer $i \geqq 1$; we define $B(k+1)$ from $B(k)$ by choosing for each $x \in B(k)$ an element $y \in S$ which is divisible by p^i for each integer $i \geqq 1$ and satisfies $py = x$.

For each integer $i \in \{1, \dots, n\}$, $G(i-1)/G(i)$ and $H(i-1)/H(i)$ are isomorphic since G and H are isomorphic. So, according to lemma 3, $G(i-1)/G(i)$ and $H(i-1)/H(i)$ have a common supplementary in $S(i-1)/S(i)$. We denote by $C(i)$ a system of representatives in $S(i-1)$ of the elements of a basis of this supplementary. We choose for each $x \in C(i)$ an element $y \in S$ such that $p^{i-1}y = x$ and we denote by $D(i)$ the set which consists of these elements y.

We are going to prove that the subgroup K of S which is generated by $\left(\bigcup_{k \geqq 1} B(k)\right) \cup \left(\bigcup_{1 \leqq i \leqq n} D(i)\right)$ is a supplementary of G and H in S. As a matter of fact, we shall only give the proof for G, because the other proof is similar.

It follows from the definition of K that, for each integer $i \in \mathbf{N}$, $K(i) = \{x \in p^i K \mid px = 0\}$ is a supplementary of $G(i)$ in $S(i)$ generated by $B \cup \left(\bigcup_{j > i} C(j)\right)$.

We show by induction on $i \geqq 1$ that $K_i = \{x \in K \mid p^i x = 0\}$ is a supplementary of $G_i = \{x \in G \mid p^i x = 0\}$ in $S_i = \{x \in S \mid p^i x = 0\}$. This result is clear for $i = 1$ since we have $K_1 = K(0)$, $G_1 = G(0)$ and $H_1 = H(0)$. Now, we suppose that it is true for some integer $i \geqq 1$ and we prove that it is also true for $i + 1$. For each $x \in S_{i+1}$, as $p^i x$ is an element of $S(i) = \langle G(i), K(i) \rangle$, there are two elements $y \in G_{i+1}$ and $z \in K_{i+1}$ such that $p^i x = p^i y + p^i z$. The element $u = x - (y + z)$, which satisfies $p^i u = 0$, belongs to $S_i = \langle G_i, K_i \rangle$ and x belongs to $\langle G_{i+1}, K_{i+1} \rangle$. For each $x \in G_{i+1} \cap K_{i+1}$, as $p^i x$ belongs to $G_1 \cap K_1 = G(0) \cap K(0)$, we have $p^i x = 0$ and x belongs to $G_i \cap K_i = \{0\}$.

$\langle G_2, S_1 \rangle / S_1$ and $\langle H_2, S_1 \rangle / S_1$ have a common supplementary in S_2/S_1.

We see from the decomposition (D) that S_2 is isomorphic to $\left(\bigoplus_{p \text{ prime}} \mathbf{Z}(p^\infty)^{D(p;S)} \right) \oplus \mathbf{Q}^{(\delta)}$ with $\delta = 0$ if S is finite and $\delta = |K^U| = |\mathbf{N}^U|$ if S is infinite. So, S_2/S_1 is isomorphic to the additive structure of a vector space over $\mathbf{Q}$. The subgroups $\langle G_2, S_1 \rangle / S_1$ and $\langle H_2, S_1 \rangle / S_1$ are subspaces of S_2/S_1 since they have supplementaries. So, $\langle G_2, S_1 \rangle / S_1 \cap \langle H_2, S_1 \rangle / S_1$ is also a subspace of S_2/S_1.

According to lemma 3, in order to prove that $\langle G_2, S_1 \rangle / S_1$ and $\langle H_2, S_1 \rangle / S_1$ have a common supplementary in S_2/S_1, it suffices to show that the groups $(\langle G_2, S_1 \rangle / S_1)/(\langle G_2, S_1 \rangle / S_1 \cap \langle H_2, S_1 \rangle / S_1)$ and $(\langle H_2, S_1 \rangle / S_1)/(\langle G_2, S_1 \rangle / S_1 \cap \langle H_2, S_1 \rangle / S_1)$ are isomorphic. As a matter of fact, we are going to prove that these groups are respectively isomorphic to $d(G'/(G' \cap H'))/t(d(G'/(G' \cap H')))$ and $d(H'/(G' \cap H'))/t(d(H'/(G' \cap H')))$, which implies that they are isomorphic since $G'/(G' \cap H')$ and $H'/(G' \cap H')$ are isomorphic.

We prove this result only for $(\langle G_2, S_1 \rangle / S_1)/(\langle G_2, S_1 \rangle / S_1 \cap \langle H_2, S_1 \rangle / S_1)$ since the proof for $(\langle H_2, S_1 \rangle / S_1)/(\langle G_2, S_1 \rangle / S_1 \cap \langle H_2, S_1 \rangle / S_1)$ is similar. We have

$$\begin{aligned}(\langle G_2, S_1 \rangle / S_1)/(\langle G_2, S_1 \rangle / S_1 \cap \langle H_2, S_1 \rangle / S_1) &\cong \langle G_2, S_1 \rangle /(\langle G_2, S_1 \rangle \cap \langle H_2, S_1 \rangle) \\ &\cong \langle \langle G_2, S_1 \rangle, \langle H_2, S_1 \rangle \rangle / \langle H_2, S_1 \rangle = \langle G_2, \langle H_2, S_1 \rangle \rangle / \langle H_2, S_1 \rangle \\ &\cong G_2/(G_2 \cap \langle H_2, S_1 \rangle) = d(G')/(d(G') \cap \langle d(H'), t(d(S')) \rangle).\end{aligned}$$

We denote by f the canonical surjection from G' to $G'/(G' \cap H')$ and we show that f induces an isomorphism from $d(G')/(d(G') \cap \langle d(H'), t(d(S')) \rangle)$ to $d(G'/(G' \cap H'))/t(d(G'/(G' \cap H')))$.

First, we prove that $f(d(G')) = d(G'/(G' \cap H'))$. If w is an element of $d(G'/(G' \cap H'))$ and if x is a representative of w in G', then, for each integer $n \geqq 1$, there are two elements $y, z \in G'$ such that $nz = y$ and $x - y \in G' \cap H'$. So, the set which consists of the formulas $\varphi_n(u) = (x - u \in G' \cap H') \wedge (\exists v)(u = nv)$ for $n \in \mathbf{N}^*$ is consistent, and therefore satisfied, in the ω_1-saturated structure $(G', G' \cap H') = (G \cap S, G \cap H \cap S)^U$. If $y \in G'$ satisfies φ_n for each $n \in \mathbf{N}^*$, then we have $y \in d(G')$ and $f(y) = w$.

Now, it suffices to prove that

$$d(G') \cap f^{-1}(t(d(G'/(G' \cap H')))) = d(G') \cap \langle d(H'), t(d(S')) \rangle.$$

We observe that $f(d(G') \cap \langle d(H'), t(d(S')) \rangle)$ is contained in $t(d(G'/(G' \cap H')))$ since, for each $x \in \langle d(H'), t(d(S')) \rangle$, there exists an integer $k \geqq 1$ such that kx belongs to $d(H')$, which implies $kf(x) = f(kx) = 0$.

Then, we show that an element $x \in d(G')$ such that $f(x) \in t(d(G'/(G' \cap H')))$ necessarily belongs to $\langle d(H'), t(d(S'))\rangle$. If $k \geqq 1$ is an integer such that $kf(x) = 0$, then kx belongs to $G' \cap H'$. As x is divisible in G', kx is divisible in S', and therefore divisible in H' since H' has a supplementary in S'. In H', there are only finitely many elements y which satisfy $ky = kx$, and one of these elements is necessarily divisible. So, there exists an element $y \in d(H')$ such that $kx = ky$. The element $x - y$, which satisfies $k(x - y) = 0$, belongs to $t(d(S'))$ since x and y respectively belong to $d(G')$ and $d(H')$.

$\langle G', S_2\rangle/S_2$ and $\langle H', S_2\rangle/S_2$ have a common supplementary in S'/S_2.

According to (D) we have $S'/S_2 \cong \prod_{p\,\text{prime}} \left(\hat{\mathsf{Z}}_p^{\mathrm{Tf}(p;S)} \oplus \left(\bigoplus_{n \geqq 1} \mathsf{Z}(p^n)^{U(p,n-1;S)}\right)\right)$, and therefore $S'/S_2 \cong \prod_{p\,\text{prime}} d_p(S'/S_2)$; this property is also true for G'/G_2, H'/H_2, A'/A_2 and B'/B_2. Moreover, we have $d_p(S'/S_2) \cap (G'/G_2) = d_p(G'/G_2)$ for each prime number p since G'/G_2 has a supplementary in S'/S_2; similar equalities hold for H'/H_2, A'/A_2 and B'/B_2. So, in order to prove that G'/G_2 and H'/H_2 have a common supplementary in S'/S_2, it suffices to show that, for each prime number p, $d_p(G'/G_2)$ and $d_p(H'/H_2)$, which respectively are supplementaries of $d_p(A'/A_2)$ and $d_p(B'/B_2)$ in $d_p(S'/S_2)$, have a common supplementary in $d_p(S'/S_2)$.

For each prime number p, $t(d_p(S'/S_2))$ has a supplementary in $d_p(S'/S_2)$ according to (D). Moreover, we have $t(d_p(S'/S_2)) \cap d_p(G'/G_2) = t(d_p(G'/G_2))$ and similar equalities hold for H'/H_2, A'/A_2 and B'/B_2. So, by lemma 2, in order to prove that $d_p(G'/G_2)$ and $d_p(H'/H_2)$ have a common supplementary in $d_p(S'/S_2)$, it suffices to show that:

1) $t(d_p(G'/G_2))$ and $t(d_p(H'/H_2))$ have a common supplementary in $t(d_p(S'/S_2))$;

2) $d_p(G'/G_2)/t(d_p(G'/G_2))$ and $d_p(H'/H_2)/t(d_p(H'/H_2))$ have a common supplementary in $d_p(S'/S_2)/t(d_p(S'/S_2))$.

As G' and H' are isomorphic, $t(d_p(G'/G_2))$ and $t(d_p(H'/H_2))$ are isomorphic. Moreover, $t(d_p(S'/S_2)) \cong \bigoplus_{n \geqq 1} \mathsf{Z}(p^n)^{U(p,n-1;S)}$ is a finite p-torsion group since $\sum_{n \geqq 1} U(p, n - 1; S)$ is finite. So, 1) follows from lemma 4.

We have $d_p(S'/S_2)/t(d_p(S'/S_2)) \cong \hat{\mathsf{Z}}_p^{\mathrm{Tf}(p;S)}$. The subgroups $d_p(G'/G_2)/t(d_p(G'/G_2))$ and $d_p(H'/H_2)/t(d_p(H'/H_2))$ are closed for the p-adic topology since they have supplementaries in $d_p(S'/S_2)/t(d_p(S'/S_2))$. They are isomorphic since G' and H' are isomorphic. So, 2) is a consequence of the following lemma:

LEMMA 5. *Let S be a free $\hat{\mathsf{Z}}_p$-module of finite dimension and let G, H be two isomorphic submodules which have supplementaries in S. Then G and H have a common supplementary in S.*

PROOF. $\hat{Z}_p$ is a principal ideal domain. The invertible elements of $\hat{Z}_p$ are the elements which are not divisible by p. The ideals of $\hat{Z}_p$ are the subgroups $p^k\hat{Z}_p$ for $k \in \mathrm{N}^*$.

According to lemma 3, G/pG and H/pH have a common supplementary M in S/pS. We consider a basis $\{x_1, \ldots, x_m\}$ of M and, for each $i \in \{1, \ldots, m\}$, a representative y_i of x_i in S. We denote by N the submodule of S which is generated by $\{y_1, \ldots, y_m\}$. We are going to prove that N is a supplementary of G in S. We can show in a similar way that N is a supplementary of H.

In order to prove that $N \cap G = \{0\}$, we consider an element $y = a_1y_1 + \ldots + a_my_m$, with $a_1, \ldots, a_m \in \hat{Z}_p$, which belongs to G, we denote by k the largest integer such that $a_1, \ldots, a_m$ belong to $p^k\hat{Z}_p$ and we write $a_i = p^k b_i$ for each $i \in \{1, \ldots, m\}$. As the element $y = p^k(b_1y_1 + \ldots + b_my_m)$ belongs to G, the element $b_1y_1 + \ldots + b_my_m$ also belongs to G, whence a contradiction since at least one of the elements $b_1, \ldots, b_m$ does not belong to $p\hat{Z}_p$.

In order to prove that S is generated by G and N, it suffices to show that S is generated by G, N and p^kS for each integer $k \geqq 1$. We consider the smallest integer $k \geqq 1$ such that S is not generated by G, N and p^kS. For each $y \in S$, there exist elements $u \in G$, $v \in N$ and $z \in S$ such that $y = u + v + p^{k-1}z$; according to the definition of N there are also elements $u' \in G$, $v' \in N$ and $z' \in S$ such that $z = u' + v' + pz'$; it follows that $y = (u + p^{k-1}u') + (v + p^{k-1}v') + p^kz'$ with $u + p^{k-1}u' \in G$ and $v + p^{k-1}v' \in N$, contrary to the definition of k.

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