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DENSITY OF THE SELF-ADJOINT ELEMENTS WITH FINITE SPECTRUM IN AN IRRATIONAL ROTATION C*-ALGEBRA

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Abstract.

It is shown that, for a dense set of values of the irrational number θ, any self-adjoint element of the irrational rotation C*-algebra A_θ can be approximated in norm by one which has finite spectrum (and, what is more, by one which has Cantor spectrum).

1. Recently, certain C*-algebras have been shown to have the property that the subset of self-adjoint elements with finite spectrum is dense (in the set of all self-adjoint elements). This was shown in [5] for the Bunce-Deddens algebras, and in [7] for the multiplier algebras of matroid C*-algebras (i.e., of AF algebras stably isomorphic to UHF algebras).

The purpose of this paper is to show that this property holds also for at least some of the irrational rotation C*-algebras. The first step is to verify a related property in the rational case.

THEOREM. *For any rational number $\theta = p/q$ with $(p, q) = 1$, the set of self-adjoint elements of the rotation algebra A_θ with q distinct eigenvalues in every irreducible representation is dense.*

2. COROLLARY. *The subset of $\mathsf{R} \setminus \mathsf{Q}$ consisting of those numbers θ such that the set of self-adjoint elements of the rotation algebra A_θ with finite spectrum is dense, is dense in $\mathsf{R} \setminus \mathsf{Q}$.*

PROOF. The proof will be given, in a more general context, in Section 6, below. It will also be shown that the subset of $\mathsf{R} \setminus \mathsf{Q}$ consisting of those numbers θ such that the set of self-adjoint elements of A_θ with Cantor spectrum is dense, is dense.

The proof is based on the fact that the family $(A_\theta)_{\theta \in [0,1]}$ is in a natural way a continuous field of C*-algebras, pointed out in [10] (see also [11], [12] and [8]; see [18] for two generalizations of this result). The continuous fields introduced

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in [10] are quite easy to describe: by Proposition 10.2.3 of [9] it is enough to describe a subspace of fields which have continuous norm and are dense in each fibre A_θ; such a subspace is the space of polynomials in the two canonical generators u and v, with $vu = e^{2\pi i\theta}uv$; continuity of the norm of such a polynomial (as a function of θ) was proved in [10].

We shall need to use two properties of the continuous field of C*-algebras (A_θ). The first is as follows. Recall (see, e.g., [8]) that if θ is rational, $\theta = p/q$ with $(p, q) = 1$, then each irreducible representation of A_θ is of dimension q; in particular, each self-adjoint element of A_θ has exactly q eigenvalues, including multiplicity, in each irreducible representation of A_θ. The property we shall need is that for each continuous field of self-adjoint elements h, each θ, and each $\varepsilon > 0$, there exists a rational number θ', with $|\theta - \theta'| < \varepsilon$, such that the eigenvalues of $h(\theta')$ in an irreducible representation of $A_{\theta'}$, considered in decreasing order, with multiplicity, vary by no more than ε as the representation varies. That this property holds follows from the definition of the continuous field (A_θ) together with the fact that, for each rational number p/q with $(p, q) = 1$, there exists a complete family of irreducible representations of $A_{p/q}$ on the fixed Hilbert space $\mathbf{C}^q$ such that the images of the canonical unitaries u and v vary in norm by at most $1/q$. Here we use the Weyl spectral variation inequality for hermitian matrices (see e.g. [2]). Such a family of representations of $A_{p/q}$ is given by mapping u and v into $z_1 U$ and $z_2 V$, respectively, where U is the diagonal matrix $\operatorname{diag}(1, \rho, \ldots, \rho^{q-1})$ $(\rho = e^{2\pi i p/q})$, V is the cyclic permutation matrix which moves the standard basis $(e_1, \ldots, e_q)$ into the basis $(e_{p+1}, \ldots, e_q,\ e_1, \ldots, e_p)$, and z_1 and z_2 are complex numbers of absolute value one such that $z_j = e^{2\pi i\phi_j}$ with $0 \leqq \phi_j < 1/q$. (See e.g. [8].)

The second property that we shall need is that a continuous field of elements has continuous canonical trace (see [10]). For a polynomial in the canonical unitaries, the canonical trace is just the constant term (which is independent of θ).

The corollary now follows from Theorem 6 below.

3. Remark. Among other things, it is shown in [13] that the unitary group of a separable simple unital approximately finite-dimensional C*-algebra, modulo its centre, is a simple group. An obvious question is whether this is also true in an irrational rotation C*-algebra. The proof given in [13] remains valid (if one recalls certain well known facts about the rotation C*-algebras), except for the proof of 9.6 of [13], which requires knowing that any self-adjoint element can be approximated by one with finite spectrum. Hence by Corollary 2, $U(A_\theta)/\text{Centre } U(A_\theta)$ is a simple group for a dense set of θ in $\mathbf{R} \setminus \mathbf{Q}$.

4. The following result subsumes Theorem 1.

THEOREM. *Let A be a separable homogeneous C^*-algebra of order $n \in \{1, 2, \ldots\}$. Suppose that the spectrum of A is of dimension at most two. Then the set of self-adjoint elements of A with n distinct eigenvalues in every irreducible representation of A is dense.*

PROOF. The following two general facts about dimension, which it seems convenient to take as a definition, allow us to pass to the case that $\hat{A}$, the spectrum of A, is a finite CW-complex. First, a locally compact metric space has dimension at most d if, and only if, it is the union of a sequence of compact subsets with dimension at most d. Second, a compact metric space has dimension at most d if, and only if, it is the projective limit of a sequence of finite CW-complexes composed of simplices of dimension at most d.

Let us note that if $\hat{A}$ is the union of a sequence of compact subsets such that the conclusion holds for each of the corresponding quotients of A, then the conclusion holds for A. (Once a self-adjoint element of A is approximated by one which, in the quotient corresponding to one of the compact subsets, has n distinct eigenvalues in every irreducible representation (of this quotient), then any other approximant close enough to this one will have the same property. Thus, assuming the result for each compact subset in the sequence, if $h = h^* \in A$, we may approximate h by $h_1 = h_1^* \in A$ such that h_1 has the desired property over the first compact subset, and then approximate h_1 by $h_2 = h_2^* \in A$ such that h_2 has the property over the second compact subset, and is sufficiently close to h_1 that it also has it over the first compact subset, and we may continue in this way. If we make the approximations sufficiently close, then they will converge to a limit $h_\infty = h_\infty^* \in A$ which is close to h and also has the desired property over every compact subset in the sequence, and, therefore, by hypothesis, over the whole spectrum.)

In particular, it follows that we may suppose that $\hat{A}$ is compact. Furthermore, since the continuous field of C^*-algebras defined by A is locally trivial, we may suppose that A is isomorphic to the C^*-algebra of all continuous $n \times n$ matrix valued functions on $\hat{A}$.

Since the conclusion is preserved under passing to inductive limits, it follows that we may suppose that $\hat{A}$ is a finite CW-complex of dimension at most two, i.e., a triangulated space. Hence by the observation above, we may suppose that $\hat{A}$ is a single simplex, i.e., a point (this case is trivial), an interval, or a triangle.

Consider first the case that the dimension of the simplex is one, i.e., that it is an interval. Let $h = h^* \in A$. Replacing h by a close approximant, and passing to a subinterval, we may suppose that h is linear, as a matrix-valued function on the interval. In this case, it is possible to make an analytic choice of eigenvalues and eigenprojections for h, that is, to choose n eigenvalues for h at each point, and corresponding one-dimensional eigenprojections, in such a way that the resulting n real-valued functions and n projection-valued functions on the interval are analytic (i.e. have entire analytic extensions); see Theorem 6.1 of [16]. Furthermore, we may suppose that the n eigenvalues of h are distinct at at least one point, so that by analyticity they are distinct at all except finitely many points. Passing to a subinterval again, we may suppose that the eigenvalues of h are distinct except at one point.

In order to perturb h to remove this singular point, at which some of the eigenvalues coincide, we may suppose that in fact only two of the eigenvalues coincide. The reduction to this case can be achieved by adding small constants to each of the n eigenvalues of h, considered as analytic functions on the interval, so that at most two of the resulting analytic functions are equal at any point of the interval. (Recall that any two analytic functions on the interval can coincide at only finitely many points.) Adding the corresponding linear combination of the n eigenprojections to h, we obtain a small perturbation of h, at most two eigenvalues of which coincide at any point of the interval. Since only finitely many of these new singular points arise, after passing to a subinterval as before we need consider only the case that there is one such point.

The case that only two eigenvalues of h coincide at the singular point is easy to deal with. We may suppose that $n = 2$, i.e. that h is a 2×2 matrix valued function. Choose a unitary $u \in A$ which interchanges the two one-dimensional eigenprojections of h; this uses only that these eigenprojections have been chosen to be continuous functions on the interval. For any $\varepsilon > 0$ the self-adjoint element $h + \varepsilon(u + u^*)$ has distinct eigenvalues at every point. (This just amounts to saying that, for arbitrary real numbers a and b, and arbitrary $c \neq 0$, the 2×2 matrix $\begin{bmatrix} a & c \\ \bar{c} & b \end{bmatrix}$ has distinct eigenvalues.)

Now consider the case that the dimension of the simplex is two, i.e., that it is a triangle. (The argument that follows is also valid in the case of dimension one.) In order to deal with this case, we shall establish the following property of the closed subset of hermitian $n \times n$ complex matrices with at least one multiple eigenvalue. Denote the subset of such matrices by H^s, and the real linear space of all hermitian $n \times n$ complex matrices by H. We shall show that H^s is a finite union of embedded submanifolds of H, each of codimension at least three.

The fact that H^s is a finite union of embedded submanifolds of H, putting aside for the moment the assertion concerning the codimensions, actually follows just from the fact that H^s is the zero set of a polynomial on H (namely, the resultant, i.e., the product of all differences of eigenvalues $\lambda_i - \lambda_j$ with i different from j, which by symmetry is a (real) polynomial in the real and imaginary parts of the matrix entries). (Inside the zero set of an arbitrary polynomial, or finite set of polynomials, the singularities form a new zero set of lower (maximal) dimension; see the proof of Theorem 5.3 of [14].)

Rather than using the properties of an arbitrary zero set, let us express H^s explicitly as a finite union of submanifolds of H. For each partition of n, i.e. each decomposition $n = n_1 + \ldots + n_k$ where $n_i \in \{1, 2, \ldots, n\}$, excluding the finest partition, $n = 1 + \ldots + 1$, consider the set of hermitian $n \times n$ complex matrices with this pattern of multiplicities of eigenvalues. It is obvious that the union of these sets, one for each partition of n (except for $n = 1 + \ldots + 1$), is equal to H^s. (It is in fact a disjoint union; one can also make the union a disjoint one for an arbitrary zero set, but not in such a natural way.)

Let us check that each of the subsets defined in this way, associated with a partition of n, is an embedded submanifold of the real linear space H. In other words, some open neighbourhood of each point (in the relative topology) should be conjugate by a homeomorphism of H to an open subset of a linear subspace of H. Of course, we also want the codimension of this subspace to be at least three. In fact, if the partition is $n = n_1 + \ldots + n_k$ then the codimension of the subspace, i.e., of the submanifold, is

$$\left(\sum_{i=1}^{k} n_i^2\right) - k.$$

(And since some n_i is at least two, this is at least three.) The coordinates perpendicular to the submanifold at a point are obtained by adding hermitian matrices which commute with the matrix at that point, and have trace zero inside each eigenprojection of the matrix at that point. For small nonzero values of these coordinates, the matrix has a different eigenvalue pattern from $(n_1, \ldots, n_k)$. The dimension of the space of such matrices inside the ith eigenprojection is $n_i^2 - 1$.

Let us now show that any continuous function from a triangle into H can be perturbed in such a way as to avoid finitely many embedded submanifolds of codimension at least three, and, therefore, in particular, to avoid H^s, as desired. It is clearly enough to show that the function can be perturbed to avoid countably many compact subsets of H, each of which is conjugate by a homeomorphism of H to a subset of some linear subspace of codimension three. Since the subsets are compact, it is actually enough to be able to avoid just one, by an arbitrarily small perturbation; one can then avoid a whole sequence, in succession, by first avoiding the first, and then avoiding the second while staying at nonzero distance to the first, and so on, by perturbations that become so small that the result converges. After a conjugation by a homeomorphism of H, it is enough to avoid a linear subspace of H of codimension three. Perturbing the function on the triangle so that it is piecewise linear, and then projecting into the orthogonal complement of the subspace, we have a piecewise linear function from a triangle into R^3. The image of such a function is a finite union of triangles and is therefore nowhere dense in R^3. In particular, after translation of the function by an arbitrarily small amount, in a suitable direction orthogonal to the subspace, the projection into R^3 must avoid 0. In other words, the function from the triangle into H, perturbed by a small amount in this way, avoids the linear subspace, as desired.

5. REMARK. Consideration of the simplest three-dimensional case, i.e. the C^*-algebra $M_2(C(B^3)) = C(B^3, M_2)$ of continuous 2×2 matrix valued functions on the unit ball in three-dimensional Euclidean space, shows that the restriction on the dimension in Theorem 4 cannot be relaxed. The self-adjoint matrix valued function $(\lambda, \mu, \nu) \mapsto \begin{bmatrix} \lambda & \mu + i\nu \\ \mu - i\nu & -\lambda \end{bmatrix}$, for instance, cannot be approximated by one with distinct eigenvalues at every point. To prove this, it is enough to show that any sufficiently small perturbation of the identity function $x \mapsto x$ on B^3 must map some point into 0. Let $x \mapsto x - g(x)$ be such a perturbation, with $\|g(x)\| \leqq 1$ for all $x \in B^3$, i.e., $g\colon B^3 \to B^3$. Then by the Brouwer fixed-point theorem, for some $x \in B^3$ we have $g(x) = x$, i.e. $x - g(x) = 0$, as stated.

6. In view of the proof of Theorem 1 (see Theorem 4), it is natural to prove Corollary 2 in the following more abstract form.

THEOREM. *Let $(A(t))$ be a continuous field of C^*-algebras over the compact Hausdorff space T. Suppose that the C^*-algebra of continuous fields of elements is separable. Suppose that for each element s of a dense subset S of T, the fibre algebra $A(s)$ is homogeneous of order $n(s)$ and has spectrum of dimension at most two. Suppose that for every continuous field h of self-adjoint elements, every $t \in T$, every neighbourhood N of t, and every $\varepsilon > 0$, there exists $s \in N \cap S$ such that the eigenvalues of $h(s)$ in the irreducible representations of $A(s)$, considered in decreasing order (with multiplicity), vary by at most ε. Then for a dense set of $t \in T$, the self-adjoint elements of $A(t)$ with finite spectrum are dense.*

Suppose, furthermore, that the order $n(s)$ of $A(s)$ is greater than any given number for s in a dense subset of S. Suppose that there is a continuous field $(\tau(t))$ of faithful tracial states. Then for a dense set of $t \in T$, the self-adjoint elements of $A(t)$ with Cantor spectrum are dense.

PROOF. Let $h = (h(t))$ be a continuous field of self-adjoint elements, and let $\varepsilon > 0$. Let us show that, under the hypotheses of the first part of the theorem, there exists a dense open subset T_0 of T such that for every $t \in T_0$ there is a self-adjoint element of $A(t)$ with finite spectrum strictly within distance ε of $h(t)$. Let us show also that, under the additional hypotheses of the second part of the theorem, T_0 may be chosen so that for every $t \in T_0$ there is a self-adjoint element of $A(t)$ with finite spectrum strictly within distance ε of $h(t)$ and such that, furthermore, the value of the trace $\tau(t)$ on each minimal spectral projection of this element is strictly less than ε.

Note first that, if we denote by T_0 the set of $t \in T$ for which approximation of $h(t)$ as above is possible (with or without the restriction on the traces of the minimal spectral projections of the approximant), then T_0 is open. (This uses only that h is a continuous field of self-adjoint elements, with respect to a continuous field of C*-algebras, and that τ is a continuous field of traces.) It follows that we need only show that T_0 as defined in this way is dense. Since T_0 is open, it is the same to show that $T_0 \cap S$ is dense.

Fix $t \in T$, and let N be a neighbourhood of t in T. By hypothesis, there exists $s \in N \cap S$ such that the eigenvalues of $h(s)$ in the irreducible representations of the (homogeneous) C*-algebra $A(s)$, considered in decreasing order, vary by at most $\varepsilon/8$. By Theorem 4, there exists $k = k^* \in A(s)$ with $\|h(s) - k\| \leqq \varepsilon/8$ and such that the eigenvalues of k in any irreducible representation of $A(s)$ are distinct. By the Weyl spectral variation inequality for hermitian matrices (see [2]), the eigenvalues of k in the irreducible representations of $A(s)$ vary by at most

$$\varepsilon/8 + 2\,\|h(s) - k\| \leqq 3\varepsilon/8.$$

It follows that k is within $3\varepsilon/8$ of a self-adjoint element k' of $A(s)$ such that the eigenvalues of k' in all irreducible representations of $A(s)$ are the same, and are distinct. (The eigenvalues of k' may be taken to be those of k in any single irreducible representation of $A(s)$.) We have

$$\|h(s) - k'\| \leqq \|h(s) - k\| + \|k - k'\| \leqq \varepsilon/8 + 3\varepsilon/8 = \varepsilon/2 < \varepsilon.$$

This shows that, when T_0 is defined as above, in the context of the first part of the theorem (i.e. without reference to the traces of spectral projections), $s \in T_0$. Since the minimal spectral projections of k' have dimension one in every irreducible representation of $A(s)$, they have trace $1/n(s)$ for every tracial state on $A(s)$ and in particular for $\tau(s)$. By hypothesis, we may choose s so that $n(s) > 1/\varepsilon$, i.e., $1/n(s) < \varepsilon$. Then also $s \in T_0$ when T_0 is defined as above in the context of the second part of the theorem. Since $s \in N$, and N is an arbitrary neighbourhood of t in T, and t is arbitrary, this shows that T_0 is dense in T.

Now choose a dense sequence $h_1, h_2, \ldots$ of continuous fields of self-adjoint elements, and a sequence $\varepsilon_1 > \varepsilon_2 > \ldots$ converging to 0. Denote by $T_n, n = 1, 2, \ldots$, the dense open subset T_0 of T constructed as above with h_n in place of h and ε_n in place of ε. The intersection $\bigcap T_n$ is dense in T, since T is a compact Hausdorff space. It is immediate that the dense set $\bigcap T_n$ fulfils the requirement of the first part of the theorem.

Let us show that the dense set $\bigcap T_n$ fulfils the requirement of the second part of the theorem. Fix $t \in \bigcap T_n$, and note that any self-adjoint element of $A(t)$ can be approximated arbitrarily closely by a self-adjoint element with finite spectrum, which can be chosen in such a way that the value of the faithful tracial state $\tau(t)$ on each minimal spectral projection is arbitrarily small. Hence by Lemma 7 which follows, any self-adjoint element of $A(t)$ can be approximated by a self-adjoint element with Cantor spectrum.

7. LEMMA. *Let A be a C*-algebra, and let ϕ be a faithful state on A. (Such a state exists, for example, if A is separable.) Suppose that any self-adjoint element of A can be approximated arbitrarily closely by a self-adjoint element with finite spectrum, and that this element can be chosen such that the value of ϕ on each of its minimal spectral projections is arbitrarily small. Then any self-adjoint element of A can be approximated by one with Cantor spectrum.*

PROOF. Let $h = h^* \in A$, and let $\varepsilon > 0$. Choose $h_1 = h_1^* \in A$ with finite spectrum such that $\|h - h_1\| \leqq \varepsilon/2$, and such that the value of ϕ on each minimal spectral projection of h_1 is strictly less then $1/2$. Denote the smallest distance between two distinct points of the spectrum of h_1 by $3s_1 = 3s(h_1)$.

Choose $h_2 = h_2^* \in A$ with finite spectrum such that

$$\|h_1 - h_2\| \leqq \min\{s_1/2, \varepsilon/2^2\},$$

and such that the value of ϕ on each minimal spectral projection of h_2 is at most $1/2^2$. Recall that the Hausdorff distance between the spectra of two self-adjoint elements of a C*-algebra is at most equal to the distance between the elements. In particular, as $\|h_1 - h_2\| \leqq s_1$, it follows that for each gap in the spectrum of h_1 at least the middle third of this gap is contained in a gap in the spectrum of h_2. Now choose h_2 much closer to h_1, close enough that for each consecutive pair of gaps in the spectrum of h_1 (including the unbounded gaps at either end), the spectral projection of h_2 corresponding to the interval between the middle thirds of these gaps (or between the middle third of one and $\pm\infty$) is close to the spectral projection of h_1 corresponding to the same interval (which is a minimal spectral projection of h_1, and so the value of ϕ on it is strictly less than $1/2$), close enough that the value of ϕ on this spectral projection of h_2 is strictly less than $1/2$. Denote the distance from h_1 within which h_2 must lie for this to happen by δ_1. Choose h_2 so that in fact

$$\|h_1 - h_2\| \leqq \min\{s_1/2, \delta_1/2, \varepsilon/2^2\}.$$

Continuing in this way, for each $n = 3, 4, \ldots$ choose inductively $h_n = h_n^* \in A$ with finite spectrum such that

$$\|h_{n-1} - h_n\| \leqq \min\{s_m/2^{n-1}, \delta_m/2^{n-1}, \varepsilon/2^n; m = 1, \ldots, n-1\},$$

where $s_m = s(h_m)$, and δ_m is defined in analogy with the case $m = 1$ as follows. Note first that

$$\|h_m - h_n\| \leqq s_m, \ m = 1, \ldots, n-1,$$

so that the middle third of each gap in the spectrum of h_m is contained in a gap in the spectrum of h_n. Choose δ_m so that whenever h_n is such that this is true, and also $\|h_m - h_n\| \leqq \delta_m$, then necessarily the value of ϕ on the spectral projection of h_n corresponding to the interval between the middle thirds of any two consecutive gaps in the spectrum of h_m is strictly less than 1/m.

The sequence (h_n) is clearly Cauchy. Denote the limit by h_∞. We have

$$\|h - h_\infty\| \leqq \varepsilon,$$

and for each $m = 1, 2, \ldots,$

$$\|h_m - h_\infty\| \leqq \min\{s_m, \delta_m\}.$$

It follows that, for each m, the middle third of each gap in the spectrum of h_m is contained in a gap of the spectrum of h_∞, and, furthermore, the value of ϕ on the spectral projection of h_∞ corresponding to the interval between the middle thirds of any two consecutive gaps in the spectrum of h_m is less than $1/m$. As we shall now show, this, together with the fact that ϕ is faithful on A, implies that the spectrum of h_∞ is a Cantor set.

Suppose that the spectrum of h_∞ contains a connected subset, I, which is either an isolated point or a whole interval. We shall deduce a contradiction. Choose a continuous function f on R with $0 \leqq f \leqq 1$ which is nonzero at some point of I and is zero on the relative complement of I in the spectrum of h_∞. For any m, there must exist two consecutive gaps in the spectrum of h_m the middle thirds of which lie on either side of I. The value of ϕ on the spectral projection of h_∞ corresponding to the interval between the middle thirds of these gaps (or between the middle third of one of them and $\pm\infty$, if the other gap abuts on $\pm\infty$) is less than $1/m$. Hence $\phi(f(h_\infty)) = 0$. But by the choice of f, $f(h_\infty) \neq 0$. This shows that I as above cannot exist. In other words, the spectrum of h_∞ is a Cantor set.

8. M. Rørdam has pointed out to us that Lemma 7 leads to the following result. While this result does not seem itself to imply the second part of Theorem 6 (although it shows that the additional hypotheses of the second part of Theorem 6 can be replaced by, for example, the hypothesis that every fibre algebra $A(t)$ with $t \in S$ is simple), it does yield the second part of Theorem 9, below.

COROLLARY. *Let A be a C^*-algebra with no nonzero elementary subquotient. (This holds if A is simple and not elementary, and if it holds for A it holds for $A \otimes B$ for any C^*-algebra B.) Suppose that the self-adjoint elements of A with finite spectrum are dense. Then the self-adjoint elements of A with Cantor spectrum are dense.*

PROOF. First, consider the case that A is separable. Choose a faithful state ϕ on A. Clearly, in view of Lemma 7, it is sufficient to show that any projection e in A majorizes a projection f with $\phi(f)/\phi(e)$ approximately equal to 1/2. Passing to eAe we may suppose that $e = 1$. (The hypotheses on A still hold; see [17].)

To find a projection f in A with $\phi(f)$ close to $1/2$, it is sufficient to do this in the weak closure of A in the representation determined by ϕ. Indeed, if $F = F^* = F^2 \in \pi_\phi(A)''$ and $\pi_\phi(h_n) \to F$ strongly, where $h_n = h_n^* \in A$ has finite spectrum, then, with χ the characteristic function of the interval $[1/2,2]$, $\chi(h_n) \in A$, and as χ is continuous on a neighbourhood of the spectrum of F, by a result of Murray and von Neumann and Kaplansky ([15], Theorem 2),

$$\pi_\phi(\chi(h_n)) = \chi(\pi_\phi(h_n)) \to F \text{ strongly.}$$

By the first hypothesis on A, which now amounts to saying that A has no nonzero finite-dimensional quotient, $\pi_\phi(A)''$ has no nonzero finite direct summand of type I. From this we conclude that $\pi_\phi(\mathrm{A})''$ contains a subfactor of type I_2. A simple computation shows that an arbitrary state on a factor of type I_2 takes the value $1/2$ on some projection. (The state with density matrix $\begin{bmatrix} \mu & 0 \\ 0 & 1-\mu \end{bmatrix}$ is equal to $1/2$ on the projection $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.)

Now, consider the case that A is not separable. To reduce the problem to the separable case, it is sufficient to show that any countable subset of A is contained in a separable sub-C*-algebra fulfilling the same conditions as A. (Cf. [3], where it is shown that a countable subset of a simple C^*-algebra is contained in a separable simple sub-C^*-algebra.) Let C be a separable sub-C^*-algebra of A, and let us enlarge C until it fulfils the conditions of the corollary. If we do this for the two conditions separately, then, constructing an increasing sequence $C \subseteq C_1 \subseteq C_2 \subseteq \cdots$ where the odd C's fulfil one and the even C's the other, we will have that the closure of the union of the C's fulfils both conditions. It is obvious how to do this for the second condition. To do it for the first condition, let us first reformulate this as follows: no hereditary sub-C^*-algebra has a nonzero finite-dimensional quotient. It is sufficient to consider singly generated hereditary sub-C^*-algebras, and, in the separable case, it is sufficient to consider the hereditary sub-C^*-algebras generated by the elements in a countable approximate unit of the Pedersen ideal.

Now, given a separable sub-C^*-algebra C, let us construct a larger one which fulfils the first condition. Choose an approximate unit $(c_1, c_2, \ldots)$ in the Pedersen ideal of C. It is sufficient to construct a larger separable C^*-algebra for which $(c_1, c_2, \ldots)$ is still an approximate unit, and such that for no n does the hereditary sub-C^*-algebra generated by c_n have a nonzero finite-dimensional quotient. Hence, it is sufficient to find a separable sub-C^*-algebra containing c_1Cc_1, for which c_1 is a strictly positive element, and which has no nonzero finite-dimensional quotient. Replacing A by c_1Ac_1, we may ignore c_1. We shall construct an increasing sequence $C \subseteq C_1 \subseteq C_2 \subseteq \ldots$ of separable sub-C^*-algebras such that C_n has no irreducible representation of dimension n or less. It follows, first, that

C_n has no nonzero representation at all of dimension n or less, and, hence, that C_∞, the closure of the union of the C's, has no nonzero finite-dimensional representation, as desired.

First, since A has no one-dimensional quotient, A is the closed two-sided ideal generated by its commutators (elements $ab - ba$). Hence, C is contained in a separable C^*-algebra C_1 which is the closed two-sided ideal generated by its commutators, and, in other words, such that C_1 has no one-dimensional irreducible representation. Next, since A has no quotient isomorphic to M_1 or to M_2, A is the closed two-sided ideal generated by the range of the polynomial P_2, where $P_2 = 0$ is the polynomial identity which holds for M_2 but not for M_3 (see [1]). It follows that C_1 is contained in a separable sub-C^*-algebra C_2 with the same property, equivalently, such that C_2 has no quotient isomorphic to M_1 or to M_2. Continuing in this way, we obtain an increasing sequence $C \subseteq C_1 \subseteq C_2 \subseteq \ldots$ of separable sub-C^*-algebras such that C_n has no quotient isomorphic to any of $M_1, M_2, \ldots, M_n$, as desired.

9. Theorem 1 can also be used, much as in the proof of Corollary 2, to give a new proof that in the Bunce-Deddens algebras the self-adjoint elements with finite spectrum are dense ([5]). This proof proceeds directly, rather than by first showing that each hereditary sub-C^*-algebra has an approximate unit consisting of projections (and then using the fact, proved in [4], that this is sufficient).

Using Theorem 4, one obtains the following result.

THEOREM. *Let $A_1 \subseteq A_2 \subseteq \ldots$ be a sequence of separable homogeneous C^*-algebras of orders $n_1, n_2, \ldots$. Suppose that each A_i has spectrum of dimension at most two. Suppose that for each i, for each self-adjoint element h of A_i, and for each $\varepsilon > 0$, there exists $j > i$ such that the eigenvalues of h in the irreducible representations of A_j, considered in decreasing order (with multiplicity), vary by at most ε. Then the inductive limit C^*-algebra, $\varinjlim A_i$, has the property that the self-adjoint elements with finite spectrum are dense.*

Suppose, furthermore, that $\varinjlim A_i$ is not elementary. Then in $\varinjlim A_i$ the self-adjoint elements with Cantor spectrum are dense.

PROOF. The proof is similar to the proof of Theorem 6, but is simpler.

Let h be a self-adjoint element of A_i, and let $\varepsilon > 0$. By hypothesis, there exists $j > i$ such that the eigenvalues of h in the irreducible representations of A_j vary by at most $\varepsilon/8$. By Theorem 4 (with A_j in place of A), there exists $k = k^* \in A_j$ with $\|h - k\| \leqq \varepsilon/8$ such that the eigenvalues of k in each irreducible representation of A_j are distinct. By the Weyl spectral variation formula for hermitian matrices (see [2]), the eigenvalues of k in the irreducible representations of A_j vary by at most

$$\varepsilon/8 + 2\|h - k\| \leqq 3\varepsilon/8.$$

It follows that k is within $3\varepsilon/8$ of a self-adjoint element k' of A_j such that the eigenvalues of k' in all irreducible representations of A_j are the same, and are distinct. (The eigenvalues of k' may be taken to be those of k in any single irreducible reprexentation of A_j.) We have

$$\|h - k'\| \leqq \|h - k\| + \|k - k'\| \leqq 3\varepsilon/8 + 3\varepsilon/8 = \varepsilon/2 < \varepsilon,$$

as desired.

The second statement follows by Corollary 8, since the hypotheses of the first part of the theorem imply that $\varinjlim A_i$ is simple. (See proof of (i) $\Rightarrow$ (ii) of Theorem 10.)

10. Theorem 9 can be applied to embeddings of the following kind (which occur in the sequence defining a Bunce-Deddens algebra).

THEOREM. *Let $A_1 \subseteq A_2 \subseteq \ldots$ be a sequence of homogeneous C^*-algebras. Suppose that the spectrum of each A_i is a compact, connected Riemannian manifold. Suppose that, for each $i = 1, 2, \ldots$, for each irreducible representation π of A_i there is exactly one irreducible representation ρ of A_{i+1} such that the restriction of ρ to A_i contains π, and that the multiplicity of π in $\rho|A_i$, if it is not zero, is one. Suppose that the mapping from $\hat{A}_i$ into $\hat{A}_{i+1}$ that is defined by the preceding hypothesis, which is necessarily continuous, and in fact a local homeomorphism, preserves the Riemannian metric, for each $i = 1, 2, \ldots$. Then the following three conditions are equivalent:*

(i) *for each self-adjoint element h of A_i, and each $\varepsilon > 0$, there exists $j > i$ such that the eigenvalues of h in the irreducible representations of A_j, considered in decreasing order (with multiplicity), vary by at most ε;*

(ii) *the inductive limit C^*-algebra, $\varinjlim A_i$, is simple;*

(iii) *the diameter of $\hat{A}_i$ converges to zero.*

PROOF. We shall use the following, more familar reformulation of the condition that $\varinjlim A_i$ is simple (which uses compactness of the spectra, but no assumption on the embeddings): for each open subset U of $\hat{A}_i$, there exists $j > i$ such that for every $\rho \in \hat{A}_j$, the restriction of ρ to A_i contains some element of U.

(It is in fact by this criterion that the Bunce-Deddens algebras are most easily seen to be simple.)

Ad (i) $\Rightarrow$ (ii). (This does not use the assumptions on the spectra or the embeddings.) Let U be an open subset of $\hat{A}_i$. Choose $h = h^* \in A_i$ with support contained in U, and let $j > i$ be such that the eigenvalues of h in the irreducible representations of A_j vary by at most $\|h\|/2$. Then $\rho(h) \neq 0$ for every $\rho \in \hat{A}_j$, whence $\rho \mid A_i$ contains some element of U.

Ad (ii) $\Rightarrow$ (iii). This is immediate.

Ad (iii) $\Rightarrow$ (i). Suppose that the diameter of $\hat{A}_i$ converges to zero. Let h be a self-adjoint element of A_i, and let $\varepsilon > 0$. Since the continuous field of C^*-algebras defined by A_i is locally trivial, by the Weyl spectral variation inequality (see [2]) the eigenvalues of $\pi(h)$, considered in decreasing order, depend continuously on $\pi \in \hat{A}_i$. Hence, by compactness, there exists $\delta > 0$ such that for each subset V of $\hat{A}_i$ of diameter at most δ, the eigenvalues of $\pi(h)$, considered in decreasing order, vary by at most ε for $\pi \in V$. Choose $j > i$ such that the diameter of $\hat{A}_j$ is at most $\delta/2$. Let us show that the eigenvalues of h in the irreducible representations of A_j, considered in decreasing order, vary by at most ε.

Note that for any two ordered pairs of real numbers (λ_1, λ_2) and (μ_1, μ_2) with $\lambda_1 \geqq \lambda_2$ and $\mu_1 \geqq \mu_2$,

$$\max(|\mu_1 - \lambda_1|, |\mu_2 - \lambda_2|) \leqq \max(|\mu_1 - \lambda_2|, |\mu_2 - \lambda_1|).$$

Hence by induction, for any two n-uples $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ with $\lambda_1 \geqq \ldots \geqq \lambda_n$ and $\mu_1 \geqq \ldots \geqq \mu_n$, the distance from λ to μ in the supremum norm is less than or equal to the distance from λ to any rearrangement of μ.

It follows from the preceding (classical) fact that to prove that the eigenvalues of $\rho(h)$ vary by at most ε when they are considered in decreasing order, it is sufficient to prove that they vary by at most ε when they are considered in just some order, possibly depending on ρ.

From the hypotheses, which also extend to the embedding $A_i \subseteq A_j$, it follows that $\hat{A}_j$ is the quotient space of $\hat{A}_i$ with respect to a finite group G acting freely and isometrically on $\hat{A}_i$. Since $\hat{A}_j$ has diameter at most $\delta/2$, there is a fundamental domain V for G of diameter at most δ. (To see this, choose $\rho \in \hat{A}_j$. Then $\hat{A}_j$ is contained in the closed ball with centre ρ and radius $\delta/2$. Choose π in the preimage of ρ in $\hat{A}_i$ (i.e., in the fibre over ρ). Then the closed ball in $\hat{A}_i$ with centre π and radius $\delta/2$ maps onto $\hat{A}_j$: any $\rho_1 \in \hat{A}_j$ is connected to ρ by a path of length at most $\delta/2$, and this lifts to a path of the same length in $\hat{A}_i$, with one end at π. This ball in $\hat{A}_i$ has diameter at most $\delta/2 + \delta/2 = \delta$, and therefore any fundamental domain V inside it (i.e. any selection of one point from each orbit under G) also has diameter at most δ.) Then also $g(V)$ has diameter at most δ for each $g \in G$.

Since $\hat{A}_i$ is the disjoint union of $g_1(V), g_2(V), \ldots, g_k(V)$ where $g_1, \ldots, g_k$ is an enumeration of the elements of G, so that every fibre has exactly one point in each $g_m(V)$, for each $\rho \in \hat{A}_j$ we have

$$\rho \mid A_i \cong \pi_1^\rho \oplus \ldots \oplus \pi_k^\rho,$$

where $\pi_m^\rho \in g_m(V)$.

It is now clear how to label the eigenvalues of $\rho(h)$. Begin with the eigenvalues of $\pi_1^\rho(h)$, in decreasing order, and then take the eigenvalues of $\pi_2^\rho(h)$, and so on, up to $\pi_k^\rho(\mathrm{h})$. Since g_m is isometric, the diameter of $g_m(V)$ is at most δ. Hence, as $\pi_m^\rho \in g_m(V)$, the variation of the eigenvalues of $\pi_m^\rho(h)$ is at most ε. In other words, the variation of the eigenvalues of $\rho(h)$, ordered in this way, is at most ε, as desired.

11. The question arises, to which subhomogeneous C^*-algebras can the results of this paper be extended? While this is not clear in the case of Theorem 10, in the case of Theorem 4, and therefore also Theorems 6 and 9, inspection shows that the proof is valid for subhomogeneous C^*-algebras defined by continuous fields of finite-dimensional C^*-algebras satisfying Fell's condition, for which the fibres are simple except at finitely many points. In particular, this includes the noncommutative spheres (or orbifolds) of [6] for rational values of the parameter. Therefore one has an analogue of Corollary 2 for irrational noncommutative spheres.

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