

EQUIVARIANT ALEXANDER-SPANIER COHOMOLOGY FOR ACTIONS OF COMPACT LIE GROUPS

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Introduction.

In [3] we constructed for a finite group G an equivariant cohomology theory, defined on all G -pairs and having properties analogous to those of ordinary Alexander-Spanier cohomology. The purpose of this paper is to extend the construction of this “equivariant Alexander-Spanier cohomology theory” to the case of a compact Lie group G .

Recall that the ordinary Alexander-Spanier cohomology groups ([7], 6.4), as well as the equivariant groups of [3], are computed from a cochain complex which is obtained by first defining certain cochain groups and then dividing out “locally zero” cochains. The main difficulty in the generalization from a finite group G to a compact Lie group G is the proper formulation of the notion of a locally zero cochain. We give this definition in section 3 below. The cochains themselves can be defined in almost the same way as in [3], cf. section 2. In these two sections it appears useful to keep in mind certain aspects of the correspondence between G -maps from a homogeneous space G/H to a G -space X , and H -fixed points of X . Therefore we devote a short section 1 to these matters; this section is essentially based on [5], 3.2.i).

The remaining two sections show that the properties of the equivariant Alexander-Spanier cohomology proved in [3] remain valid after the generalization to a compact Lie group G . In section 4 we indicate how the proofs given in [3] can be modified to show that the equivariant Eilenberg-Steenrod axioms still hold. Section 5 contains three results: a) In a paracompact G -space any closed G -subspace is taut; b) the equivariant Alexander-Spanier cohomology of a paracompact G -space X is isomorphic to the ordinary cohomology of the orbit space X/G with coefficients in a certain sheaf; c) if the paracompact G -space X is locally sufficiently nice, then the equivariant Alexander-Spanier cohomology of X is isomorphic to its equivariant singular cohomology, as defined in [4].

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Notation and terminology.

In this paper G is a compact Lie group. The notation $H \leq G$ means that H is a closed subgroup of G . Recall the discrete orbit category $\text{Or}_d G$ of G : its objects are the homogeneous spaces G/H , $H \leq G$, and its morphisms are the G -homotopy classes of G -maps between them. A contravariant G -coefficient system is a contravariant functor $\text{Or}_d G \rightarrow \text{Ab}$; we typically denote such a coefficient system by m . Finally, all G -spaces are assumed to be Hausdorff, as usual.

1. G -maps $G/H \rightarrow X$.

Let X be a G -space and $H \leq G$ a closed subgroup of G . In this preliminary section we recall some facts about G -maps $G/H \rightarrow X$; the set of these maps is denoted by $\text{Map}_G(G/H, X)$.

First of all, the formula $f \mapsto f(eH)$ defines a natural bijection

$$(1.1) \quad \text{Map}_G(G/H, X) \xrightarrow{\sim} X^H.$$

Both sides of 1.1 have a natural left action: of the Weyl group $WH = NH/H$ such that 1.1 is WH -equivariant; in $\text{Map}_G(G/H, X)$ the action is given by

$$(1.2) \quad (nH) \cdot f: gH \mapsto f(gnH), \quad n \in NH, f \in \text{Map}_G(G/H, X), g \in G.$$

Let WH_0 be the identity component of WH . From 1.1 we get the bijection $\text{Map}_G(G/H, X)/WH_0 \xrightarrow{\sim} X^H/WH_0$. It is clear from the considerations in 3.2.i) of [5] that two G -maps $f_1, f_2: G/H \rightarrow X$ are in the same WH_0 -orbit of $\text{Map}_G(G/H, X)$ if and only if they are G -homotopic by a homotopy h_t satisfying $h_t(G/H) = f_1(G/H) = f_2(G/H)$ for every $t \in [0, 1]$. In this case we say that f_1 and f_2 are G -fiber homotopic (as maps from $G/H \rightarrow * \rightarrow X \rightarrow X/G$). Hence 1.1 induces a natural bijection

$$(1.3) \quad \left\{ \begin{array}{l} G\text{-fiber homotopy classes} \\ \text{of } G\text{-maps } G/H \rightarrow X \end{array} \right\} \xrightarrow{\sim} X^H/WH_0.$$

Assume now that $u: G/K \rightarrow G/H$ is a G -map; then $u(gK) = gaH$ for every $g \in G$, where $u(eK) = aH$ and $a \in G$ satisfies $a^{-1}Ka \leq H$. The map u induces the map $X(u): X^H \rightarrow X^K$, $x \mapsto ax$ (the notation $X(u)$ is from [5]). By 1.3 it is clear that $X(u)$ further induces the arrow $\bar{X}(u)$ making the diagram

$$(1.4) \quad \begin{array}{ccc} X^H & \xrightarrow{X(u)} & X^K \\ \downarrow & & \downarrow \\ X^H/WH_0 & \xrightarrow{\bar{X}(u)} & X^K/WK_0 \end{array}$$

commutative, and in addition $\bar{X}(u)$ depends only on the G -homotopy class of u . Altogether we have now reviewed the definition of the functor

$$\bar{X}: \text{Or}_d G \rightarrow \text{Top}, \quad G/H \mapsto X^H/WH_0$$

of [5], 3.2.i).

The commutativity of 1.4 has the following immediate consequence:

LEMMA 1.5. *In the preceding notation, if $A \subset X^K$ is WK_0 -invariant, then $X(u)^{-1}(A) \subset X^H$ is WH_0 -invariant. As a special case (with $K = \{e\}$), if $A \subset X$ is G_0 -invariant, then $A \cap X^H$ is WH_0 -invariant.*

2. Definition of cochains.

Let again X be a G -space. Given $n \in \mathbb{N}$ and $H \leq G$, we denote by $V_n^H(X)$ the set of all $(n+1)$ -tuples $\varphi = (\varphi_0, \dots, \varphi_n)$, where each φ_i is a G -map $G/H \rightarrow X$, i.e. $V_n^H(X) = \text{Map}_G(G/H, X)^{n+1}$. Further we set

$$V_n(X) = \bigcup_{H \leq G} V_n^H(X) = \bigcup_{H \leq G} \text{Map}_G(G/H, X)^{n+1}.$$

Let $m: \text{Or}_d G \rightarrow \text{Ab}$ be a contravariant coefficient system and $M = \bigoplus_{H \leq G} m(G/H)$.

We can now define the n^{th} cochain group $C^n(X; m)$ of X with coefficients m in the same way as in [3], namely

$$C^n(X; m) = \{c: V_n(X) \rightarrow M \mid c(\varphi) \in m(G/H) \text{ if } \varphi \in V_n^H(X)\}.$$

These form the cochain complex $C^*(X; m)$ with the usual coboundary operator.

A cochain $c \in C^n(X; m)$ is *equivariant*, if the following condition holds: given $\varphi = (\varphi_0, \dots, \varphi_n) \in V_n^H(X)$ and $\alpha = (\alpha_0, \dots, \alpha_n) \in V_n^K(G/H)$ such that the G -maps $\alpha_i: G/K \rightarrow G/H$ are all G -homotopic,

$$c(\varphi \circ \alpha) = m(\alpha_0)(c(\varphi)),$$

where of course $\varphi \circ \alpha = (\varphi_0 \circ \alpha_0, \dots, \varphi_n \circ \alpha_n)$; note that $m(\alpha_0) = \dots = m(\alpha_n)$. The equivariant cochains form a cochain subcomplex $C_G^*(X; m) \subset C^*(X; m)$.

We point out the following functoriality property: any equivariant function $f: X \rightarrow Y$, not necessarily continuous, between G -spaces induces a cochain map $f^*: C^*(Y; m) \rightarrow C^*(X; m)$ by the formula

$$(f^*(c))(\varphi_0, \dots, \varphi_n) = c(f \circ \varphi_0, \dots, f \circ \varphi_n); \quad c \in C^n(Y; m), \quad \varphi \in V_n(X).$$

Namely, the composite $f \circ \varphi: G/H \rightarrow Y$ is a G -map if $\varphi: G/H \rightarrow X$ is one, even if f is not continuous. It is also clear that $f^* C_G^*(Y; m) \subset C_G^*(X; m)$.

Next we give another description of the cochain complex $C_G^*(X; m)$. We begin with

LEMMA 2.1. *If $c \in C_G^n(X; m)$, $\varphi, \varphi' \in V_n^H(X)$ and φ_i is G -fiber homotopic to φ'_i for every $i \in \{0, \dots, n\}$, then $c(\varphi) = c(\varphi')$.*

PROOF. By 1.3 there are elements $n_i H \in WH_0$ such that $\varphi'_i(gH) = \varphi_i(gn_i H)$ for $i \in \{0, \dots, n\}$, $g \in G$. This means that $\varphi'_i = \varphi_i \circ \alpha_i$, where $\alpha_i: G/H \rightarrow G/H$ is the G -map $gH \mapsto gn_i H$, and $\alpha_i \approx_{\mathcal{F}} \text{id}$ for all $i \in \{0, \dots, n\}$. By the equivariance of c we have

$$c(\varphi') = c(\varphi \circ \alpha) = m(\alpha_0)(c(\varphi)) = m(\text{id})(c(\varphi)) = c(\varphi).$$

This shows that in the definition of $C^n(X; m)$ above, instead of $V_n(X) =$

$$\bigcup_{H \leq G} V_n^H(X), \text{ we could have used } \bar{V}_n(X) = \bigcup_{H \leq G} \bar{V}_n^H(X), \text{ where}$$

$$\bar{V}_n^H(X) = \{(n+1)\text{-tuples of } G\text{-fiber homotopy classes of } G\text{-maps } G/H \rightarrow X\}.$$

Let C_* be the functor which to a topological space A associates the chain complex $C_*(A)$, $C_n(A) =$ free abelian group with basis A^{n+1} . Because

$$\bar{V}_n^H(X) \cong (X^H / WH_0)^{n+1}$$

by 1.3, we get the following identification (where $Z\text{-Or}_d G$ is the abelian category of contravariant coefficient systems):

$$\text{PROPOSITION 2.2. } C_G^*(X; m) = \text{Hom}_{Z\text{-Or}_d G}(C_*(\bar{X}), m).$$

In the rest of this paper, however, we prefer to use the original definition of $C_G^*(X; m)$ to keep our notation consistent with [3].

3. Locally zero cochains.

To get an adequate notions of locally zero cochains, when G is a compact Lie group, we must replace the open G -coverings used in [3] with a more complicated concept.

DEFINITION 3.1. An *open G -covering* of the G -space X is a pair $(\mathcal{U}, \mathcal{P})$, where \mathcal{U} is a covering of X by G_0 -invariant open sets and \mathcal{P} is a function assigning to every $U \in \mathcal{U}$ and $H \leq G$ a partition $\mathcal{P}_H(U)$ of $U \cap X^H$ into disjoint WH_0 -invariant open subsets. Furthermore we require:

- i) \mathcal{U} is G -invariant, i.e. $g \in G, U \in \mathcal{U} \Rightarrow gU \in \mathcal{U}$;
- ii) $\mathcal{P}_{\{e\}}(U) = \{U\}$ for every $U \in \mathcal{U}$;
- iii) $U \in \mathcal{U}, H \leq G, g \in G \Rightarrow g \cdot \mathcal{P}_H(U) = \mathcal{P}_{gHg^{-1}}(gU)$;
- iv) $H \leq K \leq G, U \in \mathcal{U} \Rightarrow \mathcal{P}_K(U)$ is a refinement of $\mathcal{P}_H(U) \cap X^K$;
- v) Assume $U \in \mathcal{U}$; then x has an open neighborhood V_x in X with the following property: whenever $x \in X^H$, i.e. $H \leq G_x$, the set $V_x \cap X^H$ is contained in some set of $\mathcal{P}_H(U)$ (in particular $V_x \subset U$).

REMARK 3.2. i) The sets $U \cap X^H$ for $U \in \mathcal{U}$ are WH_0 -invariant by 1.5.

ii) If a) X is a locally smooth G -manifold, or b) G is finite, then condition v) is automatically satisfied.

PROOF a) In the notation of 3.1.v), let $K = G_x$. Because X is also a locally smooth K -manifold, x has an open K -neighborhood V_x such that V_x is K -homeomorphic to a linear representation of K . Now $V_x \cap X^H$ is connected and hence contained in some set of $\mathcal{P}_H(U)$, for every $H \leq K$.

b) Let again $x \in U \in \mathcal{U}$. For each $H \leq G_x$ choose an open set $V_H \subset U$ such that $x \in V_H \cap X^H \in \mathcal{P}_H(U)$. Then $V_x = \bigcap_{H \leq G_x} V_H$ is the required open neighborhood of x , the intersection being finite, because G is finite.

EXAMPLE 3.3. For a locally smooth G -manifold X , take $\mathcal{U} =$ set of components of X and $\mathcal{P}_H(U) =$ set of components of $U \cap X^H$ ($U \in \mathcal{U}, H \leq G$). Then $(\mathcal{U}, \mathcal{P})$ is an open G -covering of X .

Using this concept of a G -covering we can now define locally zero cochains. Note that if $H \leq G$ and $\varphi \in V_n^H(X)$, then $\{\varphi_0(eH), \dots, \varphi_n(eH)\}$ is contained in X^H by 1.1.

DEFINITION 3.4. A cochain $c \in C^n(X; m)$ is *locally zero*, if there exists an open G -covering $(\mathcal{U}, \mathcal{P})$ of X with the property that $c(\varphi) = 0$ whenever $\varphi \in V_n^H(X)$ ($H \leq G$) and $\{\varphi_0(eH), \dots, \varphi_n(eH)\}$ is contained in a set of $\mathcal{P}_H(U)$ for some $U \in \mathcal{U}$.

In the situation of 3.4 we also say that c is locally zero with respect to (w.r.t.) $(\mathcal{U}, \mathcal{P})$.

It is clear that two open G -coverings $(\mathcal{U}, \mathcal{P})$ and $(\mathcal{U}', \mathcal{P}')$ of X have a common refinement (intersect the sets of $(\mathcal{U}, \mathcal{P})$ with those of $(\mathcal{U}', \mathcal{P}')$). Therefore the locally zero n -cochains form a subgroup $C_0^n(X; m)$ of $C^n(X; m)$, and in fact $C_0^*(X; m)$ is a subcomplex of $C^*(X; m)$. We also denote

$$C_{G,0}^*(X; m) = C_G^*(X; m) \cap C_0^*(X; m),$$

$$\bar{C}^*(X; m) = C^*(X; m)/C_0^*(X; m) \text{ and } \bar{C}_G^*(X; m) = C_G^*(X; m)/C_{G,0}^*(X; m)$$

as in [3].

If $f: X \rightarrow Y$ is a (continuous) G -map, then $f^* C_0^*(Y; m) \subset C_0^*(X; m)$; namely, if $c \in C^n(Y; m)$ is locally zero w.r.t. $(\mathcal{V}, \mathcal{Q})$, then $f^*(c) \in C^n(X; m)$ is locally zero w.r.t. $(f^{-1}\mathcal{V}, f^{-1}\mathcal{Q})$, where

$$f^{-1}\mathcal{V} = \{f^{-1}(V) \mid V \in \mathcal{V}\},$$

$$(f^{-1}\mathcal{Q})_H = \{f^{-1}(V_H) \mid V_H \in \mathcal{Q}_H(V)\} \quad (H \leq G, V \in \mathcal{V}).$$

Thus we get $f^*: \bar{C}^*(Y; m) \rightarrow \bar{C}^*(X; m)$ and $f^*: \bar{C}_G^*(Y; m) \rightarrow \bar{C}_G^*(X; m)$.

To end this section, we show that in case G is finite, definition 3.4 agrees with the definition of locally zero cochains given in [3].

PROPOSITION 3.5. *Suppose G is a finite group. Then a cochain $c \in C^n(X; m)$ is locally zero in the sense of 3.4, if and only if it is locally zero in the sense of [3].*

PROOF. The “if”-part is trivial, for an open G -covering \mathcal{U} of X in the sense of [3] can also be regarded as an open G -covering $(\mathcal{U}, \mathcal{P})$, with $\mathcal{P}_H(U) = \{U \cap X^H\}$ ($U \in \mathcal{U}, H \leq G$).

To prove the “only if”-part, assume that c is locally zero w.r.t. $(\mathcal{U}, \mathcal{P})$. Let $x \in X$ and pick $U \in \mathcal{U}$ such that $x \in U$. Choose an open neighborhood V_x of x as in 3.1.v) (cf. 3.2.ii) b)). Because $\bigcup_{H \not\leq G_x} X^H$ is closed (since X is Hausdorff) and $x \notin \bigcup_{H \not\leq G_x} X^H$, we may assume that $V_x \cap X^H = \emptyset$ for $H \not\leq G_x$. Further we may assume that V_x is G_x -invariant (take $\bigcap_{g \in G_x} gV_x$ if necessary). Then, if $y = gx$ ($g \in G$) is in the orbit of x , define $V_y = gV_x$.

Now we have constructed an open neighborhood V_y of every point y in the orbit of x . Performing this construction for every orbit we obtain an open G -covering $\mathcal{V} = \{V_x | x \in X\}$ of X in the sense of [3], and c is locally zero w.r.t. \mathcal{V} in the sense of [3].

Note that here the Hausdorff condition on X was needed, while in [3] it played no role until section 5.

4. Definition of the cohomology theory.

Let X be a G -space and $A \subset X$ a G -subspace. Let $i: A \hookrightarrow X$ be the inclusion. We give the same definition as in [3]:

DEFINITION 4.1. $\bar{C}_G^*(X, A; m) = \ker [i^*: \bar{C}_G^*(X; m) \rightarrow \bar{C}_G^*(A; m)]$ is the *equivariant Alexander-Spanier cochain complex* of the G -pair (X, A) with coefficients m . Its cohomology groups $\bar{H}_G^n(X, A; m) = H^n(\bar{C}_G^*(X, A; m))$ are the *equivariant Alexander-Spanier cohomology groups* of (X, A) with coefficients m .

THEOREM 4.2. *The functors \bar{H}_G^n satisfy all the Eilenberg-Steenrod axioms for an equivariant cohomology theory, including the dimension axiom.*

As in [3], *exactness* is immediate from the definition of \bar{H}_G^* , and the proof of the *excision axiom* needs only marginal modifications. As for the *dimension axiom*, the essential point is that lemma 2.2 of [3] holds in this more general setting, too:

LEMMA 4.3. *Assume $H \leq G$ and $c \in C_G^n(G/H; m)$. Then c is locally zero if and only if $c(\text{id}_{G/H}, \dots, \text{id}_{G/H}) = 0$.*

PROOF. Let $c(\text{id}, \dots, \text{id}) = 0$. The equivariance of c then implies that $c(\varphi_0, \dots, \varphi_n) = 0$, if the G -maps $\varphi_i: G/K \rightarrow G/H$ are G -homotopic. Let $(\mathcal{U}, \mathcal{P})$ be the open G -covering of G/H described in 3.3. Now c is locally zero w.r.t. $(\mathcal{U}, \mathcal{P})$,

for if $\varphi = (\varphi_0, \dots, \varphi_n) \in V_n^K(G/H)$ is such that $\{\varphi_0(eK), \dots, \varphi_n(eK)\}$ is contained in a set of $\mathcal{P}_K(U)$, $U \in \mathcal{U}$, then the G -maps $\varphi_i: G/K \rightarrow G/H$ are G -homotopic, and so $c(\varphi) = 0$.

The reader may convince himself that the proof of the *homotopy axiom* given in section 4 of [3] can also be modified to hold in the present situation. However, we want to be more specific about two particular modifications needed; here we restrict to the absolute case to keep notation more simple, and leave the case of G -pairs in peace.

Firstly, given a G -space X and an open G -covering $(\mathcal{U}, \mathcal{P})$ of X , we define a simplicial complex $X(\mathcal{U}, \mathcal{P})$ as follows: its set of vertices is $V_0(X)$, and the vertices $\varphi_0, \dots, \varphi_n$ span a simplex, if $\varphi_0, \dots, \varphi_n \in V_0^H(X)$ for some $H \leq G$ and $\{\varphi_0(eH), \dots, \varphi_n(eH)\}$ is contained in some set of $\mathcal{P}_H(U)$ for some $U \in \mathcal{U}$. The complex $X(\mathcal{U}, \mathcal{P})$ is clearly a natural generalization of the complex $X(\mathcal{U})$ considered in [3].

Secondly, let X be a G -space, $I = [0, 1]$ the unit interval with trivial G -action and $(\mathcal{U}, \mathcal{P})$ an open G -covering of $X \times I$. The core of the proof of the homotopy axiom is, as in [3], to construct a suitable open G -covering $(\mathcal{V}, \mathcal{Q})$ of X and a suitable chain homotopy $C_*(X(\mathcal{V}, \mathcal{Q})) \rightarrow C_{*+1}((X \times I)(\mathcal{U}, \mathcal{P}))$. We want to explain the construction of $(\mathcal{V}, \mathcal{Q})$.

Let $x \in X$. By the compactness of I we can find an open neighborhood V_x of x in X satisfying

$$(4.4) \quad \text{there is an } n = n_x \in \mathbf{N} \text{ with the property that for each } k \in \{0, 1, \dots, 2^n - 1\}, \\ V_x \times [k/2^n, (k + 1)/2^n] \subset U_{x,k} \text{ for some } U_{x,k} \in \mathcal{U}.$$

Because the sets $U_{x,k}$ are G_0 -invariant, $G_0 \cdot V_x$ also satisfies 4.4; thus we may assume that V_x is G_0 -invariant. In addition, V_x may be assumed G_x -invariant (replace it with a smaller neighborhood, if necessary, by [1], Exercise 1.9 applied to the group $\langle G_0, G_x \rangle$). Finally, if $y = gx$ ($g \in G$) is in the orbit of x , define $V_y = gV_x$.

In this way we obtain a G -invariant covering $\mathcal{V} = \{V_x | x \in X\}$ of X by G_0 -invariant open subsets. Let $H \leq G$. Then

$$(V_x \cap X^H) \times [k/2^n, (k + 1)/2^n] \subset U_{x,k} \cap (X^H \times I) \quad (n = n_x),$$

and the partitions $\mathcal{P}_H(U_{x,k})$ of $U_{x,k} \cap (X^H \times I)$ ($0 \leq k \leq 2^n - 1$) induce a partition $\mathcal{Q}_H(V_x)$ of $V_x \cap X^H$; this is due to the fact that, by connectedness, every slice $\{y\} \times [k/2^n, (k + 1)/2^n]$ is contained in some set of $\mathcal{P}_H(U_{x,k})$. This finishes the construction of the required G -covering $(\mathcal{V}, \mathcal{Q})$ of X .

5. Some properties.

In this section we explain, how the proofs of the results of sections 5, 6 and 7 of [3] carry over to the present situation.

a) *Tautness*

PROPOSITION 5.1. *Suppose X is a paracompact G -space and $A \subset X$ a closed G -subspace. Then the canonical homomorphism*

$$\varinjlim_N \bar{H}_G^n(N; m) \rightarrow \bar{H}_G^n(A; m)$$

is an isomorphism (N runs through the G -neighborhoods of A).

As in [3], the proof of 5.1 is similar to the classical case ([7] 6.6.2), and is based on the following two lemmas:

LEMMA 5.2. *Let X be a paracompact G -space and $(\mathcal{U}, \mathcal{P})$ an open G -covering of X . Then there exists an open G -covering $(\mathcal{V}, \mathcal{Q})$ of X such that for each $H \leq G$, the covering $\mathcal{V}_H = \bigcup_{V \in \mathcal{V}} \mathcal{Q}_H(V)$ is a star refinement of $\mathcal{U}_H = \bigcup_{U \in \mathcal{U}} \mathcal{P}_H(U)$.*

PROOF. We first show that there is a locally finite open refinement \mathcal{W}' of \mathcal{U} such that \mathcal{W}' is G -invariant and consists of G_0 -invariant subsets of X .

The sets U/G_0 ($U \in \mathcal{U}$) form an open covering of X/G_0 . Because X/G_0 is paracompact by the theorem of E. Michael ([2], p. 165), this covering has an open locally finite refinement. Taking inverse images in X we obtain an open locally finite refinement \mathcal{W} of \mathcal{U} such that the sets of \mathcal{W} are G_0 -invariant. Then, because G/G_0 is finite,

$$\mathcal{W}' = \bigcap_{gG_0 \in G/G_0} g\mathcal{W}$$

is furthermore G -invariant (compare [1], p. 133).

Let $U' \in \mathcal{W}'$ and choose some $U \in \mathcal{U}$ such that $U' \subset U$. If $H \leq G$, we define

$$\mathcal{P}'_H(U') = \left(\bigcap_{gG_0} \mathcal{P}_H(gU) \right) \cap U',$$

the first (finite) intersection being taken over those $gG_0 \in G/G_0$ for which $gU' = U'$ (note: if \mathcal{A} and \mathcal{B} are collections of subsets of a certain set Y , we denote $\mathcal{A} \cap \mathcal{B} = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$). For a set gU' in the orbit of U' in \mathcal{W}' , we can now define $\mathcal{P}'_H(gU') = g \cdot \mathcal{P}'_g^{-1} \mathcal{P}'_H(U')$. In this way we obtain an open G -covering $(\mathcal{W}', \mathcal{P}')$ of X , and if we replace $(\mathcal{U}, \mathcal{P})$ with $(\mathcal{W}', \mathcal{P}')$, we can (and henceforth will) assume that the original \mathcal{U} is locally finite.

Next we construct, using the method of [2], p. 167, a star refinement \mathcal{V} of \mathcal{U} such that \mathcal{V} is G -invariant and consists of G_0 -invariant open subsets of X . Fix $V \in \mathcal{V}$, $V \neq \emptyset$. The set

$$\mathcal{U}_V = \{U \in \mathcal{U} \mid V \subset U\}$$

is finite, because \mathcal{U} is locally finite. For $H \leq G$ define

$$\mathcal{Q}_H(\mathcal{V}) = \left(\bigcap_{U \in \mathcal{U}_V} \mathcal{P}_H(U) \right) \cap V.$$

Then $(\mathcal{V}, \mathcal{Q})$ is an open G -covering of X .

To prove that $(\mathcal{V}, \mathcal{Q})$ has the required property, let $H \leq G$ and $W \in \mathcal{Q}_H(V)$ for some $V \in \mathcal{V}$. Choose $U \in \mathcal{U}$ such that $\text{st}(V, \mathcal{V}) \subset U$. Then $U \in \mathcal{U}_V$. Assume that $W' \in \mathcal{Q}_H(V')$, $V' \in \mathcal{V}$, and $W \cap W' \neq \emptyset$. Now $V \cap V' \neq \emptyset$, $V' \subset \text{st}(V, \mathcal{V}) \subset U$ and so $U \in \mathcal{U}_{V'}$. By the definition of $\mathcal{Q}_H(V)$ and $\mathcal{Q}_H(V')$, W and W' are each contained in a set of $\mathcal{P}_H(U)$, and in fact $W \cap W' \neq \emptyset$ implies that they are both contained in the same set of $\mathcal{P}_H(U)$. It follows that $\text{st}(W, \mathcal{V}_H)$ is contained in this set of $\mathcal{P}_H(U)$.

LEMMA 5.3. *Let X be a completely regular (e.g. paracompact) G -space and $A \subset X$ a G -subspace. Given an open G -covering $(\mathcal{V}, \mathcal{Q})$ of X , there is an open G -neighborhood N of A and an equivariant function $f: N \rightarrow A$ (not necessarily continuous) satisfying*

- i) $f(x) = x$ for $x \in A$, and
- ii) if $H \leq G$ and $W \in \mathcal{V}_H$, then $f(W \cap N) \subset \text{st}(W, \mathcal{V}_H)$.

PROOF. Let

$$N = \{x \in X \mid \text{there is a } V \in \mathcal{V} \text{ with } x \in V, \text{ and a } G\text{-map } f_x: Gx \rightarrow A \text{ such that } f_x(x) \in W \cap A \text{ whenever } x \in X^H \text{ (} H \leq G \text{) and } x \in W \in \mathcal{Q}_H(V)\}.$$

Clearly $A \subset N$ and N is a G -subset. We show that N is open. Let $x \in N$, and choose $V \in \mathcal{V}$ and $f_x: Gx \rightarrow A$ as above. Also, let $V_x \subset V$ be an open neighborhood of x as in v) of 3.1. Pick a slice $S_x \subset V_x$ at x ([1], II 5.4). For every $y \in S_x$ there is a G -map $h_y: Gy \rightarrow Gx$ with $h_y(y) = x$. Now the set G_0S_x is an open neighborhood of x , and $G_0S_x \subset G_0V = V$. We claim that $G_0S_x \subset N$.

Let $z \in G_0S_x$, $z = gy$ ($g \in G_0, y \in S_x$). Then $z \in V$ and $f_x \circ h_y: Gy \rightarrow Gx \rightarrow A$ is a G -map. If $z \in X^K$, $z \in W \in \mathcal{Q}_K(V)$, then $y \in X^{g^{-1}Kg}$, $y \in g^{-1}W \in \mathcal{Q}_{g^{-1}Kg}(g^{-1}V) = \mathcal{Q}_{g^{-1}Kg}(V)$. Because $G_y \leq G_x$ and $y \in S_x \subset V_x$, also $x \in g^{-1}W$, and thus

$$(f_x \circ h_y)(z) = g \cdot f_x(x) \in g \cdot ((g^{-1}W) \cap A) = W \cap A.$$

Therefore $z \in N$. This completes the proof that N is open.

We now construct $f: N \rightarrow A$. Set $f(x) = x$ for $x \in A$. To define $f|N \setminus A$, let S be a set of representatives for the G -orbits of $N \setminus A$. Given $y \in S$, we choose a $V_y \in \mathcal{V}$ with $y \in V_y$ and a G -map $f_y: Gy \rightarrow A$ as in the definition of N . We define $f|Gy = f_y$.

By definition, f satisfies i). To prove ii), assume that $H \leq G$, $W \in \mathcal{Q}_H(V)$, $V \in \mathcal{V}$ and $x \in W \cap N$; we claim that $f(x) \in \text{st}(W, \mathcal{V}_H)$. This is obvious if $x \in A$. Let then $x \in N \setminus A$, $x = gy$ with $g \in G$, $y \in S$. Because $x \in X^H$, we have $y \in X^{g^{-1}Hg}$; assume

$y \in W' \in \mathcal{Q}_{g^{-1}Hg}(V_y)$ ($V_y \in \mathcal{V}$ was chosen in the preceding paragraph). Then $gW' \in \mathcal{Q}_H(gV_y)$ and $x = gy \in W \cap gW'$, whence $gW' \subset \text{st}(W, \mathcal{V}_H)$. Finally $f(x) = g \cdot f(y) = g \cdot f_y(y) \in g \cdot (W' \cap A) \subset gW' \subset \text{st}(W, \mathcal{V}_H)$, so $f(x) \in \text{st}(W, \mathcal{V}_H)$.

b) *Interpretation as sheaf cohomology of X/G*

Let X be a paracompact G -space. Because the canonical projection $\pi: X \rightarrow X/G$ is a closed surjection, the theorem of E. Michael referred to above implies that X/G is also paracompact. Exactly as in section 6 of [3] we obtain the exact sequence

$$(5.4) \quad \bar{C}_G^0 \xrightarrow{d} \bar{C}_G^1 \xrightarrow{d} \bar{C}_G^2 \rightarrow \dots$$

of fine sheaves on X/G , where \bar{C}_G^n is the sheaf associated to the presheaf $U \mapsto \bar{C}_G^n(\pi^{-1}U; m)$, $U \subset X/G$ open. Recall in particular that the exactness of 5.4 is a consequence of the tautness result 5.1. The proof of 6.3 in [3] can also be modified to show that the global sections of \bar{C}_G^n are

$$\Gamma(X/G, \bar{C}_G^n) = \bar{C}_G^n(X; m).$$

Define $A = \ker[\bar{C}_G^0 \xrightarrow{d} \bar{C}_G^1]$, a sheaf on X/G . As in [3], the above remarks suffice to prove

THEOREM 5.5. $H^n(X/G; A) \cong \bar{H}_G^n(X; m)$ for all $n \in \mathbb{N}$.

We say that a 0-cochain $c \in C_G^0(X; m)$ is *locally constant*, if there exists an open G -covering $(\mathcal{V}, \mathcal{P})$ of X with the property that $c(\varphi) = c(\varphi')$ for C -maps $\varphi, \varphi': G/H \rightarrow X$ ($H \leq G$) whenever $\{\varphi(eH), \varphi'(eH)\}$ is contained in a set of $\mathcal{P}_H(V)$ for some $V \in \mathcal{V}$. As in 6.5.a) of [3], the sections of A on an open set $U \subset X/G$ are given by

$$(5.6) \quad \Gamma(U, A) = \{c \in C_G^0(\pi^{-1}U; m) \mid c \text{ is locally constant}\}.$$

As to the stalks of A , the result of 6.5.b) of [3] holds, but the proof needs some modifications:

PROPOSITION 5.7. *If $y \in X/G$, $x \in \pi^{-1}(y)$ and $H = G_x \leq G$, then the stalk A_y of A at y is isomorphic to $m(G/H)$.*

PROOF. We have the canonical homomorphism

$$\gamma: A_y = \varinjlim_U \Gamma(U, A) \rightarrow m(G/H), y \in U \subset X/G, U \text{ open,}$$

defined as follows: Let $\varphi_x: G/H \rightarrow Gx$ be the G -homeomorphism $gH \mapsto gx$. If $U \subset X/G$ is an open neighborhood of y , we denote the composite $G/H \xrightarrow{\varphi_x} Gx \hookrightarrow \pi^{-1}U$ by the same symbol φ_x . Now γ is the direct limit of the homomorphisms $\gamma_U: \Gamma(U, A) \rightarrow m(G/H)$, $\gamma_U(c) = c(\varphi_x)$. We claim that γ is an isomorphism.

Let W be a tube around the orbit $\pi^{-1}(y) = Gx$ and $r: W \rightarrow Gx$ a G -retraction. For the surjectivity of γ , let $a \in m(G/H)$. We define $c \in C_G^0(W; m)$ by $c(\varphi) = m(r \circ \varphi)(a) \in m(G/K)$ for a G -map $\varphi: G/K \rightarrow W$. Let $(\mathcal{U}, \mathcal{P})$ be the open G -covering of $Gx \cong G/H$ described in 3.3; then c is locally constant w.r.t. the open G -covering $(r^{-1}\mathcal{U}, r^{-1}\mathcal{P})$ of W . Thus $c \in \Gamma(\pi W, A)$, and clearly $\gamma_{\pi W}: c \mapsto a$, proving surjectivity.

For injectivity, let $c \in \Gamma(U, A)$, $U \subset X/G$ an open neighborhood of y , and assume that $\gamma_U(c) = c(\varphi_x) = 0$. By assumption, c is locally w.r.t. an open G -covering $(\mathcal{V}, \mathcal{P})$ of $\pi^{-1}U$. Choose $V \in \mathcal{V}$ such that $x \in V$. By 3.1.v) there is an open neighborhood V_x of x , which we may assume H -invariant, with the property that $V_x \cap X^K$ is contained in some set of $\mathcal{P}_K(V)$, whenever $K \leq H$.

Let $S_x = r^{-1}(x)$ be the slice at x corresponding to the above tube W and retraction r . Then $G(S_x \cap V_x)$ is also a tube around Gx ; to simplify notation we denote this smaller tube again by W . Now we have $r^{-1}(x) \subset V_x$. We intend to show that $c|_{\pi W} = 0 \in \Gamma(\pi W, A)$.

Let $\varphi: G/K \rightarrow W$ be a G -map, and let $\varphi': G/K \rightarrow W$ be the composite

$$\varphi': G/K \xrightarrow{\varphi} W \xrightarrow{r} Gx \hookrightarrow W.$$

We have $\varphi'(eK) = gx \in gV_x$ for some $g \in G$, $g^{-1}Kg \leq H$ and $\varphi(eK) \in r^{-1}(gx) = g \cdot r^{-1}(x) \subset gV_x$. By the choice of V_x , $gV_x \cap X^K$ is contained in a set of $\mathcal{P}_K(gV)$. Therefore $\{\varphi(eK), \varphi'(eK)\}$ is contained in a set of $\mathcal{P}_K(gV)$, and the local constantness of c w.r.t. $(\mathcal{V}, \mathcal{P})$ implies that $c(\varphi) = c(\varphi')$.

Finally, $\varphi' = \varphi_x \circ \alpha$, where $\alpha: G/K \rightarrow G/H$ is the G -map given by $g_1K \mapsto g_1gH$ ($g_1 \in G; g$ as in the preceding paragraph), and thus $c(\varphi) = c(\varphi') = m(\alpha)(c(\varphi_x)) = 0$.

c) *Comparison with equivariant singular cohomology*

The result of section 7 of [3] (Theorem 7.2) is true in the present context, too:

THEOREM 5.8. *Suppose that X is a paracompact G -space and every orbit $Gx \subset X$ is taut with respect to equivariant singular cohomology $H_G^*(\cdot; m)$. Then there is a natural isomorphism*

$$\tilde{H}_G^*(X; m) \cong H_G^*(X; m).$$

In fact section 7 of [3] can be read almost word for word to get a proof of 5.8. The following two comments are in order, however:

Firstly, we must define an equivariant singular cochain $c \in S_G^n(X; m)$ to be *locally zero*, if there is an open G -covering $(\mathcal{V}, \mathcal{P})$ of X with the property that $c(\sigma) = 0$ for any equivariant singular simplex $\sigma: G/H \times \Delta^n \rightarrow X$ such that $\sigma(\{eH\} \times \Delta^n)$ is contained in a set of $\mathcal{P}_H(V)$ for some $V \in \mathcal{V}$. Again the proof of Prop. I.6.4 in [4] essentially shows that the cohomology of $S_G^*(X; m)$ does not change, if we divide out the locally zero cochains.

Secondly, in the article [6] of Piacenza referred to in [3], it is assumed that G is

a discrete group. However, the result we need, i.e. that $H_G^*(X; m)$ can be computed from the exact sequence $S_G^0 \rightarrow S_G^1 \rightarrow S_G^2 \rightarrow \dots$ of fine sheaves on X/G , clearly holds under the assumptions of 5.8, too.

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