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CONFORMAL WELDING OF RECTIFIABLE CURVES

CHRISTOPHER J. BISHOP

1. Introduction.

Suppose D_1 and D_2 are two Jordan domains on the Riemann sphere, $\bar{\mathbf{C}}$, and that $\psi\colon \Gamma_1 \to \Gamma_2$ is a homeomorphism of their boundaries. We say that a conformal welding (or conformal sewing) exists if there is a Jordan curve Γ in $\bar{\mathbf{C}}$ with complementary domains Ω_1 and Ω_2 and conformal mappings $\Phi_i\colon D_i \to \Omega_i$ for $i = 1, 2$ such that $\psi = \Phi_2^{-1} \circ \Phi_1$ (this makes sense because both mappings extend to be homeomorphisms from Γ_i to Γ). Another way of expressing this is to say that the topological sphere created by identifying Γ_1 and Γ_2 by ψ has a conformal structure which is consistent with the conformal structures of D_1 and D_2. If the welding exists we say it is unique if there is only one such conformal structure on the identified sphere (this the same as saying Γ is unique to Möbius transformations). In general we are interested in knowing when a welding exists, if it is unique and how the geometry of Γ depends on the D_1, D_2 and ψ.

In this note we will be concerned with the special case when Γ_1 and Γ_2 are rectifiable curves of equal length and $\psi\colon \Gamma_1 \to \Gamma_2$ is isometric (i.e., preserves arclength). It is known that the conformal welding need not exist in this case [10], but one might hope that if it did exist that Γ would be well behaved, say rectifiable (e.g., see [2, Question 6.61]). However, this is not the case.

THEOREM. *There exist rectifiable domains D_1 and D_2 and an isometry ψ of their boundaries so that the conformal welding exists, but the corresponding curve Γ has positive area.*

In fact, we can take $D_1 = D_2$ and ψ to be an orientation reversing isometry. If E is a closed plane set with positive area then we can always find a homeomorphism Φ of the sphere which is conformal off E but which is not a Möbius transformation. Composing this map with the conformal mappings Φ_1 and Φ_2 therefore shows:

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COROLLARY 1. *The conformal welding corresponding to an isometric identification of rectifiable domains need not be unique.*

A locally rectifiable curve Γ is called chord-arc if there exists a $C \geqq 1$ such that for any two points z and w on Γ the shortest arc on Γ between z and w has length at most $C|z - w|$. Γ is a quasicircle if the diameter of the shortest arc between z and w is at most $C|z - w|$. It follows from known results (e.g. [6], [11] and [18]) that if ψ is an isometry between chord-arc domains then the conformal welding exists and is unique and that the corresponding curve Γ must be a quasicircle. It is also easy to show that the Hausdorff dimension ([5]) of a quasicircle is strictly less than 2, so that the example in the theorem cannot be a chord-arc domain. However, the proof of the theorem will also give.

COROLLARY 2. *For any $1 \leqq d < 2$ there exist chord-arc domains and an isometry ψ so that the corresponding Γ has Hausdorff dimension greater than d.*

If the domains are chord-arc with constants close to 1, the corresponding Γ must also be chord-arc (hence locally rectifiable) [6]. Thus Corollary 2 is only possible for curves with large chord-arc constant. In the next section we will review a few facts about quasiconformal mappings which will allow us to reduce to constructing quasiconformal mappings instead of conformal ones. In Section 3 we will give the construction and deduce the corollaries. For further background on conformal welding see [14] and [17]. Welding of rectifiable domains via isometries is discussed in the papers [9] and [10]. Some other constructions involving conformal welding and rectifiability are given in [3] and [16]. Finally, I would like to thank the referee for many helpful comments and corrections to the original manuscript.

2. Quasiconformal Mappings.

By Lemma 1 of [14], to show that a conformal welding exists it is enough to build quasiconformal mappings which induce the homeomorphism ψ. To explain this and to see why we can reduce the theorem to this case, we will briefly review some of the basic definitions and results about quasiconformal (q.c.) mappings. For more details the reader is referred to Ahlfors' book [1].

Suppose $\Omega \subset \mathbf{C}$ is a domain and φ is a homeomorphism of Ω into $\mathbf{C}$. φ is called quasiconformal with constant $K \geqq 1$ (or K-q.c.) if φ is absolutely continuous on almost every horizontal and vertical line in Ω and

$$|\bar{\partial}\varphi| \leqq \frac{K-1}{K+1}|\partial\varphi|$$

where

$$\bar{\partial}\varphi = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x} + i\frac{\partial\varphi}{\partial y}\right), \quad \partial\varphi = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y}\right).$$

The ratio

$$\mu_\varphi = \frac{\bar{\partial}\varphi}{\partial\varphi}$$

satisfies $\|\mu_\varphi\|_\infty < 1$ and is called the complex dilatation of φ. One important property of q.c. mappings is that they are absolutely continuous with respect to area, i.e., $|E| = 0$ iff $|\varphi(E)| = 0$. Another is the measurable Riemann mapping theorem: given any measurable μ on $\mathbf{C}$ with $\|\mu\|_\infty = k < 1$ there is a K-q.c. mapping φ with dilatation μ and $K = (1 + k)/(1 - k)$. Moreover, φ is unique up to Möbius transformations.

Now suppose we have curves Γ_1, Γ_2 and Γ and a homeomorphism ψ as described in the introduction, but instead of conformal mappings we are given q.c. mappings Φ_1 and Φ_2 with dilatations μ_1 and μ_2, such that $\psi = \Phi_2^{-1} \circ \Phi_1$. Define a dilatation μ by

$$\mu(z) = \begin{cases} -\mu_{\Phi_1}(w), & z = \Phi_1(w), \quad w \in D_1 \\ -\mu_{\Phi_2}(w), & z = \Phi_2(w), \quad w \in D_2 \\ 0, & z \in \Gamma. \end{cases}$$

Clearly $\|\mu\|_\infty \leqq \max(\|\mu_1\|_\infty, \|\mu_2\|_\infty) < 1$ so there exists a q.c. homeomorphism φ with dilatation μ. One can also show (see [1, page 10]) that

$$\mu_{(\varphi \circ \Phi_i)} = 0$$

on D_i for $i = 1, 2$ so that the homeomorphisms $\tilde{\Phi}_1 = \varphi \circ \Phi_1$ and $\tilde{\Phi}_2 = \varphi \circ \Phi_2$ are conformal from D_1 and D_2 to $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$, the complementary domains of the curve $\tilde{\Gamma} = \varphi(\Gamma)$. Also,

$$\psi = (\Phi_2)^{-1} \circ \Phi_1 = (\tilde{\Phi}_2)^{-1} \circ \tilde{\Phi}_1$$

so that $\tilde{\Gamma}$ corresponds to a conformal welding of the curves Γ_1 and Γ_2 via ψ. Since φ is absolutely continuous with respect to area, $\tilde{\Gamma}$ has positive area iff Γ does. Thus to prove the theorem, it suffices to construct quasiconformal mappings Φ_i such that $\psi = \Phi_2^{-1} \circ \Phi_1$. This is much easier than constructing the conformal mappings directly, and will be done in the next section.

To prove Corollary 1 we define a dilatation μ by $\mu(z) = 1/2$ if $z \in \Gamma$ and 0 otherwise. Let φ be a q.c. homeomorphism with this dilatation and set $\tilde{\Gamma} = \varphi(\Gamma)$. Since Γ has positive area, φ is not a Möbius transformation. Thus Γ and $\tilde{\Gamma}$ both correspond to conformal weldings of D_1, D_2 via ψ, but they are not related by a Möbius transformation. Thus the theorem implies Corollary 1. (This argument shows a welding is not unique whenever Γ has positive area. The converse is false; one can have nonuniqueness even when $\dim(\Gamma) = 1$ [12], [4].)

Corollary 2 is slightly more complicated since quasiconformal mappings do not, in general, preserve Hausdorff dimension. However, a result from Gehring and Väisälä's paper [8] gives the inequality

$$\dim(\varphi(E)) \geqq \frac{\dim(E)}{K}$$

where K is the q.c. constant of φ (this is a consequence of Mori's theorem, see [1]). Therefore to prove Corollary 2 it will be enough to build the curve Γ and the associated homeomorphisms Φ_i, $i = 1, 2$ so that the Hausdorff dimension of Γ is large (close to 2) and the q.c. constant of Φ_i is small (close to 1).

3. The Construction.

We will construct Γ_1, Γ_2, D_1, D_2, Φ_1, Φ_2 and Γ as limits of uniformly convergent sequences $\{\Gamma_1^n\}, \{\Gamma_2^n\}, \ldots, \{\Gamma^n\}$.

We begin by setting $D_1^0 = (-4,4) \times (0,4)$ and $D_2^0 = (-4,4) \times (-4,0)$ and letting Γ_1^0 and Γ_2^0 be their boundaries. Let $\Gamma = \mathbb{R}$ and let $\Phi_1^0 \colon D_1^0 \to \{\mathrm{Im}(z) > 0\}$ be quasiconformal and equal to the identity on $(-2,2) \times (0,2)$. We define Φ_2^0 by symmetry. We fix intervals $J_0 = [-1,1] \subset \Gamma_1^0$, $K_0 = [-1,1] \subset \Gamma_2^0$ and $I_0 = [-1,1] \subset \Gamma^0$.

Now suppose we have constructed Γ_i^n, D_i^n, Φ_i^n for $i = 1, 2$ and Γ^n such that $\Phi_i^n\colon \Gamma_i^n \to \Gamma^n$ are homeomorphisms with q.c. extensions to D_i^n and such that $\psi^n = (\Phi_2^n)^{-1} \circ \Phi_1^n$ is a length preserving homeomorphism from Γ_1^n to Γ_2^n. Also suppose that we are given collections of line segments $\mathscr{I}_n$, $\mathscr{J}_n$ and $\mathscr{K}_n$ in Γ^n, Γ_1^n and Γ_2^n respectively such that for each $I \in \mathscr{I}_n$ there are corresponding line segments $J \in \mathscr{J}_n$ and $K \in \mathscr{K}_n$ such that $\Phi_1^n(J) = I$ and $\Phi_2^n(K) = I$ and that Φ_1^n and Φ_2^n are affine on J and K respectively. Note that since ψ^n is an isometry we must have $\ell(J) = \ell(K)$ (ℓ denotes length). Our induction hypothesis also includes the assumption that for each $J \in \mathscr{J}_n$, Φ_1^n is affine and conformal on a rectangle R in D_1^n which has J as one side and has width $\beta_n \ell(J)/2$ (β_n will be chosen below). Moreover the reflection of this rectangle across J is contained in the complement of D_1^n. Similarly for Φ_2^n and $K \in \mathscr{K}_n$.

Now fix a triple of line segments I, J and K as above. We will describe how to modify the curves Γ_1^n, Γ_2^n and Γ^n by replacing the line segments I, J and K by more complicated polygonal paths. We will also define new mappings Φ_i^n so that the desired properties still hold. Making this modification on every line segment in the collections $\mathscr{I}$, $\mathscr{J}$ and $\mathscr{K}$ will give us Γ^{n+1}, Γ_1^{n+1} and Γ_2^{n+1}.

To ease notation we will drop the n's and merely write $\Gamma = \Gamma^n$, $\Gamma_1 = \Gamma_1^n$ and $\Gamma_2 = \Gamma_2^n$. Let β, ε and η be positive numbers which will be chosen later. It will be

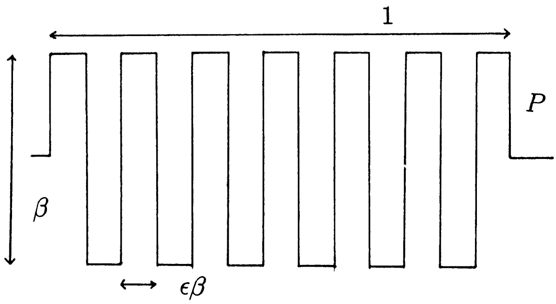

Figure 1

convenient to assume however that $\varepsilon^{-1} = 2N$ is an even integer and that $(\varepsilon\beta)^{-1} = 2M + 1$ is an odd integer. We also wish to fix an integer $L < N$.

First we describe the modification of Γ along I. Rescale so that $\ell(I) = 1$. We modify Γ by removing I and replacing it by the polygonal curve illustrated in Figure 1. P consists of M "towers" each of height β and of width $\varepsilon\beta$. The new curve is still Jordan by our induction hypothesis.

The replacements for the line segments J and K are much more complicated. Roughly speaking we will replace J and K by polygonal arcs Q_1 and Q_2 which consist of M "hourglasses" (see Figures 2, 4 and 6) instead of rectangular towers as in P. Each of these "hourglasses" can be mapped quasiconformally to the towers in P by a mapping with a uniformly small q.c. constant. However, the size of the hourglasses will be much smaller than that of the corresponding towers, and hence we are adding much less length to Γ_1 and Γ_2 than we are to Γ. This explains why Γ can have infinite length, but Γ_1 and Γ_2 will remain rectifiable. The tricky part of the construction is to arrange for the homeomorphism $\Phi_2^{-1} \circ \Phi_1$ to preserve length.

We start by defining what we mean by an hourglass. For non-negative integers k, l and m and a positive η we define a polygonal region $H(k, l, m, \eta)$ as follows (see Figure 2). It consists of $2k$ squares of side length 1, $2m$ squares of side length $(1 - \eta)^l$ and $2l$ trapezoids each similar to a symmetric trapezoid with base lengths 1 and $1 - \eta$ and height $1 - \eta$. Of course, each of the remaining sides has length $\lambda = \sqrt{1 - 2\eta + 5\eta^2/4}$ (which is < 1 if η is small enough). For each $0 \leqq j < l$ there are exactly two trapezoids so that the longer base has length $(1 - \eta)^j$.

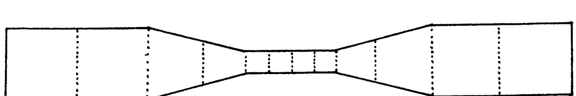
$H(2, 2, 2, \eta)$

Figure 2

These $2(k+l+m)$ quadrilaterals are placed end-to-end to obtain the 12-gon "hourglass" pictured in Figure 2. Note that $H(k, 0, 0, \eta)$ is merely a $2k$ by 1 rectangle and that $H(k, l, m, \eta)$ can easily be mapped to the rectangle $H(k+l+m, 0, 0, \eta)$ by a homeomorphism which is just an affine mapping on each of quadrilaterals in $H(k, l, m, \eta)$ and maps that quadrilateral onto the corresponding square in $H(k+l+m, 0, 0, \eta)$. In fact, this map is conformal except on the trapezoids an explicit calculation shows it is quasiconformal with dilatation μ bounded by 3η.

We would like to build Q_1 and Q_2 by attaching scaled copies of these hourglasses in a row and defining the maps Φ_i in terms of the q.c. maps described above. Unfortunately it seems impossible to do this and still get the isometric properties without introducing some further complications.

The first of these complications involves defining a "modified hourglass" $\tilde{H}(k, l, m, \eta)$. To obtain $\tilde{H}$ we will take a copy of $H(k, l, m, \eta)$ and slightly perturb some of its sides. If $l = 0$ we make no changes. If $l \neq 0$ we replace all but the largest trapezoids in H by the region in the center of Figure 3. More precisely, if the trapezoid T has base length $(1-\eta)^j$ then the top edge of T is simply replaced by a polygonal curve of length $(1-\eta)^{j-1}\lambda$. If this curve is chosen correctly the modified trapezoid $\tilde{T}$ can be still be mapped to a square by a q.c. mapping which is affine on the 3 original sides and which preserves length on the new side. The dilatation of this mapping can be taken less than $C\sqrt{\eta}$. To see this, consider the middle picture in Figure 3. We are replacing a line segment of length $\lambda(1-\eta)^j$ by a polygonal arc of length $\lambda(1-\eta)^{j-1}$. Therefore the new polygonal arc must be $(1-\eta)^{-1} \sim 1+\eta$ times as long as the original line segment. Thus the "bump" shown in the figure can be taken to have length about $(1-\eta)^j/2$ and height proportional to $\sqrt{\eta}(1-\eta)^j$ by the Pythagorean theorem. From this it is easy to construct the desired mapping. For example, triangulate the perturbed trapezoid by connecting each vertex of the polygon to a point in the center and then build a map which is affine of each piece. The dilatation on triangles corresponding to the original sides will be about C_η, and on the remaining pieces it is no more than

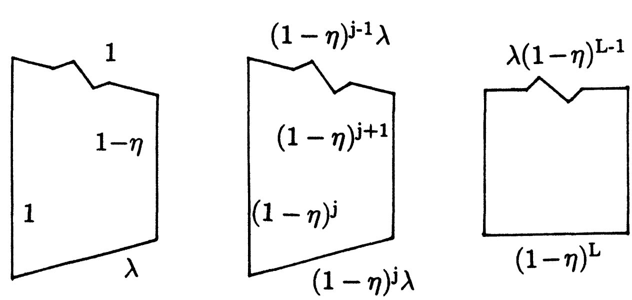

Figure 3

$C\sqrt{\eta}$. On the two largest trapezoids in H we also replace the top edge by something longer, but now the new edge has length 1 (see left side of Figure 3). As above, the dilatation can be taken to be bounded by $C\sqrt{\eta}$.

If $m = 0$ these are the only changes we need to make. However if $m \neq 0$, we make one additional change. Moreover, we will only need $l = L$. In this case H has two "small" squares of size $(1-\eta)^L$ which are adjacent to trapezoids. We replace the top edge of each of these squares by a polygonal arc whose length is $(1-\eta)^{L-1}\lambda$, i.e., the length of one of the sides of the adjacent trapezoid. This can be done in such a way that the modified square can be mapped by a q.c. homeomorphism to a square with dilatation less that $C\sqrt{\eta}$ and which is affine on the 3 original sides and which preserves arclength on the modified edge. Thus $\tilde{H}$ has been built so that each quadrilateral in H has had one side replaced by an arc with the same length as the corresponding edge in the hourglass $H(k+1, L, m-1, \eta)$ if $m > 0$ or $H(k+1, l-1, 0, \eta)$ if $m = 0$. Moreover, the resulting "hourglass" can be mapped in a q.c. manner (with small dilatation) to the rectangle $H(k+l+m, 0, 0, \eta)$ in a way which preserves arclength (up to a constant multiple) on each side of a quadrilateral (different constants for different sides and quadrilaterals of course). See Figure 4.

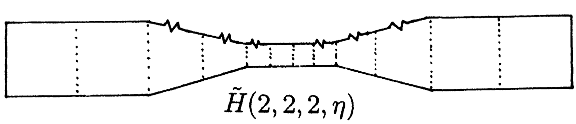

Figure 4

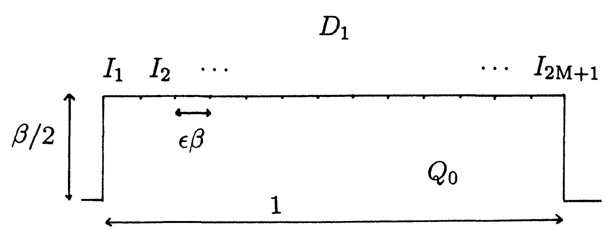

Figure 5

We are now ready to build the path Q_1. Scale J so that $\ell(J) = 1$. Let Q_0 be the curve in Figure 5.

Divide the edge of length 1 parallel to the original segment J into $2M+1$ intervals $\{I_j\}$ each of length $\varepsilon\beta$. We define Q_1 by attaching a hourglass to every other interval. More precisely, for $j = 1, \ldots, M$ we attach a copy of $H_j = H(k(j),$

$l(j)$, $m(j)$, η) (scaled by a factor of $\varepsilon\beta$) to I_{2_j}, where $k(j)$, $l(j)$ and $m(j)$ are integer valued functions defined as follows.

$$k(j) = \begin{cases} N - j + 1, & 1 \leqq j \leqq N \\ 1, & N \leqq j \leqq M - N \\ j - M + N, & M - N + 1 \leqq j \leqq M \end{cases}$$

$$l(j) = \begin{cases} j - 1, & 1 \leqq j \leqq L + 1 \\ L, & N \leqq j \leqq M - L - 1 \\ M - j, & M - L \leqq j \leqq M \end{cases}$$

$$m(j) = N - k(j) - l(j)$$

This determines Q_1 and hence the modification we make to the curve Γ_1 on the line segment J. The curve Q_1 is sketched in Figure 6 for the case $N = 3$, $M = 3$ and $L = 1$.

Because of the construction of the hourglasses there is a piecewise affine homeomorphism of Q_1 to P which extends to a piecewise affine q.c. mapping of each hourglass to the corresponding rectangle in the complement of Γ. This mapping agrees with the map Φ_1 along the intervals $\{I_j\}$ in the definition of Q_1 (because Φ_1 is affine and conformal in this region by induction) and so we obtain a new q.c. mapping $\tilde{\Phi}_1$ from the modified domain $\tilde{D}_1$ to the complement of the

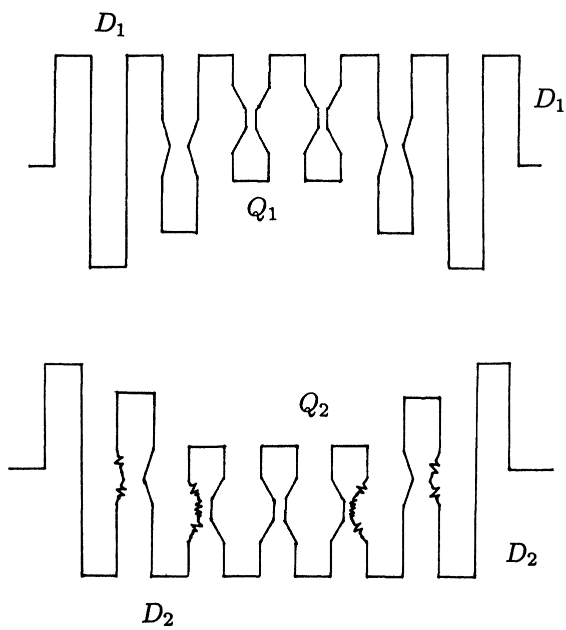

Figure 6

modified curve $\tilde{\Gamma}$. The dilatation of this mapping is no more than the maximum of $C\sqrt{\eta}$ and the norm of the dilatation of the original Φ_1.

To build Q_2 we do exactly the same thing except that certain of the hourglasses in the construction need to be replaced by the modified hourglasses $\tilde{H}$. More precisely, for $j = 1$ we attach a scaled copy of $H(M, 0, 0, \eta)$ to I_1. For $j = 2, \ldots N$ we attach $\tilde{H}_j = \tilde{H}(k(j), l(j), m(j), \eta)$ to the interval I_{2j-1}. The rest of the curve is defined by making Q_2 symmetric with respect to the perpendicular bisector of K. The curve Q_2 is pictured in Figure 6. (Note that Figure 6 is drawn so D_1 lies above Q_1 and D_2 lies below Q_2.)

As before we replace the line segment K in Γ_2 with an appropriately scaled copy of Q_2 to obtain a new curve $\tilde{\Gamma}_2$ bounding a domain $\tilde{D}_2$. Also as before there is a homeomorphism of $\tilde{\Gamma}_2$ to $\tilde{\Gamma}$ which extends to be a q.c. homeomorphism $\tilde{\Phi}_2$ of $\tilde{D}_1$ to one side of $\tilde{\Gamma}$. This homeomorphism agrees with Φ_2 along the ends of the hourglasses and has dilatation no larger than the maximum of $C\sqrt{\eta}$ and the dilatation of Φ_2.

Finally, and most importantly, it follows from the construction that $\tilde{\psi} = (\tilde{\Phi}_2)^{-1} \circ \tilde{\Phi}_1$ preserves arclength. This involves considering several cases, but we leave the verification to the reader since Figure 6 makes the argument quite clear. The main point is to observe that if the sides of two quadrilaterals are identified by the map $\tilde{\psi}$ then these two sides must have the same length (this is what the modifications in $\tilde{H}$ were all about). On each of the quadrilaterals the maps $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ were chosen to preserve arclength up to a multiplicative constant, so the constant for corresponding edges must be the same. Hence ψ preserves arclength.

Thus we have described how to modify Γ^n, Γ_i^n and Γ_2^n on the intervals I, J and K. We now carry out this procedure on all the intervals in the collections $\mathscr{I}_n$, $\mathscr{J}_n$ and $\mathscr{K}_n$. This gives us the $(n+1)$th generation curves $\Gamma_1^{n+1}, \ldots$. To complete the induction step we need only define the collections of line segments $\mathscr{I}_{n+1}$, $\mathscr{J}_{n+1}$ and $\mathscr{K}_{n+1}$. To each line segment $J \in \mathscr{J}_n$ consider the M hourglasses in the copy of Q_1 replacing J. Of these $M - 2N$ are of the form $H(1, L, N - L - 1, \eta)$ and in each of these are two line segments of length $2(N - L - 1)(1 - \eta)^L$. We take all of these segments to be in the collection $\mathscr{J}_{n+1}$. Thus for each $J \in \mathscr{J}_n$ we obtain

$$\frac{|\mathscr{J}_{n+1}|}{|\mathscr{J}_n|} = 2(M - 2N) \tag{3.1}$$

arcs $\tilde{J} \in \mathscr{J}_{n+1}$ and each has length

$$\ell(\tilde{J}) = (1 - \eta)^L \varepsilon\beta 2(N - L - 1)\ell(J). \tag{3.2}$$

We can now define the collections $\mathscr{K}_{n+1}$ and $\mathscr{I}_{n+1}$ simply as the collection $\{\Phi_2^{n+1}(J)\}$ and $\{\psi^{n+1}(J)\}$ for $J \in \mathscr{J}_{n+1}$. It is easy to check that these are indeed line segments and satisfy the desired conditions.

This completes the induction step. So given sequences of parameters $\{\beta_n\}$, $\{\varepsilon_n\}$, $\{\eta_n\}$ and $\{L_n\}$ we can construct a corresponding sequence of curves and maps. We easily pass to a limit to obtain curves Γ_1, Γ_2 and Γ and the mappings Φ_1, Φ_2 and ψ. All that remains to show is that Γ_1 (and hence Γ_2) is rectifiable and that Γ has positive area. For this construction we may take $\{\eta_n\}$ to be a constant sequence, say $1/10 = \eta = \eta_1 = \eta_2 = \dots$.

First we will show Γ_1 is rectifiable by uniformly bounding the length of the $\{\Gamma_i^n\}$. Note that

$$\begin{aligned}\ell(\Gamma_1^{n+1}\backslash\Gamma_1^n) &\leqq \ell(Q_1)\sum_{\mathscr{I}_n}\ell(J)\\ &\leqq (2N+1)(2M+2)(1-\eta)^L 2N\sum_{\mathscr{I}_{n-1}}\ell(J)\\ &\leqq (4/(\beta_n\varepsilon_n^3))(1-\eta)^{L_n}\sum_{\mathscr{I}_{n-1}}\ell J)\\ &\leqq \tfrac{1}{2}\ell(\Gamma_1^n\backslash\Gamma_1^{n-1}).\end{aligned}$$

The last inequality holds for given ε_n and β_n if we choose

$$L_n \geqq \frac{\log 8 + |\log \varepsilon_n^3\beta_n|}{|\log(1-\eta)|}. \tag{3.3}$$

We will choose $\beta_n \sim \varepsilon_n$, so if ε_n is small enough we can satisfy (3.3) and also choose L_n so that

$$L_n \leqq \frac{1}{2\sqrt{\varepsilon_n}}. \tag{3.4}$$

The inequality above implies

$$\ell(\Gamma_1^n) = \sum_n \ell(\Gamma_1^{n+1}\backslash\Gamma_1^n) \leqq \ell(\Gamma_1^0)\sum_n 2^{-n} < \infty$$

as desired. Thus Γ_1 (and hence Γ_2) is rectifiable as long as (3.3) holds.

Next we claim that Γ will have positive area if we choose the parameters $\{\varepsilon_n\}$ and $\{\beta_n\}$ correctly. To see this let F_n be the union of all the $\ell(I)$ by $\beta_n\ell(I)$ rectangles centered along the line segments $I \in \mathscr{I}_n$. Then $\cap_n F_n \subset \Gamma$ and $F_{n+1} \subset F_n$ so it suffices to show $|F_n| > 0$, independent of n. However, using (3.1), (3.2) and (3.4) gives

$$\begin{aligned}|F_{n+1}| &= \sum_{\mathscr{I}_{n+1}} \beta_{n+1}\, l(I)^2\\ &\geqq \beta_{n+1}(M_n - 2N_n)(2(N_n - L_n - 1)\varepsilon_n\beta_n)^2\sum_{\mathscr{I}_n}\ell(I)^2\\ &\geqq \beta_{n+1}((\varepsilon_n\beta_n)^{-1} - 2\varepsilon_n^{-1})((\varepsilon_n^{-1} - \varepsilon_n^{-1/2} - 2)\varepsilon_n\beta_n)^2\beta_n^{-1}\,|F_n|\end{aligned}$$

$$\geqq \beta_{n+1}(\varepsilon_n\beta_n)^{-1}(1-\beta_n)(1-\sqrt{\varepsilon_n}-2\varepsilon_n)^2\beta_n|F_n|$$
$$= \frac{\beta_{n+1}}{\varepsilon_n}(1-\beta_n)(1-\sqrt{\varepsilon_n}-2\varepsilon_n)^2|F_n|$$
$$\geqq (1-2^{-n})^4|F_n|$$

The last inequality holds if we choose

$$\varepsilon_n \leqq 4^{-n}$$
$$\varepsilon_n > \beta_{n+1} \leqq (1-2^{-n})\varepsilon_n.$$

and so that $(\varepsilon_n\beta_n)^{-1}$ is an odd integer as required. Since $\Pi(1-2^{-n}) > 0$, Γ has positive area.

To prove Corollary 2 one first notes that Γ_1 and Γ_2 will be chord-arc curves as long as (3.4) holds and ε_n and β_n are uniformly bounded away from 0. Fix a value of $d = 2 - v$ in Corollary 2 and let η be fixed small enough that the maps Φ_1 and Φ_2 constructed above are q.c. with constant $K \leqq 1 + v/2$. Fix an ε and let $\varepsilon_n = \varepsilon$ for all n and $\beta_n \sim (1-\varepsilon)\varepsilon$ for all n. Then the Hausdorff dimension of the resulting Γ is close to 2 if ε is close to 0. We will not prove this, but simply note that it is not much different than the calculation above which showed Γ had positive area. Thus by the result of Gehring and Väisälä mentioned in Section 2, we can get $\dim(\Gamma) > d$ by taking ε small enough.

4. Remarks.

First of all, the construction could have been organized so as to give a single rectifiable domain D and an orientation reversing isometry of its boundary instead of two distinct domains. The only changes would be to construct Q_1 using both modified and unmodified hourglasses in such a way that the orientation reversing isometry from Q_1 to itself had the desired q.c. extension to D_1. This is only slightly more difficult than what we did.

The isometry ψ we constructed is very special among all the isometries from Γ_1 to Γ_2. Infact, by adapting an argument of Oikawa [14] we could construct the curves so that the ψ we construct is the only isometry from Γ_1 to Γ_2 for which a conformal welding exists.

For chord-arc curves, however, a welding exists for any isometry with the correct orientation. How special is the isometry in Corolary 3 among all the isometries from Γ_1 to Γ_2? These isometries are naturally parameterized by points of Γ_2 (there is one sending some fixed point of Γ_1 to each point of Γ_2) and so it makes sense to ask, for example, if the set of isometries for which the corresponding curve $\Gamma = \Gamma_\psi$ has dimension greater than one is small, say length zero. A easier problem would be to prove that the welding of a chord-arc curve with a smooth curve, say a circle, always gives a chord-arc curve.

We should also mention that the construction in this note can be used to give a negative answer to a question asked in [15]. Other counterexamples are given in [3] and [13].

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DEPARTMENT OF MATHEMATICS
UCLA
405 HILGARD AV.
LOS ANGELES, CALIFORNIA 90024
U.S.A.