MATH. SCAND. 67 (1990), 15–22

4-CRITICAL 4-VALENT PLANAR GRAPHS CONSTRUCTED WITH CROWNS

G. KOESTER

Abstract.

A construction of arbitrarily large 4-critical 4-valent planar graphs with aid of socalled "crowns" is given which also proves a conjecture of B. Grünbaum. Furthermore some coloring properties of the aid graphs are derived

Introduction.

All graphs considered are finite, connected, planar, undirected, and have no loops. A vertex is r-valent if it is incident with r edges. A graph is r-valent if each of its vertices is r-valent. A (vertex) coloring is an integer valued function on the vertices of a graph. It is a proper coloring if adjacent vertices have different colors (values). A graph G has the chromatic number $\chi(G) = k$ if there is a proper coloring of G with k different colors (a k-coloring) but none with fewer colors. G is k-critical if $\chi(G) = k$ and $\chi(G') < k$ for every proper subgraph G' of G.

4-critical 4-valent graphs were known in the past only for the nonplanar case [1]. G. A. Dirac and T. Gallai even conjectured [1] that every 4-critical planar graph contains 3-valent vertices. But the author found in 1984 [3] a 4-critical 4-valent planar graph G^* (see Fig. 1). In this paper a construction is given which generates infinite families of 4-critical 4-valent planar graphs and which is based on G^* and on some symmetric planar graphs here called "crowns". This also proves a recent conjecture of B. Grünbaum [4] that there exist arbitrarily large 4-critical 4-valent planar graphs. Some lemmas concerning crown colorings support the proofs of the results.

In the sequel, vertex indices shall be taken modulo s (s integer, $s > 1$; $x_{s+1} = x_1$, etc.). An edge joining two vertices x_1, x_2 is denoted (x_1, x_2), $c(x)$ denotes the color of x. In a colored vertex sequence an *r-block* is a maximal subsequence of r consecutive vertices of the same color ($r > 1$). Each graph G is assumed to be properly embedded in an infinite plane with a given outside region (G is said to be *plane*). G^0 arises from G by removal of the edges which bound the

Received May 1, 1989; in revised form October 11, 1989.

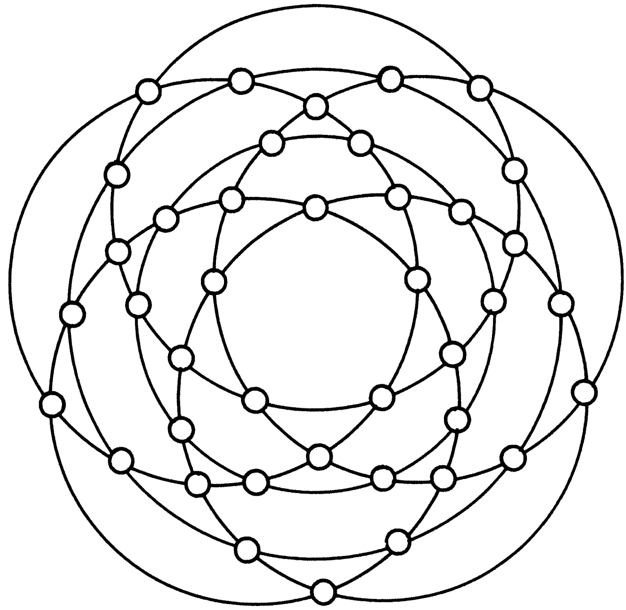

Figure 1. The 4-critical graph G^*.

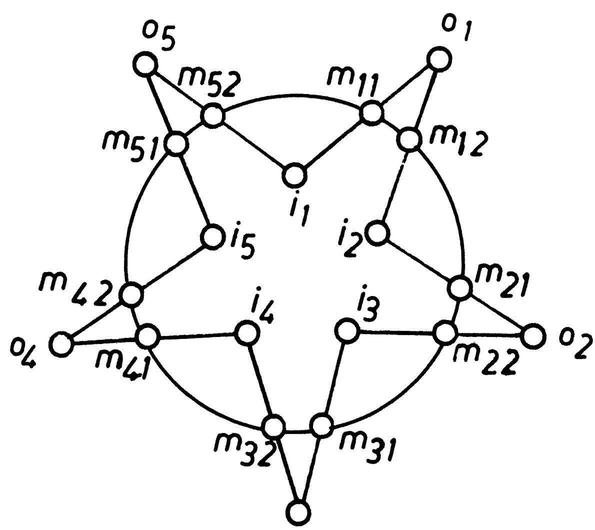

Figure 2. The 5-crown C_5.

outside region of G. Let G have an outside region such that its boundary contains vertices, some of which are labelled. Then $[G]$ denotes the graph which arises from G by joining consecutive labelled vertices by new edges, drawn along the boundary of the outside region of G (if we label all vertices which bound the outside region of G then $G = [G^0]$).

The operation of "crowning".

DEFINITION 1. An *s-crown* C_s is a plane graph, the vertex-set $V(C_s)$ of which is the disjoint union of the following subsets: $O = \{o_j\}$ (outside vertices), $I = \{i_j\}$ (inside vertices), $M = \{m_{jk}\}$ (midside vertices). o_j is adjacent to m_{j1}, m_{j2}; i_j to $m_{j-1,2}$, m_{j1}; m_{j1} to $m_{j-1,2}$, m_{j2}, i_j, o_j; m_{j2} to m_{j1}, $m_{j+1,1}$, i_{j+1}, o_j ($j = 1,\dots,s$; $k = 1,2$; Fig. 2 shows C_5 as an example).

DEFINITION 2. LET A_G be an s-gon (with vertices $a_1,\dots,a_s$ in cyclic order) that bounds the outside region of plane graph G. The plane graph given by

$$F = G \circ C_s, \tag{1}$$

arises from G^0 and $[C_s]$ (where $o_1,\dots,o_s$ in C_s are labelled) by identifying the boundaries of

- a closed disc containing G^0, such that its boundary contains only the vertices $a_1,\dots,a_s$ from G^0
- $[C_s]$ minus an open disc from its inside region (the one having $i_1,\dots,i_s$ on its boundary), such that the boundary of what remains contains only vertices $i_1,\dots,i_s$ from $[C_s]$

in such a way that a_j is identified with i_j ($j = 1,\dots,s$). F shall be called an *s-crowning* of G. If G is 4-valent then so is F. An n-fold s-crowning

$$F = (\dots((G \circ C_s) \circ C_s)\dots) \circ C_s$$

shall be abbreviated by

$$F = G \circ C_s^n. \tag{2}$$

REMARK 1. Let L_s be a 4-valent plane graph, the vertex-set $V(L_s)$ of which is the disjoint union of the following subsets: $U = \{u_j\}$ (inside vertices) and $A = \{a_j\}$ (outside vertices). u_j is adjacent to u_{j-1}, u_{j+1}, a_{j-1}, a_j; a_j to a_{j-1}, a_{j+1}, u_j, u_{j+1} ($j = 1,\dots,s$; for $s = 5$ see Fig. 3). Then G^* (from Fig. 1) is a 2-fold 5-crowning of L_5:

$$G^* = L_5 \circ C_5^2. \tag{3}$$

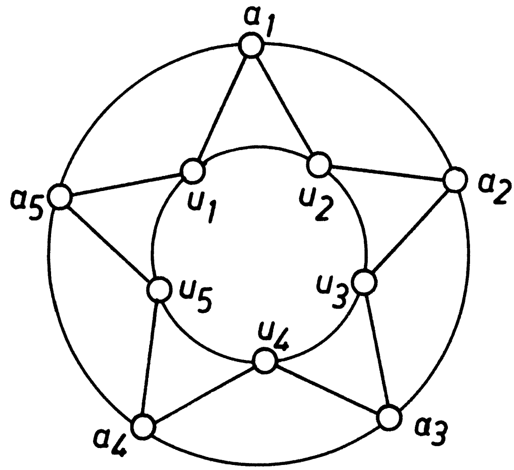

Figure 3. The graph L_5.

The results.

PROPOSITION.

(a) *Let G be a* 4*-critical plane graph bounded on the outside by a triangle. Then*

$$F = G \circ C_3 \tag{4}$$

is also 4*-critical.*

(*b*) *Let*

$$F_s^{(k)} = L_s \circ C_s^k. \tag{5}$$

Then $F_5^{(k)}$ *is* 4*-critical for* $k > 1$ (for L_s see Remark 1 and Fig. 3).

REMARK 2. With a suitable embedding in the plane, repeated 3-crowning of the graph G^* (Fig. 1) generates an infinite family of 4-critical 4-valent plane graphs. Similarly, $F_5^{(k)}$ with $k > 1$ is another such family, and its smallest member is also G^*.

Lemmas and proofs.

Let $c(I)$ be a given k-coloring ($k < 4$) of the inside vertices of C_s. Then $c(C, I)$ denotes a 3-coloring of C_s which extends $c(I)$. The existence of some $c(C, I)$ which extends any $c(I)$ is established in Lemma 2. Let $c(A)$, $c(B)$ be k-colorings of vertex sequences $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$ resp. $c(A)$, $c(B)$ are said to be *isochromal* if they can be made to coincide by an index translation (reflexion) or a color permutation or a combination of both transitions. A coloring of an edge sequence $A = \{a_1, \ldots, a_n\}$ is denoted $(c(a_1), \ldots, c(a_n))$.

REMARK 3. For any 3-coloring of C_s and for any $j \in \{1,\ldots,s\}$:
if $c(i_j) = c(i_{j+1}) = c$ then

$$c(o_j) = c. \tag{6}$$

LEMMA 1. (color invariance).

(a) *At every 3-coloring of C_s using colors 1, 2 & 3, the numbers of inside and outside vertices colored with j ($j = 1,2,3$) are equal.*

(b) *If $c(I)$ is an inside coloring of C_s using at most two colors then for each $c(C,I)$ one of the following holds*

$$c(o_j) = c(i_j) \quad \text{for} \quad j = 1,\ldots,s \quad \text{or} \tag{7'}$$

$$c(o_j) = c(i_{j+1}) \quad \text{for} \quad j = 1,\ldots,s. \tag{7''}$$

PROOF. Let I_j, M_j, O_j be the subsets of inside, midside, and outside vertices resp. which are colored by j ($j = 1,2,3$). Then $|I_j| + |M_j| = |O_j| + |M_j| = s$, and hence

$$|I_j| = |O_j|, \tag{8}$$

which proves (a).

In case (b) let $c(I)$ have colors 1, 2 (1). Then all m_{j1} or all m_{j2} ($j = 1,\ldots,s$) must have color 3, and from this follows (7″) or (7′) resp.

If $c(A)$ is a k-coloring of a vertex sequence A so that the number of vertices colored by j ($j \in 1,\ldots,k\}$) is greater than the number of the remaining vertices of A then $c(A)$ shall be called a *dominant coloring*. Then A contains at least one block. From Lemma 1 follows immediately:

COROLLARY. *If $c(I)$ is a dominant k-coloring of the inside vertices of C_s ($k < 4$) then each $c(C,I)$ restricts to a dominant outside k-coloring $c(O)$.*

LEMMA 2. *Let $c(I)$ be a k-coloring ($k < 4$) of the inside vertices of C_s.*

(a) *For every $c(I)$ there exists some $c(C,I)$.*

(b) *If I has no block then there is a $c(C, I)$ which restricts to an outside k-coloring $c(O)$, and $c(O)$ is isochromal with $c(I)$.*

PROOF. We introduce a "diagonal coloring" of C_s (which forces a $c(C,I)$) by the following (r integer > 0; $j = 1,\ldots,s$): If

$$c(i_j) \neq c(i_{j+1}) = \ldots = c(i_{j+r}) \neq c(i_{j+r+1}) \quad \text{then} \quad c(o_{j+r}) = c(m_{j2}) = c(i_j). \tag{9}$$

(6) and (9) cause a unique outside coloring $c(O)$. It remains to determine the midside coloring $c(M)$. In the antiblock case ($r = 1$) the 4 neighbours of $m_{j+1,1}$ are colored with exactly 2 different colors and therefore $c(m_{j+1,1})$ is uniquely determined. In the block case ($r > 1$) $c(o_{j+r})$ determines the colors of $m_{j+1,1}$,

$m_{j+1,2}, \ldots, m_{j+r,1}$ which must be different from the block color. In the blockfree case (b) clearly (9) gives on outside coloring $c(O)$ which is isochromal with $c(I)$.

LEMMA 3. *Let G be a plane graph and $F = G \circ C_s$. If $G(G^0)$ has chromatic number 3 then so has F (F^0).*

PROOF. If G^0 has chromatic number 3 then so has F^0 because of the existence of a $c(C, I)$ for every 3-coloring $c(I)$ of the inside vertices of C_s (Lemma 2 (a)). From Lemma 2 (b) follows that F has chromatic number at most 3 if G has. Clearly F has chromatic number at lest 3, since it contains triangles.

PROOF OF THE PROPOSITION. For the k-criticality of a graph H it is sufficient that

$$\chi(H) = k \quad (H \text{ is } k\text{-chromatic}), \text{ and} \tag{10}$$

$$\chi(H - e) = k - 1 \tag{11}$$

holds for each edge e of H. A path P in a k-chromatic graph H shall be called a *critical path*, it for some coloring of the graph $H - P$ using $k - 1$ colors, there are two colors c_1, c_2, so that $c_1(c_2)$ does not occur among the neighbours outside P of all (all interior) vertices of P. For each edge e of a critical path P of H holds (11). In the following let $k = 4$.

(a) Let $H = F$ (F from (4)). Since G in (4) is 4-critical, any 3-coloring of G^0 restricts to a dominant coloring of the outside vertices a_1, a_2, a_3 of G (isochromal with (1, 1, 2)). Let $c(a_1) = 1, c(a_2) = 1$, and $c(a_3) = 2$. From Lemma 1 (b) follow for o_j, m_{jk} in C_3 ($j = 1, 2, 3; k = 1, 2$):

$$c(o_1) = c(o_2) = 1,\ c(o_3) = 2 \quad \text{or} \tag{12}$$

$$c(o_1) = c(o_3) = 1,\ c(o_2) = 2, \tag{13}$$

which gives (10). If (12) holds then we have $c(M) = (2,3; 2,3; 1,3)$ as midside coloring ((13) forces an analogous midside coloring). From Lemma 2 (b) follows (11) if e is an edge of G^0. From the above follows (11) for e (o_1, o_2). Moreover, the vertex sequence $\{o_1, o_2, m_{22}, m_{31}, m_{32}, i_1\}$ forms a critical path P of F. From the symmetry of C_3 now follows (11) for each edge e of F and hence the 4-criticality of F in (4).

(b) From Lemma 3 follows that each $(F_s^{(k)})^0$ $(s > 1,\ k \geqq 0)$ is 3-colorable. Each 3-coloring of L_5^0 restricts to a dominant coloring of A isochromal with $(1, 1, 1, 2, 3)$. For $H = F_5^{(k)}$ follows (10) from the Corollary. The proof of the 4-criticality proceeds by induction. We start with $G^* = F_5^{(2)}$ which is shown to be 4-critical by [3]. Assume $F_5^{(k-1)}$ is 4-critical for some $k > 2$. From the 4-criticality and from Lemma 1 (color invariance) follows that there are 3-colorings of $(F_5^{(k-1)})^0$ which restrict to outside colorings isochromal with $(1, 1, 2, 1, 3)$. Such a 3-coloring we extend to a 3-coloring of $(F_5^{(k)})^0$:

(1, 1, 2, 1, 3)	inside	of C_5,
(3,2; 3,1; 3,2; 3,2; 1,2)	midside	of C_5,
(1, 2, 1, 1, 3)	outside	of C_5.

The vertices o_4, o_3, m_{31}, m_{22}, m_{21}, i_2, form a critical path P' of $F_5^{(k)}$. From the symmetry of $F_5^{(k)}$ and from the 4-criticality of $F_5^{(k-1)}$ follows (11) for each edge e of $F_5^{(k)}$.

Concluding remarks.

1. Without detailed proofs we state that:

(14) $$\chi(F_s^{(k)}) = 4 \quad \text{for} \quad s = 2, 4, 5 \quad \text{and} \quad k >= 0,$$

(15) $$\chi(F_s^{(k)}) = 3 \quad \text{for} \quad s = 3, s > 5 \quad \text{and} \quad k > 0.$$

(14) follows from the above Lemmas and the Corollary. To show (15) it is sufficient to establish 3-colorings of $F_s^{(1)}$ for $s = 3$ and $s > 5$. From Lemma 3 then follows (15) for $k > 1$.

2. Neither $F_2^{(k)}$ nor $F_4^{(k)}$ is 4-critical for any k. The first contains double-edges and the second cannot be 4-critical because every 3-coloring of $(F_4^{(k)})^0$ restricts to an outside 2-coloring with 2 2-blocks or to a 1-coloring (Lemma 2).

3. An old conjecture of H. Grötzsch ws stated by H. Sachs as follows: "Let G be a finite 4-regular plane graph generated by a set of simple closed curves (= Jordan curves) (i.e., every vertex of G is an intersection point of exactly two of the generating curves of G (which are not allowed to touch)). H. Grötzsch conjectured that $\chi(G) = 3$." The author found two counterexamples, namely the graph $F_4^{(1)}$ which was published in 1984 [2] and the graph $F_5^{(2)} = G^*$ [3]. It is easy to verify that $F_s^{(s-3)}$ for $s > 2$ is of the Grötzschian type, but there are no more 4-chromatic graphs among them because of statement (15).

ACKNOWLEDGEMENT. The author thanks T. Jensen (Waterloo, Can.), B. Toft (Odense), and the referee for their interest in this paper, for many helpful remarks and corrections. The idea of "critical pathes" is due to W. Wessel (Berlin) (oral communication, see also [3]), and the present formulation is suggested by the referee.

REFERENCES

1. T. Gallai, *Critical graphs*, in "*Theory of Graphs and its Applications*," Proc. Symp. Smolenice, 1963, Publ. House Czechoslovak Acad. Sci., Prague 1964, 43–45.
2. G. Koester, *Bemerkung zu einem Problem von H. Grötzsch*, Wiss. Z. Univ. Halle XXXIII '84 M.H. 5 (1984), 129.

3. G. Koester, *Note to a problem of T. Gallai and G. A. Dirac*, Combinatorica 5 (1985), 227–228.
4. B. Grünbaum, *The edge-density of 4-critical planar graphs*, Combinatorica 8 (1988), 137–139.

BREDENSAND 10
2100 HAMBURG 90
GERMANY