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EXISTENCE OF MEROMORPHIC SOLUTIONS OF ALGEBRAIC DIFFERENTIAL EQUATIONS

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1. Introduction.

In this paper we consider algebraic differential equations of the form

$$\Omega(z,w) = \sum_{k=0}^{n} A_k(z)w^k, \tag{1.1}$$

where each $A_k(z)$ is meromorphic ($A_n \not\equiv 0$), and where $\Omega(z,w)$ is a differential polynomial in w and its derivatives with meromorphic coefficients, i.e.,

$$\Omega(z,w) = \sum_{\lambda \in I} B_\lambda(z) w^{i_0}(w')^{i_1} \ldots (w^{(p)})^{i_p}, \tag{1.2}$$

where λ denotes the multi-index $\lambda = (i_0, \ldots, i_p)$, where each $B_\lambda(z)$ is meromorphic, and where the index set I is of course finite. As a special case of (1.1) we will, in particular, consider first order equations of the form

$$w' = \frac{\sum_{k=0}^{n} A_k(z)w^k}{\sum_{k=0}^{m} B_k(z)w^k}, \tag{1.3}$$

where each $A_k(z)$, $B_k(z)$ is meromorphic (and $A_n \not\equiv 0$, $B_m \not\equiv 0$). We assume that the right-hand side of (1.3) is irreducible as a rational function in w. Throughout this paper, the term "meromorphic" always means meromorphic in the whole complex plane.

The classical Malmquist-Yosida theorem (see, e.g., [11] and [16]) states that if a differential equation of the form

$$(w')^p = R(z,w), \tag{1.4}$$

where p is a positive integer and $R(z,w)$ is a rational function in z and w, possesses a transcendental meromorphic solution $w = f(z)$, then (1.4) must actually be of

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the more restricted form

$$(w')^p = R_0(z) + R_1(z)w + \dots + R_n(z)w^n \tag{1.5}$$

where $n \leqq 2p$ (and where each $R_k(z)$ is rational). The possible types of equations (1.5) with transcendental meromorphic solutions, and the possible orders of growth of such solutions, have been settled completely in the articles by Steinmetz [15] and Bank and Kaufman [1].

Several authors have generalized the Malmquist-Yosida theorem by proving that, if an equation of the form (1.1) possesses a meromorphic solution whose growth dominates the growth of all of the coefficients, then the equation must actually belong to some smaller subclass of equations of the form (1.1), see, e.g., [4], [13] and the references in [4] and [8].

If the equations in these smaller subclasses are loosely called "Malmquist" equations, then the question of whether or not a "non-Malmquist" equation can actually admit a meromorphic solution seems to have been treated very little in the literature, see however [5], [6], [8] and [10]. We believe that the first question to be asked about a "non-Malmquist" equation of the form (1.1) is whether or not a meromorphic solution exists. More generally, it is natural to ask how many distinct meromorphic solutions can a "non-Malmquist" equation of the form (1.1) possess. These are the main questions we address in this paper.

2. Statement of the main results.

The following results, as well as their proofs, need some familiarity with the Nevanlinna theory, see, e.g., [9] for notations and basic results. In particular, $S(r,f)$ denotes any quantity satisfying $S(r,f) = o(T(r,f))$ as $r \to +\infty$, possibly outside a set of values r of finite linear measure.

For convenience in stating our results, we will give the following definition separately:

DEFINITION 2.1. Consider the algebraic differential equation (1.1), and suppose that q is a fixed integer such that $0 \leqq q \leqq n$. We say that the *A_q-hypothesis* holds, if there exist $v \not\equiv 0$ meromorphic and h nonconstant entire such that

$$A_q = ve^h, \tag{2.1}$$

where

$$T(r,v) = S(r,e^h), \tag{2.2}$$

and where for all $k \neq q$, $0 \leqq k \leqq n$, and all $\lambda \in I$ we have

$$T(r,A_k) = S(r,A_q), \qquad T(r,B_\lambda) = S(r,A_q). \tag{2.3}$$

Of course when we speak about the A_q-hypothesis for an equation (1.3), we first rewrite it in the form

$$\sum_{k=0}^{m} B_k(z)w^k w' = \sum_{k=0}^{n} A_k(z)w^k. \tag{2.4}$$

Our first results give conditions under which an algebraic differential equation cannot possess a meromorphic solution.

THEOREM 2.2. *Consider an equation* (1.1) *such that*

$$n > \Delta = \max_{\lambda \in I}\left(\sum_{\alpha=0}^{p} (\alpha + 1)i_\alpha\right) \tag{2.5}$$

and assume that the A_q-hypothesis holds for some q that satisfies $\Delta \leqq q \leqq n - 1$. Suppose also that $A_0 \not\equiv 0$, $B_{(0,\dots,0)} \equiv 0$. Then (1.1) *does not possess a meromorphic solution.*

When we consider the special case (1.3) in Theorem 2.2, we have $\Delta = m + 2$ from (2.4). Therefore we obtain an immediate corollary:

THEOREM 2.3. *Consider an equation* (1.3) *such that $n \geqq m + 3$ and assume that the A_q-hypothesis holds with $m + 2 \leqq q \leqq n - 1$. Suppose also that $A_0 \not\equiv 0$. Then* (1.3) *does not possess a meromorphic solution.*

In the next section we give examples to show that Theorem 2.3 is sharp. In Section 6 we give several results concerning (1.3) that come out of the proof of Theorem 2.2. For example, Theorem 6.2 gives a maximum number for the possible number of distinct meromorphic solutions that certain equations of the form (1.3) could possess.

Now we consider the special case of (1.3) of "non-Malmquist" polynomial equations, i.e., equations of the form

$$w' = \sum_{k=0}^{n} A_k(z)w^k, \qquad n \geqq 3, \tag{2.6}$$

where each A_k is meromorphic ($A_n \not\equiv 0$). With additional assumptions, we can extend Theorem 2.3 slightly with

THEOREM 2.4. *Suppose the A_n-hypothesis holds for* (2.6). *Suppose also that $A_0 \not\equiv 0$, $A_{n-1} \not\equiv 0$. Then* (2.6) *does not possess a meromorphic solution.*

None of the hypotheses in Theorem 2.4 can be deleted (see §8). Finally, in Section 9 we consider (2.6) in the special case of polynomial coefficients, mostly to obtain upper bounds for both the number of distinct meromorphic solutions, and also for the number of linearly independent meromorphic solutions.

3. Sharpness of Theorem 2.3.

EXAMPLE 3.1. We cannot have $q = m + 1$ in Theorem 2.3. In fact, let $m \geqq 0$ be any integer. Then $w(z) = e^z + 1$ satisfies the equation

$$w' = \frac{-1 + e^{2z}w^{m+1} + 2w^{m+2} - w^{m+3}}{1 + w + \ldots + w^{m-1} + w^m}.$$

This is a straightforward verification by observing that $w' = w - 1$ and that $e^{2z} = w^2 - 2w + 1$.

EXAMPLE 3.2. Also, we cannot have $q = n$ in Theorem 2.3. For example, for all $n \geqq 4$, $w(z) = e^{-z}$ satisfies

$$w' = \frac{1 - w - w^2 - e^{nz}w^n}{1 + w}.$$

EXAMPLE 3.3 The assumption $A_0 \not\equiv 0$ in Theorem 2.3 cannot be deleted, since, for example, $w(z) = e^z$ satisfies

$$w' = \frac{w + w^2 + e^z w^{n-1} - w^n}{1 + w}$$

for any $n \geqq 4$.

EXAMPLE 3.4. In Theorem 2.3, the two assumptions $n \geqq m + 3$ and $m + 2 \leqq q \leqq n - 1$ cannot be replaced by the two assumptions $n \geqq m + 2$ and $m + 1 \leqq q \leqq n - 1$, since, for example, $w(z) = e^z + 1$ satisfies

$$w' = \frac{1 + e^z w^2 - w^3}{-1 - w}.$$

4. Lemmas.

In addition to the basic results of the Nevanlinna theory, we need the following three lemmas:

LEMMA A. *Let f be a transcendental meromorphic solution of the equation*

$$f^n P(f) = Q(f)$$

where $P(f)$, $Q(f)$ are polynomials in f and its derivatives with meromorphic coefficients, say a_j. If the total degree of Q is at most n, then

$$m(r, P(f)) \leqq \sum_j m(r, a_j) + S(r, f).$$

Lemma A is essentially due to J. Clunie. In fact, the proof of Lemma A is a simple modification of the proof of Lemma 3.3 in [9], see also [3] and [2], Lemma 1.

The next lemma may be found in [14].

LEMMA B. *There does not exist a transcendental meromorphic function f which satisfies an identity*

$$a_0(z) + a_1(z)f(z) + \ldots + a_p(z)(f(z))^p \equiv 0,$$

where $p \geqq 1$, $a_p \not\equiv 0$, *and* $a_0, \ldots, a_p$ *are meromorphic functions such that*

$$\sum_{j=0}^{p} T(r, a_j) = o(T(r, f))$$

as $r \to \infty$ *through some sequence* $r_1, r_2, r_3, \ldots$ *of r-values.*

The last lemma is contained in a more general result due to Mok'honko in [12], see also [4], p. 279–280.

LEMMA C. *Let* f, $a_0, \ldots, a_p$ *be meromorphic functions with* $a_p \not\equiv 0$, $p \geqq 1$, *and define*

$$G = a_0 + a_1 f + \ldots + a_p f^p.$$

Then

$$T(r, G) = pT(r, f) + \psi(r),$$

where

$$\psi(r) = O\left(\sum_{j=0}^{p} T(r, a_j)\right) \qquad \text{as } r \to \infty.$$

5. Proof of Theorem 2.2.

Suppose that $w = f(z)$ is a meromorphic solution of (1.1), and write (1.1) in the form

$$A_q = f^{-q}(\Omega(z, f) - A_0 - \ldots - A_{q-1}f^{q-1} - A_{q+1}f^{q+1} - \ldots - A_n f^n). \tag{5.1}$$

By elementary Nevanlinna theory and by (2.3), we can deduce from (5.1) that there exists a constant $D > 0$ such that

$$T(r, A_q) \leqq DT(r, f) + S(r, f). \tag{5.2}$$

Hence, from (2.3) and (5.2), we see that for all $k \neq q$, $0 \leqq k \leqq n$, and all $\lambda \in I$ we have

$$T(r, A_k) = S(r, f), \qquad T(r, B_\lambda) = S(r, f). \tag{5.3}$$

Now, from (1.1) and $n > \Delta$, any pole of f must either be a zero of A_n, a pole of some A_k $(k \neq n)$, or a pole of some B_λ. By taking multiplicities into account, it is easy to see that

$$N(r,f) \leqq N\left(r,\frac{1}{A_n}\right) + \sum_{k=0}^{n-1} N(r,A_k) + \sum_{\lambda\in I} N(r,B_\lambda). \tag{5.4}$$

From (2.1) and (2.2), $N(r,A_q) = S(r,A_q)$; hence $N(r,A_q) = S(r,f)$ from (5.2). Then from (5.3) and (5.4),

$$N(r,f) = S(r,f). \tag{5.5}$$

Regarding (5.5), see also [10], (5) on p. 81.

We make two observations here. First, it follows from (2.1), (2.2) and (5.2) that f is necessarily transcendental. Second, from (2.1), (2.2), the lemma for the logarithmic derivative, and (5.2), we obtain

$$T\left(r,\frac{A_q'}{A_q}\right) = S(r,f). \tag{5.6}$$

Taking now the logarithmic derivative of both sides of (5.1), then rearranging the terms and noting that $n > \Delta$, we obtain

$$f^n\left((n-q)A_nf' + \left(A_n' - A_n\frac{A_q'}{A_q}\right)f\right) = Q_n, \tag{5.7}$$

where Q_n is a differential polynomial in f and its derivatives with total degree $\leqq$n. The coefficients of Q_n are polynomials in A_k, A_k', where $k \neq q$, in A_q'/A_q and in B_λ, B_λ', $\lambda\in I$. Thus, by (5.3) and (5.6), all the coefficients, say c_μ, of Q_n satisfy $T(r,c_\mu) = S(r,f)$. Let us define

$$H = (n-q)A_nf' + \left(A_n' - A_n\frac{A_q'}{A_q}\right)f. \tag{5.8}$$

Suppose now that $H \equiv 0$. Since $n \neq q$, we may integrate (5.8) to obtain

$$A_qf^q \equiv CA_nf^n, \tag{5.9}$$

where $C \neq 0$ is some constant. Substituting (5.9) into (1.1) we obtain

$$\Omega(z,f) = A_0 + A_1f + \ldots + (C+1)A_nf^n, \tag{5.10}$$

where the term A_qf^q is now missing from the right-hand side. By repeated differentiation of (5.8), we see from (5.3) and (5.6) that

$$f^{(k)} = \beta_k(z)f \tag{5.11}$$

for all $k\in\mathbf{N}$, where $T(r,\beta_k) = S(r,f)$. Substituting (5.11) into (5.10), we eliminate all derivatives from the left-hand side, resulting in an algebraic equation of the form

$$\sum_{\lambda\in I} D_\lambda f^{i_0+\ldots+i_p} = A_0 + \ldots + (C+1)A_nf^n, \tag{5.12}$$

where $T(r, D_\lambda) = S(r, f)$ for all $\lambda \in I$, and $D_{(0,\ldots,0)} \equiv 0$. Lemma B applied to (5.12) results now in the contradiction $A_0 \equiv 0$.

Thus from now on, we may suppose $H \not\equiv 0$. Applying the Clunie lemma, Lemma A, to (5.7) we obtain $m(r, H) = S(r, f)$. On the other hand, it follows from (5.3), (5.5), and (5.6) that $N(r, H) = S(r, f)$. Hence

$$T(r, H) = S(r, f). \tag{5.13}$$

Now from (5.8), we get

$$\frac{1}{f} = \frac{1}{H}\left((n - q)A_n \frac{f'}{f} + A_n' - A_n \frac{A_q'}{A_q}\right). \tag{5.14}$$

Hence, by (5.13), (5.3) and (5.6), (5.14) gives

$$m\left(r, \frac{1}{f}\right) = S(r, f),$$

and therefore

$$T(r, f) = N\left(r, \frac{1}{f}\right) + S(r, f). \tag{5.15}$$

Suppose now that z_1 is a zero of f of multiplicity $\nu \geqq 2$. Then it follows from (5.8) that H/A_n must have a zero of multiplicity $\geqq \nu - 1$ at z_1. Let $N_2(r, 1/f)$ denote the counting function for multiple zeros of f. Then we obtain from (5.3) and (5.13) that

$$N_2\left(r, \frac{1}{f}\right) \leqq 2N\left(r, \frac{A_n}{H}\right) = S(r, f). \tag{5.16}$$

Let now $N_1(r, 1/f)$ denote the counting function for simple zeros of f. By (5.15) and (5.16), we have

$$T(r, f) = N_1\left(r, \frac{1}{f}\right) + S(r, f). \tag{5.17}$$

From (5.8), we again obtain by repeated differentiation

$$f^{(k)} = \beta_k(z)f + \alpha_k(z) \tag{5.18}$$

for all $k \in \mathrm{N}$, where $T(r, \alpha_k) = S(r, f)$, $T(r, \beta_k) = S(r, f)$ by (5.13), (5.6), and (5.3). Substituting (5.18) into (1.1) we obtain an algebraic equation

$$\sum_{k=0}^{n} C_k(z)f^k = 0, \tag{5.19}$$

where $C_k = A_k + D_k$ with $T(r, D_k) = S(r, f)$ holding for $k = 0, \ldots, n$. Moreover, $D_k \equiv 0$ for all $k \geqq q$, since $B_{(0,\ldots,0)} \equiv 0$ and $q \geqq \Delta$, see (2.5).

Now suppose that z_0 is a simple zero of f that is not a pole of any of the coefficients C_k. Then we must have $C_0(z_0) = 0$. Since $T(r, C_0) = S(r,f)$ and $N(r, C_k) = S(r,f)$ for all $k = 0, \ldots, n$, we obtain that either $C_0 \equiv 0$ or

$$N_1\left(r, \frac{1}{f}\right) \leqq N\left(r, \frac{1}{C_0}\right) + S(r,f) = S(r,f).$$

In view of (5.15), we must have $C_0 \equiv 0$. Therefore (5.19) takes the form

$$\sum_{k=0}^{n-1} C_{k+1}(z) f^k = 0.$$

We may now repeat this same reasoning up to

$$\sum_{k=0}^{n-q} C_{k+q}(z) f^k = 0,$$

i.e., we have

$$\sum_{k=0}^{n-q} A_{k+q}(z) f^k = 0. \tag{5.20}$$

Now let z_0 again be a simple zero of f. Then from (5.20), either A_k has a pole at z_0 for some $k > q$ or $A_q(z_0) = 0$. Hence, from (5.3) and (5.6), we obtain

$$N_1\left(r, \frac{1}{f}\right) = S(r,f),$$

which contradicts (5.17). This contradiction proves Theorem 2.2.

6. Corollaries of the preceding proof.

In the preceding proof, the assumptions of Theorem 2.2 were used mostly in a few critical points. Therefore, the reasoning given in this proof may be used to obtain related results to Theorem 2.2, with only part of the assumptions being assumed to hold. For simplicity, we restrict ourselves to results related to the special case of Theorem 2.3.

PROPOSITION 6.1. *Consider the equation* (1.3) *with* $n \geqq m + 3$ *and* $A_0 \equiv 0$. *Suppose the* A_q*-hypothesis holds with* $1 \leqq q \leqq n - 1$. *If* $f \not\equiv 0$ *is a meromorphic solution of* (1.3), *then necessarily*

$$A_q + A_n f^{n-q} = 0.$$

PROOF. We can apply the proof in the preceding section up to (5.8). Assuming $H \not\equiv 0$ holds, then also (5.13) and (5.15) hold. As well, (5.16) and (5.17) remain true. Solving now f' from (5.8) and substituting into (1.3), we obtain

$$\frac{1}{n-q}\left(\left(\frac{A_q'}{A_q}-\frac{A_n'}{A_n}\right)f+\frac{H}{A_n}\right)\left(\sum_{k=0}^{m} B_k f^k\right)=\sum_{k=1}^{n} A_k f^k. \tag{6.1}$$

Let now z_0 be a simple zero of f such that z_0 is not a pole of any A_k, B_k, and such that $A_n(z_0) \neq 0$, $A_q(z_0) \neq 0$. Then obviously $H(z_0)B_0(z_0) = 0$ by (6.1), and therefore we either have $HB_0 \equiv 0$ or

$$N_1\left(r,\frac{1}{f}\right) \leqq N\left(r,\frac{1}{HB_0}\right) + S(r,f) = S(r,f). \tag{6.2}$$

Since (6.2) and (5.17) give a contradiction, we must have $HB_0 \equiv 0$. Since $A_0 \equiv B_0 \equiv 0$ would contradict our irreducibility assumption about (1.3), it follows that $H \equiv 0$. Therefore,

$$A_q f^q \equiv C A_n f^n$$

for some $C \in \mathbf{C}$, see (5.9). Substituting this into (6.1) gives

$$\beta f \sum_{k=0}^{m} B_k f^k = A_1 f + \dots + (C+1) A_n f^n \tag{6.3}$$

where the term $A_q f^q$ is missing on the right-hand side and where all coefficients on both sides have a characteristic function of the form $S(r,f)$. Since f is transcendental, and $n \geqq m+3$, we may apply Lemma B to (6.3) to conclude that $C = -1$, which proves the assertion.

THEOREM 6.2. *Consider the equation* (1.3) *with* $n \geqq m+3$, *and suppose the* A_q-*hypothesis holds for some* q.

(a) *If* $A_0 \not\equiv 0$ *and* $1 \leqq q \leqq m+1$, *then* (1.3) *admits at most* $n-q$ *distinct meromorphic solutions.*

(b) *If* $A_0 \equiv 0$ *and* $1 \leqq q \leqq n-1$, *then* (1.3) *admits at most* $n-q+1$ *distinct meromorphic solutions.*

PROOF. (a) In this case, we may follow the proof of Theorem 2.2 up to (5.19), where here we can only say that (5.19) holds with $C_k = A_k + D_k$ where $T(r, D_k) = S(r,f)$ for $k = 0, \dots, n$. Then by the same reasoning that was used after (5.19), we see that $C_0 = \dots = C_{q-1} \equiv 0$; hence we get a polynomial equation of degree $n-q$ in f,

$$\sum_{k=0}^{n-q} C_{k+q}(z) f^k = 0.$$

Since the collection of all meromorphic functions is a field, there can exist at most $n-q$ distinct meromorphic solutions to this polynomial equation by basic field theory, see [7], p. 515.

(b) This follows immediately by Proposition 6.1, observing that $w \equiv 0$ also satisfies (1.3) in this case.

The next three examples and remark illustrate the sharpness of Theorem 6.2.

EXAMPLE 6.3. Let $n \geqq 3$ be given, and let C be any constant such that $C^{n-1} = 1$. Then there exist complex constants $b_0, \ldots, b_{n-3}$, where $b_{n-3} \neq 0$, such that $w(z) = Ce^z + 1$ satisfies

$$w' = \frac{n - 2 - e^{(n-1)z}w + (1 - n)w^{n-1} + w^n}{b_0 + b_1 w + \ldots + b_{n-3}w^{n-3}} \tag{6.4}$$

for every such $C \in \mathbf{C}$. Hence we have the maximal number of meromorphic solutions for case (a) of Theorem 6.2. The coefficients $b_0, \ldots, b_{n-3}$ can be determined by substituting $w(z) = Ce^z + 1$ into (6.4), and we will leave the details of this elementary calculation to the reader. We mention that

$$b_{n-3} = -\tfrac{1}{2}n^2 + \tfrac{3}{2}n - 1 \neq 0.$$

EXAMPLE 6.4. Let $n \geqq 3$ be given, and let C be any constant such that $C^{n-2} = 1$. Then $w(z) = Ce^z$ satisfies

$$w' = w + e^{(n-2)z}w^2 - w^n \tag{6.5}$$

for every such $C \in \mathbf{C}$. Since $w \equiv 0$ also satisfies (6.5), we have the maximal number of meromorphic solutions for case (b) of Theorem 6.2.

EXAMPLE 6.5. Let $n \geqq 3$ be given, and let C be any constant such that $C^{n-1} = 1$. Then $w(z) = C\exp(z/n)$ satisfies

$$w' = \frac{1}{n}\exp\left\{\frac{1-n}{n}z\right\}w^n.$$

Hence we cannot allow $q = n$ in Theorem 6.2 (b).

REMARK. It is easy to construct examples with $n = 2$, $m = 0$, which show that we cannot replace the assumption $n \geqq m + 3$ with $n \geqq m + 2$ in either (a) or (b) of Theorem 6.2.

PROPOSITION 6.6. *Consider the equation* (1.3) *with* $n \geqq m + 3$, *and suppose the* A_q*-hypothesis holds for some* q. *Let* f *be a meromorphic solution of* (1.3). *Then we have:*

(a) $N(r, f) = S(r, f)$.

(b) *If* $A_0 \not\equiv 0$ *and* $1 \leqq q \leqq m + 1$, *then* $T(r, f) = N(r, 1/f) + S(r, f)$.

(c) *If* $A_0 \equiv 0$, $1 \leqq q \leqq n - 1$ *and* $f \not\equiv 0$, *then* $N(r, 1/f) = S(r, f)$.

PROOF. (a) and (b) will follow by applying the proof of Theorem 2.2, for (a) up to (5.5) and for (b) up to (5.15).

(c) follows immediately from Proposition 6.1.

REMARK. The condition $1 \leqq q \leqq m+1$ in (b) above cannot be improved. See Theorem 2.3, Example 3.2, and the fact that $w(z) = e^z$ satisfies

$$w' = \frac{e^{4z} + w + w^2 - w^4}{1 + w}.$$

PROPOSITION 6.7. *Consider the equation* (1.3) *with* $n \geqq m+3$ *and where the* A_q*-hypothesis holds with* $1 \leqq q \leqq n-1$. *Let* $f \not\equiv 0$ *be a meromorphic solution of* (1.3). *Then*

$$T(r,f) = \frac{1}{n-q} T(r, A_q) + S(r,f).$$

PROOF. If $A_0 \equiv 0$, then the assertion follows immediately from Proposition 6.1.

Suppose now that $A_0 \not\equiv 0$. We may use the same reasoning as in part (a) of the proof of Theorem 6.2 to obtain

$$\sum_{k=0}^{n-q} C_{k+q}(z) f^k = 0. \tag{6.6}$$

By applying Lemma C to (6.6), the assertion follows immediately.

7. A B_q-hypothesis in (1.3).

It is natural to ask whether there is some general nonexistence theorem like Theorem 2.3 in the case when for some q, B_q dominates the other coefficients in (1.3) in the way described in Definition 2.1 for the A_q-hypothesis. The examples below seem to indicate that the answer might be negative when one asks this question about nontrivial meromorphic solutions in the case when $A_0 \equiv 0$, because of the variety of different values that n, m, and q take in these examples.

In fact $w(z) = e^z$ satisfies all the following equations, except for (4) which is satisfied by $w(z) = e^z + 1$:

(1) $$w' = \frac{w + w^{m+1} + w^n}{1 + e^{(n-q-1)z} w^q + w^m}, \qquad n \geqq m+2,\ 1 \leqq q \leqq m-1.$$

(2) $$w' = \frac{w + w^n}{1 + w^{n-1} + e^{(m-q)z} w^q - w^m}, \qquad m \geqq n+1,\ n \leqq q \leqq m-1.$$

(3) $$w' = \frac{2w + 2w^{m+1}}{2 + e^z w^{m-1} + w^m}, \qquad n = m+1,\ q = m-1.$$

(4) $$w' = \frac{-1 + w^2}{1 - e^z w + w^2}, \qquad n = m = 2,\ q = 1.$$

(5) $$w' = \frac{w + w^n}{1 + w^{n-1} + w^{m-1} - e^{-z}w^m}, \qquad m \geqq n + 1, q = m.$$

(6) $$w' = \frac{w + w^n}{1 + e^{(n-m-1)z}w^m}, \qquad n \geqq m + 2, q = m.$$

(7) $$w' = \frac{w + w^m}{1 + e^{-z}w^m}, \qquad m = n = q \geqq 1.$$

(8) $$w' = \frac{w^n}{e^{mz} + w^{n-1} - w^m}, \qquad m \geqq n \geqq 2, q = 0.$$

(9) $$w' = \frac{w^{m+1} + w^n}{e^{(n-1)z} + w^m}, \qquad n \geqq m + 2 \geqq 3, q = 0.$$

8. Proof of Theorem 2.4.

In Theorem 2.4, neither of the two assumptions $A_0 \not\equiv 0$ or $A_{n-1} \not\equiv 0$ can be removed, and the assumption $n \geqq 3$ cannot be replaced by $n \geqq 2$. This is illustrated by the following three equations, which are all satisfied by $w(z) = e^z$:

(1) $$w' = w + w^2 - e^{-z}w^3.$$

(2) $$w' = 1 + w - e^{-nz}w^n, \qquad n \geqq 3.$$

(3) $$w' = 1 + w - e^{-2z}w^2.$$

Of course, in the case when $A_0 \equiv 0$, $w \equiv 0$ always satisfies (2.6).

Proof of Theorem 2.4. A lot of the reasoning will be similar to the proof of Theorem 2.2.

Suppose that f is a meromorphic solution of (2.6). Writing (2.6) in the form

(8.1) $$A_n = f^{-n}(f' - A_0 - A_1 f - \dots - A_{n-1}f^{n-1}),$$

taking the logarithmic derivative of both sides of (8.1), rearranging terms and noting that $A_{n-1} \not\equiv 0$, we obtain

(8.2) $$f^{n-1}\left(f' + \left(\frac{A_n'}{A_n} - \frac{A_{n-1}'}{A_{n-1}}\right)f\right) = Q(z),$$

where $Q(z)$ is a differential polynomial in f, f', f'' of total degree $\leqq n - 1$, with coefficients γ such that $T(r, \gamma) = S(r, f)$. Denote again

(8.3) $$H = f' + \left(\frac{A_n'}{A_n} - \frac{A_{n-1}'}{A_{n-1}}\right)f.$$

Consider first the case $H \equiv 0$. Integrating then (8.3) we get

$$f = CA_{n-1}/A_n \tag{8.4}$$

for some $C \in \mathbf{C}$. Since $A_0 \not\equiv 0$, we must have $f \not\equiv 0$, hence $C \neq 0$. By (8.4), we may write (2.6) in the form

$$f' = A_0 + \ldots + A_{n-2}f^{n-2} + A_{n-1}(1 + C)f^{n-1}.$$

If $C \neq -1$, then Lemma A and the fact that $N(r,f) = S(r,f)$ by Proposition 6.6 (a) would imply $T(r,f) = S(r,f)$. Hence we must have $C = -1$ and

$$f' = A_0 + \ldots + A_{n-2}f^{n-2}. \tag{8.5}$$

Now if $A_k \not\equiv 0$ for some k, $2 \leqq k \leqq n-2$, then Lemma A and $N(r,f) = S(r,f)$ again would imply $T(r,f) = S(r,f)$. Therefore (8.5) must reduce to

$$f' = A_0 + A_1 f. \tag{8.6}$$

Combining now (8.6), (8.3) with $H \equiv 0$, and (8.4) with $C = -1$, we obtain

$$\frac{1}{A_0}\left(\frac{A'_{n-1}}{A_{n-1}} - \frac{A'_n}{A_n} - A_1\right) = \frac{1}{f} = -\frac{A_n}{A_{n-1}},$$

hence

$$A_n = -\frac{A_{n-1}}{A_0}\left(\frac{A'_{n-1}}{A_{n-1}} - \frac{A'_n}{A_n} - A_1\right).$$

This implies $T(r, A_n) = S(r, A_n)$ from the A_n-hypothesis, a contradiction.

Let us now assume that $H \not\equiv 0$. From Lemma A applied to (8.2), we get $m(r,H) = S(r,f)$. Since $N(r,f) = S(r,f)$ and $T(r, A'_n/A_n) = S(r,f)$, we also have $N(r,H) = S(r,f)$. Hence

$$T(r,H) = S(r,f). \tag{8.7}$$

Suppose now that $z_0 \in \mathbf{C}$ is such that $f(z_0) = 0$. If z_0 is a multiple zero of f, of multiplicity $\nu \geqq 2$, then z_0 is a zero of H of multiplicity $\geqq \nu - 1$ from (8.3). If z_0 is a simple zero of f, then either (i) $A_k(z_0) \neq 0, \infty$ for all k and $H(z_0) = f'(z_0) = A_0(z_0)$ from (2.6) and (8.3), or (ii) $A_k(z_0) = 0$ or ∞ for some k. Assume that $H \not\equiv A_0$. Then from this discussion, (8.7), and the A_n-hypothesis, it follows that

$$N\left(r,\frac{1}{f}\right) \leqq 2N\left(r,\frac{1}{H}\right) + N\left(r,\frac{1}{H - A_0}\right) + S(r,f) = S(r,f).$$

Since

$$m\left(r,\frac{1}{f}\right) \leqq m\left(r,\frac{1}{H}\right) + m\left(r,\frac{H}{f}\right) = S(r,f), \tag{8.8}$$

we have

$$T\left(r, \frac{1}{f}\right) = S(r, f),$$

a contradiction. Therefore $H \equiv A_0$ and (8.3) becomes

$$f' + \left(\frac{A_n'}{A_n} - \frac{A_{n-1}'}{A_{n-1}}\right)f = A_0. \tag{8.9}$$

Substituting (8.9) into (2.6), we get

$$A_n + \left(A_1 - \frac{A_n'}{A_n} + \frac{A_{n-1}'}{A_{n-1}}\right)f^{-(n-1)} = -A_2 f^{-(n-2)} - \dots - A_{n-1}f^{-1}. \tag{8.10}$$

Considering counting functions for poles on both sides of (8.10), we deduce that

$$N\left(r, \frac{1}{f}\right) = S(r, f).$$

By (8.8), this results in

$$T\left(r, \frac{1}{f}\right) = S(r, f),$$

a contradiction. The assertion follows.

9. Polynomial non-Riccati equations with polynomial coefficients.

We close this paper with some remarks concerning the equation (2.6) with polynomial coefficients, i.e., equations of the form

$$w' = \sum_{k=0}^{n} P_k(z)w^k, \qquad n \geqq 3, \tag{9.1}$$

where each P_k is a polynomial ($P_n \not\equiv 0$). Looking at the proofs in Sections 5 and 8, we see that their key idea is that all meromorphic solutions of the original differential equation satisfy a second, simpler differential equation. Combining these two differential equations, we obtained an algebraic equation that the meromorphic solutions would have to satisfy. This same idea may be applied to (9.1). Theorem 3 in [8] tells us that (9.1) may possess at most finitely many distinct meromorphic solutions. Of course, it is well known that (9.1) with $n = 2$ can possess an infinite number of distinct meromorphic solutions. He Yuzan proved in [10], Theorem 2, the existence of a constant $K = K(n, \deg P_0, \dots, \deg P_n)$ such that the number of distinct meromorphic solutions of (9.1) is at most K. A new proof for this result is contained in

THEOREM 9.1. *Let* $d_k = \deg P_k$, $k = 0, \ldots, n$, *denote the degrees of the polynomials* P_k *in* (9.1), *and set*

$$q = d_n + \max_{0 \leqq k \leqq n-1} \frac{(d_k - d_n)^+}{n-k}. \tag{9.2}$$

Then we have:

(i) (9.1) *possesses at most* $(q+1)n - q$ *distinct meromorphic solutions.*

(ii) (9.1) *possesses at most* $q + 1$ *linearly independent meromorphic solutions.*

PROOF. Let $w = f(z)$ be an arbitrary meromorphic solution of (9.1). By the Malmquist theorem f is rational, hence we may assume that $f = R/S$ where R and S are irreducible polynomials with degrees d_R, d_S, respectively. Denote $d = \max(d_R, d_S)$. Any pole of f of multiplicity μ has to be a zero of multiplicity $\geqq \mu$ of P_n by (9.1). Therefore, $P_n f$ is a polynomial, say Q, and we must have $d_S \leqq d_n$. Although Q depends on the solution f, we can give an upper bound for the degree of Q that is independent of f.

First suppose that $d_R > d_S$. Then $f(z) \to \infty$ as $|z| \to \infty$. Write now (9.1) in the form

$$f' - \sum_{k=0}^{n-1} P_k f^k = P_n f^n. \tag{9.3}$$

Clearly, $P_n f^n$ behaves like $z^{d_n + n(d_R - d_S)}$ as $|z| \to \infty$. To get equality near $z = \infty$ in (9.3), at least one of the numbers

$$d_R - d_S - 1, d_0, d_1 + d_R - d_S, \ldots, d_{n-1} + (n-1)(d_R - d_S)$$

must be $\geqq d_n + n(d_R - d_S)$. Hence for at least one k, $0 \leqq k \leqq n-1$, we have

$$0 < d_R - d_S \leqq \frac{d_k - d_n}{n-k},$$

and we get from (9.2),

$$d_R \leqq d_S + \max_{0 \leqq k \leqq n-1} \frac{(d_k - d_n)^+}{n-k} = d_S + q - d_n.$$

For the degree of Q we now obtain

$$\deg Q = d_R + d_n - d_S \leqq q.$$

In the case when $d_R \leqq d_S$, we obviously have $\deg Q \leqq d_n \leqq q$. Thus in all cases we have

$$\deg Q \leqq q. \tag{9.4}$$

By (9.4), $Q^{(q+1)}$ vanishes identically, independent of f. Differentiating repeatedly we get

$$Q' = f'P_n + fP_n' = P_n \sum_{k=0}^{n} P_k f^k + P_n' f = \sum_{k=0}^{n} P_{k,1} f^k,$$

$$Q'' = \sum_{k=0}^{n} P_{k,1}' f^k + \sum_{k=0}^{n} k P_{k,1} f^{k-1} \sum_{k=0}^{n} P_k f^k = \sum_{k=0}^{2n-1} P_{k,2} f^k,$$

$$\ldots,$$

$$Q^{(i+1)} = \sum_{k=0}^{in-(i-1)} P_{k,i}' f^k + \sum_{k=0}^{in-(i-1)} k P_{k,i} f^{k-1} \sum_{k=0}^{n} P_k f^k = \sum_{k=0}^{(i+1)n-i} P_{k,i+1} f^k,$$

$$\ldots,$$

where each $P_{k,i+1}$ is a polynomial, and where $P_{(i+1)n-i,i+1} \not\equiv 0$ for any i. Since $Q^{(q+1)} \equiv 0$, it follows that the algebraic equation

$$\sum_{k=0}^{(q+1)n-q} P_{k,q+1}(z) f^k = 0 \tag{9.5}$$

with polynomial coefficients holds for all rational solutions f of (9.1). By basic field theory, (9.5) may have at most $(q+1)n - q$ distinct solutions in the field of rational functions. This proves part (i).

From $Q = P_n f$ and $Q^{(q+1)} \equiv 0$, it also follows that any rational solution f of (9.1) must satisfy the linear differential equation

$$P_n(z) f^{(q+1)} + (q+1) P_n'(z) f^{(q)} + \ldots + P_n^{(q+1)}(z) f = 0.$$

Part (ii) immediately follows.

REMARK. The above reasoning can be applied, with minor modifications, in the more general case where (9.1) has rational coefficients. The upper bounds in (i) and (ii) of Theorem 9.1 will then be modified. See also [5] and [6].

EXAMPLE 9.2 [8]. If $Q_1, Q_2, \ldots, Q_n$ are distinct polynomials such that $Q_i - Q_j$ is a constant for all i and j, then the differential equation

$$w' = Q_1' + (w - Q_1)(w - Q_2)\ldots(w - Q_n) \tag{9.6}$$

is satisfied by $w = Q_1, Q_2, \ldots, Q_n$.

As an example, the differential equation

$$w' = -z^3 + 3z^2 w - 3zw^2 + w^3 \tag{9.7}$$

admits the three distinct meromorphic solutions

$$w_i = z + \beta_i, \qquad i = 1, 2, 3, \qquad \text{where } \beta_i^3 = 1.$$

Thus we have two linearly independent meromorphic solutions of (9.7), which is the maximum number that is allowed by Theorem 9.1 (ii). Observe that the upper bound given by Theorem 9.1 (i) for the number of meromorphic solutions of (9.7) equals 5.

EXAMPLE 9.3. Gao Shi'an [5] gave the differential equation

$$u' = -1 + 3zu - 3z^2u^2 + z(z^2 - 1)u^3 \tag{9.8}$$

which has the four rational solutions

$$u_1(z) = \frac{1}{z}, \quad u_2(z) = \frac{1}{z-1}, \quad u_3(z) = \frac{1}{z+1}, \quad u_4(z) = \frac{z}{z^2-1}.$$

Note that $u_4 = \frac{1}{2}(u_2 + u_3)$, hence we have three linearly independent meromorphic solutions, which is one less than the upper bound that is given by Theorem 9.1 (ii). Observe that the upper bound given by Theorem 9.1 (i) for the number of meromorphic solutions of (9.8) equals 9!

REMARKS. Neither of the upper bounds given in (i) and (ii) of Theorem 9.1 are the best possible. We can obtain better bounds in special cases (see Theorems 9.4 and 9.6 below). We mention that a similar example to Example 9.3, where the number of meromorphic solutions of (9.1) exceeds n, can be found on page 84 of [10].

THEOREM 9.4. *Let $K \neq 0, c \in \mathbf{C}, q \geqq 0$ and $n \geqq 3$ be given, and let $P_0, \ldots, P_{n-1}$ be polynomials. Then the differential equation*

$$f' = P_0 + P_1 f + \ldots + P_{n-1} f^{n-1} + K(z-c)^q f^n \tag{9.9}$$

possesses at most n distinct meromorphic solutions.

EXAMPLE 9.5. Let $n \geqq 3$ be given, and let C be any constant such that $C^{n-1} = 1$. Then $f(z) = C/z$ satisfies

$$f' = -z^{n-2} f^n. \tag{9.10}$$

Since $f \equiv 0$ also satisfies (9.10), there exists n distinct meromorphic solutions of (9.10), which is the maximum number that is allowed by Theorem 9.4.

REMARK. Similar examples are given in Example 9.2.

PROOF OF THEOREM 9.4. For any meromorphic f solving (9.9), $K(z-c)^q f$ has to be a polynomial by the same argument as in the proof of Theorem 9.1. Now if $f_i \neq f_j$ are two meromorphic solutions of (9.9), then it can be deduced from (9.9) that $w = K(z-c)^q(f_i - f_j)$ satisfies

$$(K(z-c)^q)^{n-2} w' = Q_1(z)w + \ldots + Q_{n-1}(z)w^{n-1} + w^n, \tag{9.11}$$

where $Q_1, \ldots, Q_{n-1}$ are polynomials (see [8], p. 291). From (9.11), it follows that

$z_0 = c$ can be the only possible zero of $w(z)$. Hence, for some constant $B_{ij} \neq 0$,

$$K(z-c)^q(f_i - f_j) = B_{ij}(z-c)^{m_{i,j}}.$$

Let now $f_1, \ldots, f_{n+1}$ be distinct meromorphic solutions of (9.9) and let f_i, f_j, f_k be any three of them. Then obviously

$$B_{ij}(z-c)^{m_{i,j}} - B_{kj}(z-c)^{m_{k,j}} = B_{ik}(z-c)^{m_{i,k}}.$$

It follows that there exists a single m such that

$$w_j = K(z-c)^q(f_{j+1} - f_1) = B_j(z-c)^m \tag{9.12}$$

for all $j = 1, \ldots, n$, where $B_1, \ldots, B_n$ are distinct nonzero constants. Since all of the $w_1, \ldots, w_n$ satisfy (9.11), we may substitute (9.12) into (9.11), and this yields

$$Q_1 - mK^{n-2}(z-c)^{q(n-2)-1} + B_jQ_2(z-c)^m + B_j^2Q_3(z-c)^{2m}$$
$$+ \ldots + B_j^{n-2}Q_{n-1}(z-c)^{(n-2)m} + B_j^{n-1}(z-c)^{(n-1)m} = 0, \quad j = 1, \ldots, n.$$

By selecting any point $z \neq c$, we obtain an $n \times n$ linear system of equations

$$\phi_0(z) + B_j\phi_1(z) + \ldots + B_j^{n-1}\phi_{n-1}(z) = 0$$

with a nontrivial solution $(\phi_0(z), \ldots, \phi_{n-1}(z))$. Thus by Cramer's rule, the coefficient determinant must vanish. Since this determinant is a Vandermonde determinant, we have a contradiction.

THEOREM 9.6. *Let $k \geqq 0$, $q \geqq 0$, $C \neq 0$, $n \geqq 3$ and $a \neq b$ be given, and let $P_0, \ldots, P_{n-1}$ be polynomials. Then the differential equation*

$$f' = P_0 + P_1 f + \ldots + P_{n-1}f^{n-1} + C(z-a)^k(z-b)^q f^n \tag{9.13}$$

possesses at most three linearly independent meromorphic solutions.

PROOF. Denote $P_n(z) := C(z-a)^k(z-b)^q$, and suppose that f_1, f_2, f_3, f_4 are four linearly independent meromorphic solutions of (9.13). Considering $w_{i,j} = P_n(f_i - f_j)$ for $i \neq j$, we get, by the same argument as in the preceding proof, that for some constants $K_{ij} \neq 0$, $\alpha_{ij} \geqq 0$, $\beta_{ij} \geqq 0$, we have

$$w_{i,j}(z) = K_{ij}(z-a)^{\alpha_{ij}}(z-b)^{\beta_{ij}}.$$

Changing slightly the notation, we have therefore

$$\begin{cases} P_n(f_1 - f_2) = K_1(z-a)^{\alpha_1}(z-b)^{\beta_1} \\ P_n(f_1 - f_3) = K_2(z-a)^{\alpha_2}(z-b)^{\beta_2} \\ P_n(f_3 - f_2) = K_3(z-a)^{\alpha_3}(z-b)^{\beta_3} \end{cases} \tag{9.14}$$

where K_1, K_2, K_3 are all $\neq 0$. Denote $\alpha = \min(\alpha_1, \alpha_2, \alpha_3)$ and $\beta = \min(\beta_1, \beta_2, \beta_3)$. By (9.14), we have

$$K_1(z-a)^{\alpha_1}(z-b)^{\beta_1} - K_2(z-a)^{\alpha_2}(z-b)^{\beta_2} = K_3(z-a)^{\alpha_3}(z-b)^{\beta_3}. \tag{9.15}$$

Checking the multiplicities of the zeros (at $z = a$ and $z = b$) on both sides of (9.15), we deduce that at least two of the integers $\alpha_1, \alpha_2, \alpha_3$ must equal α, and at least two of $\beta_1, \beta_2, \beta_3$ must equal β. Assume, without restricting generality, that $\alpha_1 = \alpha$, $\beta_1 = \beta$.

If $\alpha_3 = \alpha$, $\beta_3 = \beta$, then by (9.14),

$$\frac{f_1 - f_2}{f_3 - f_2} = \frac{K_1}{K_3},$$

which contradicts the linear independence of f_1, f_2, f_3. Thus we have either $\alpha_3 > \alpha$ or $\beta_3 > \beta$. Suppose $\alpha_3 > \alpha$. Then $\alpha_2 = \alpha$. If now $\beta_2 = \beta$, a contradiction of the linear independence of f_1, f_2, f_3 follows. Hence $\beta_2 > \beta$ and therefore $\beta_3 = \beta$, and from (9.15),

$$K_1 - K_2(z - b)^{\beta_2 - \beta} = K_3(z - a)^{\alpha_3 - \alpha}.$$

This implies immediately that $\beta_2 - \beta = \alpha_3 - \alpha = 1$. Then (9.14) reduces into

$$(9.16)\qquad \begin{cases} P_n(f_1 - f_2) = K_1(z - a)^{\alpha}(z - b)^{\beta} \\ P_n(f_1 - f_3) = K_2(z - a)^{\alpha}(z - b)^{\beta + 1} \\ P_n(f_3 - f_2) = K_3(z - a)^{\alpha + 1}(z - b)^{\beta}. \end{cases}$$

Similarly, if $\beta_3 > \beta$, then

$$(9.17)\qquad \begin{cases} P_n(f_1 - f_2) = K_1(z - a)^{\alpha}(z - b)^{\beta} \\ P_n(f_1 - f_3) = K_2(z - a)^{\alpha + 1}(z - b)^{\beta} \\ P_n(f_3 - f_2) = K_3(z - a)^{\alpha}(z - b)^{\beta + 1}. \end{cases}$$

Let us consider now f_1, f_2, f_4 instead of f_1, f_2, f_3. For some $K_4 \neq 0$, $K_5 \neq 0$, and some $\alpha_4, \alpha_5, \beta_4, \beta_5$, we have

$$(9.18)\qquad \begin{cases} P_n(f_1 - f_2) = K_1(z - a)^{\alpha}(z - b)^{\beta} \\ P_n(f_1 - f_4) = K_4(z - a)^{\alpha_4}(z - b)^{\beta_4} \\ P_n(f_4 - f_2) = K_5(z - a)^{\alpha_5}(z - b)^{\beta_5}. \end{cases}$$

If now $\alpha = \min(\alpha, \alpha_4, \alpha_5)$ and $\beta = \min(\beta, \beta_4, \beta_5)$, then the reasoning used above to obtain (9.16) and (9.17) implies that either $\alpha_4 = \alpha + 1$ and $\beta_4 = \beta$, or $\alpha_4 = \alpha$ and $\beta_4 = \beta + 1$. Therefore, by (9.16) and (9.17), we see that either $(f_1 - f_4)(f_1 - f_3)^{-1}$ or $(f_1 - f_4)(f_3 - f_2)^{-1}$ is a nonzero constant. This contradicts the linear independence of f_1, f_2, f_3, f_4.

Therefore, again by the reasoning used to obtain (9.16) and (9.17), we see that exactly one of the following four situations must hold:

(1) $\qquad \alpha_4 = \alpha_5 = \alpha - 1, \qquad \beta_4 = \beta, \qquad \beta_5 = \beta + 1;$

(2) $\qquad \alpha_4 = \alpha_5 = \alpha - 1, \qquad \beta_5 = \beta, \qquad \beta_4 = \beta + 1;$

(3) $\beta_4 = \beta_5 = \beta - 1, \qquad \alpha_4 = \alpha, \qquad \alpha_5 = \alpha + 1;$

(4) $\beta_4 = \beta_5 = \beta - 1, \qquad \alpha_5 = \alpha, \qquad \alpha_4 = \alpha + 1.$

Without restricting generality, we may assume that we have the situation (1). Then (9.18) gives

$$\begin{cases} P_n(f_1 - f_4) = K_4(z-a)^{\alpha-1}(z-b)^{\beta} \\ P_n(f_1 - f_2) = K_1(z-a)^{\alpha}(z-b)^{\beta} \\ P_n(f_4 - f_2) = K_5(z-a)^{\alpha-1}(z-b)^{\beta+1}. \end{cases} \tag{9.19}$$

We showed earlier that either (9.16) or (9.17) holds. If (9.16) holds, then from (9.16) and (9.19), we obtain

$$\begin{cases} P_n(f_2 - f_4) = -K_5(z-a)^{\alpha-1}(z-b)^{\beta+1} \\ P_n(f_2 - f_3) = -K_3(z-a)^{\alpha+1}(z-b)^{\beta} \\ P_n(f_3 - f_4) = K_6(z-a)^{\alpha_6}(z-b)^{\beta_6}, \end{cases}$$

which contradicts our usual reasoning since the exponents $\alpha - 1, \alpha + 1$ differ by 2. Hence, we must have (9.17). Combining (9.17) with (9.19), we obtain

$$\begin{cases} P_n(f_2 - f_3) = -K_3(z-a)^{\alpha}(z-b)^{\beta+1} \\ P_n(f_2 - f_4) = -K_5(z-a)^{\alpha-1}(z-b)^{\beta+1} \\ P_n(f_4 - f_3) = K_6(z-a)^{\alpha-1}(z-b)^{\beta+2}. \end{cases} \tag{9.20}$$

From (9.17), (9.19) and (9.20) we now get

$$\begin{cases} P_n(f_4 - f_3) = K_6(z-a)^{\alpha-1}(z-b)^{\beta+2} \\ P_n(f_4 - f_1) = -K_4(z-a)^{\alpha-1}(z-b)^{\beta} \\ P_n(f_1 - f_3) = K_2(z-a)^{\alpha+1}(z-b)^{\beta}, \end{cases}$$

a contradiction, proving the assertion.

REMARK. Example 9.2, Example 9.3, [8, Example 9.1], and the proof of Theorem 9.6, suggest the following question: Does the maximum number of linearly independent meromorphic solutions of (9.1) depend only on the number of *distinct* zeros of $P_n(z)$?

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