MATH. SCAND. 67 (1990), 97–118

ON A COMPLETE INITIAL-BOUNDARY VALUE PROBLEM FOR PARABOLIC PSEUDO-DIFFERENTIAL OPERATORS

VEIKKO T. PURMONEN

1. Introduction.

In this paper we discuss parabolic pseudo-differential initial-boundary value problems, stated in Sobolev spaces of sections of vector bundles. The notation and terminology are adopted from [GG] and [P], to which we refer for more details. Accordingly, let $\bar{\Omega}$ be a compact and connected, n-dimensional ($n \geqq 2$) C^∞ manifold with interior Ω and boundary Γ. In addition, set $Q = \Omega \times \mathsf{R}_+$ and $S = \Gamma \times \mathsf{R}_+$ with $\mathsf{R}_+ = \{t \in \mathsf{R}: t > 0\}$. Let $\bar{\Omega}$ be smoothly imbedded into a compact and connected, n-dimensional Riemannian C^∞ manifold Σ without boundary, and let x and x' denote points in Σ and Γ, respectively, and choose a normal coordinate x_n near Γ such that $x = (x', x_n)$. We suppose that E and F_k are smooth vector bundles such that $E = \hat{E}|_\Omega$ and $F_k = \hat{F}_k|_\Gamma$, where $\hat{E}$ and $\hat{F}_k$ are Hermitean complex C^∞ vector bundles over Σ with fiber dimensions $N \geqq 1$ and $M_k \geqq 0$, respectively; here the case $M_k = 0$ is included for notational convenience (see [GG, p. 46]). Furthermore, for example, let E^t denote the trivial extension of E to a bundle over $\bar{Q}$.

In order to state our problem, let m be a positive integer and d an even positive integer, the forthcoming parabolic weight. For a multi-index $v \in \mathsf{Z}^{md}$ we write $v = (v_{\alpha d+\beta})_{\alpha,\beta} = (v_{\alpha d+\beta})_{0 \leqq \alpha < m, 0 \leqq \beta < d}$, and let $\mu \in \mathsf{N}^{md}$ be the multi-index which has the property $\mu_{\alpha d+\beta} = \alpha + 1$ for all $\alpha = 0,\dots,m-1, \beta = 0,\dots,d-1$. For $l \in \mathsf{Z}$ let $\bar{l}$ stand for the multi-index $v \in \mathsf{Z}^{md}$ with $v_{\alpha d+\beta} = l$ for all $\alpha = 0,\dots,m-1$, $\beta = 0,\dots,d-1$. Now, if $I_{F_{\alpha d+\beta}}$ denotes the identity on $F_{\alpha d+\beta}$, we set $I^l_{\alpha d+\beta}(z) = z^l I_{F_{\alpha d+\beta}}$ for $l \geqq 0$ and $I^l_{\alpha d+\beta}(z) = 0$ for $l < 0$. Set further $F = F_0 \oplus \dots \oplus F_{md-1}$, and define a diagonal operator $I^v(z)$ from F to F by

$$I^v(z) = \operatorname{diag}(I_0^{v_0}(z), \dots, I_{md-1}^{v_{md-1}}(z)).$$

Analogously we introduce the operator

$$I^v(\partial_t) = \operatorname{diag}(I_0^{v_0}(\partial_t), \dots, I_{md-1}^{v_{md-1}}(\partial_t)).$$

Received November 2, 1989

Here z is a complex parameter which is related to $\partial_t = \partial/\partial t$ by the Laplace transformation.

The operator (system)

$$A(\partial_t) = \begin{bmatrix} \partial_t^m & 0 \\ 0 & I^\mu(\partial_t) \end{bmatrix} + \sum_{j=0}^{m-1} A^{(m-j)} \begin{bmatrix} \partial_t^j & 0 \\ 0 & I^{\mu-(\overline{m-j})}(\partial_t) \end{bmatrix}$$

is called *parabolic*, if the operator

$$A(z): \begin{matrix} C^\infty(\bar{\Omega}, E) \\ \oplus \\ C^\infty(\Gamma, F) \end{matrix} \to \begin{matrix} C^\infty(\bar{\Omega}, E) \\ \oplus \\ C^\infty(\Gamma, F) \end{matrix},$$

which depends polynomially on the complex parameter z and is of the form

$$A(z) = \begin{bmatrix} z^m & 0 \\ 0 & I^\mu(z) \end{bmatrix} + \sum_{j=0}^{m-1} A^{(m-j)} \begin{bmatrix} z^j & 0 \\ 0 & I^{\mu-(\overline{m-j})}(z) \end{bmatrix},$$

is parameter-elliptic on every ray $z = \rho e^{i\theta}$ with $\rho \geqq 0$, $-\pi/2 \leqq \theta \leqq \pi/2$ (see [GG, Sections 1.5, 3.1]). Here

$$A^{(m-j)} = \begin{bmatrix} P_\Omega^{(m-j)} + G^{(m-j)} & K^{(m-j)} \\ T^{(m-j)} & R^{(m-j)} \end{bmatrix}$$

is a Green operator (system) of order $(m-j)d$, the parabolic weight d being an even positive integer, which means that (see [GG, Chapter 2])

(i) $P_\Omega^{(m-j)} = r_\Omega P^{(m-j)} e_\Omega$ and $P^{(m-j)}$ is a classical pseudo-differential operator of order $(m-j)d$ from $\hat{E}$ to $\hat{E}$ with the *transmission property at* Γ (the operators r_Ω and e_Ω give the restriction and extension by zero, respectively);

(ii) $G^{(m-j)}$ is a singular Green operator of order $(m-j)d$ and class $r \leqq (m-j)d$ from E to E;

(iii) $K^{(m-j)} = (K_{\alpha d+\beta}^{(m-j)})_{0 \leqq \alpha < m, 0 \leqq \beta < d}$, where $K_{\alpha d+\beta}^{(m-j)}$ is a Poisson operator of order $md - \alpha d - \beta + (m-j)d$ from $F_{\alpha d+\beta}$ to E and $K_{\alpha d+\beta}^{(m-j)} = 0$ for $\alpha < m-1-j$;

(iv) $T^{(m-j)} = (T_{\alpha d+\beta}^{(m-j)})_{0 \leqq \alpha < m, 0 \leqq \beta < d}$, where $T_{\alpha d+\beta}^{(m-j)}$ is a trace operator of order $r = \alpha d + \beta - jd$ and class $r+1$ from E to $F_{\alpha d+\beta}$ when $\alpha \geqq j$, and $T_{\alpha d+\beta}^{(m-j)} = 0$ for $\alpha < j$;

(v) $R^{(m-j)} = (R_{\alpha d+\beta, \alpha' d+\beta'}^{(m-j)})_{0 \leqq \alpha, \alpha' < m, 0 \leqq \beta, \beta' < d}$, where $R_{\alpha d+\beta, \alpha' d+\beta'}^{(m-j)}$ is a pseudo-differential operator of order $(m-j)d + (\alpha - \alpha')d + \beta - \beta'$ from $F_{\alpha' d+\beta'}$ to $F_{\alpha d+\beta}$ and is 0 when $\alpha < j$ or $\alpha' < m-1-j$.

If the operator $A(\partial_t)$ is parabolic, then the following *initial-boundary value problem* (1.1–3) is called *parabolic*:

$$A(\partial_t) \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \tag{1.1}$$

(1.2) $$\gamma_t \partial_t^j u = h_j \quad \text{for } j = 0, \ldots, m-1,$$

and, for $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$,

(1.3) $$\gamma_t \partial_t^j w_{\alpha d+\beta} = \eta_{\alpha d+\beta,j} \quad \text{for } j = 0, \ldots, \alpha,$$

where γ_t is the usual trace operator with respect to t, $\gamma_t v = v|_{t=0}$. Here we suppose that, for $s \geqq 0$,

$$f \in H^{(s)}(Q, \rho, E^t),$$

$$g = (g_{\alpha d+\beta})_{\alpha,\beta} \in \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta}),$$

$$h = (h_j)_j \in \bigoplus_{j=0}^{m-1} H^{s+md-jd-d/2}(\Omega, E),$$

$$\eta = (\eta_{\alpha d+\beta,j})_{\alpha,\beta,j} \in \bigoplus_{\alpha,\beta} \bigoplus_{j=0}^{\alpha} H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma, F^t_{\alpha d+\beta}),$$

and we seek a solution $(u, w) = \begin{bmatrix} u \\ w \end{bmatrix}$ such that

$$u \in H^{(s+md)}(Q, \rho, E^t),$$

$$w = (w_{\alpha d+\beta})_{\alpha,\beta} \in \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta}).$$

The Sobolev spaces we use here are defined and denoted, with slight simplification, as in [P] (see also the references there). Thus, the Sobolev space of order s of sections of E is denoted by $H^s(\Omega, E)$ (instead of $H^s(\bar{\Omega}, E)$ in [P]), and in this notation we have

$$H^{(s)}(Q, \rho, E^t) = H^0(\mathbb{R}_+, \rho; H^s(\Omega, E)) \cap H^s(\mathbb{R}_+, \rho; H^0(\Omega, E)),$$

where $\rho \geqq 0$ and, for a Hilbert space H,

$$H^r(\mathbb{R}_+, \rho; H) = \{v \in \mathcal{D}'(\mathbb{R}_+; H) : e^{-\rho t} v \in H^r(\mathbb{R}_+; H)\}.$$

The spaces $H^s(\Gamma, F_{\alpha d+\beta})$ and $H^{(s)}(S, \rho, F^t_{\alpha d+\beta})$ have analogous meanings. Furthermore, for the sake of brevity, we write $\bigoplus_{\alpha,\beta}$ instead of $\bigoplus_{\alpha=0}^{m-1} \bigoplus_{\beta=0}^{d-1}$.

Our treatment of the problem (1.1–3) in the case of homogeneous initial values is based on the application of the Laplace transformation with respect to t. This leads us to a polynomially parameter-dependent elliptic boundary value problem. In Section 2 we briefly give an isomorphism result (Theorem 2.3) for the parameter-elliptic operator $A(z)$ of such a problem. An essential part of the result is a consequence of the general theory on parameter-dependent boundary problems developed by Gerd Grubb in [GG].

In [P] we have been concerned with questions of solvability of parabolic problems without the boundary function (section) w, that is, problems of the form

$$\partial_t^m u + \sum_{j=0}^{m-1} (P_\Omega^{(m-j)} + G^{(m-j)})\partial_t^j u = f,$$

$$\sum_{j=0}^{m-1} T_k^{(m-j)} \partial_t^j u = g_k \quad \text{for } k = 0, \dots, md - 1,$$

$$\gamma_t \partial_t^j u = h_j \quad \text{for } j = 0, \dots, m - 1.$$

This problem can have a solution only for such data f, g_k, h_j which satisfy certain intrinsic compatibility conditions (cf. [P]; see also [GG–S]). It is our purpose in the present paper to show that this "deficiency" appears no more in the problem (1.1–3). In Section 3 we prove that the problem (1.1–3) always has at least one solution (Theorem 3.3). The uniqueness of a solution follows from the special a priori estimate given in Theorem 3.5. The general a priori estimate is derived in Theorem 3.8. In the considerations certain values of s are exceptional (cf. [P]). However, by making use of a method originating in [GG–S], we are able to treat these exceptional values, too.

2. Polynomially parameter-elliptic problems.

2.1. It follows from [GG, Corollary 2.5.6] (see also [P, Theorem 2.7]) that for any $s \geqq 0$ the parameter-dependent operator $A(z)$ extends by continuity to a continuous operator

$$(2.1)\quad A(z)\colon \begin{array}{c} H_z^{s+md}(\Omega, E) \\ \oplus \\ \oplus_{\alpha,\beta} H_z^{s+md+d-\beta-1/2}(\Gamma, F_{\alpha d+\beta}) \end{array} \to \begin{array}{c} H_z^{s}(\Omega, E) \\ \oplus \\ \oplus_{\alpha,\beta} H_z^{s+md-\alpha d-\beta-1/2}(\Gamma, F_{\alpha d+\beta}) \end{array},$$

whose norm is uniformly bounded for all z with $\operatorname{Re} z \geqq 0$. Here, for example, $H_z^s(\Omega, E)$ denotes the space $H^s(\Omega, E)$ with a norm depending on z in proportion to the parabolic weight d (see [P]).

Let us now suppose that the operator $A(\partial_t)$ is parabolic. By arguing as in [GG, Sections 3.3, 3.4] and [P, Section 2.8], one can conclude from [GG, Section 3.2] that for any $s \geqq 0$ there exists $\rho > 0$ such that for every z with $\operatorname{Re} z \geqq \rho$ the inverse $A(z)^{-1}$ of the operator $A(z)$ in (2.1) exists, depends analytically on z in the corresponding operator norm, and satisfies the estimate

$$\left\| A(z)^{-1} \begin{bmatrix} F \\ G \end{bmatrix} \right\|_{H_z^{s+md}(\Omega,E) \oplus \bigoplus_{\alpha,\beta} H_z^{s+md+d-\beta-1/2}(\Gamma,F_{\alpha d+\beta})}$$

$$\leqq C \|(F, G)\|_{H_z^s(\Omega,E) \oplus \bigoplus_{\alpha,\beta} H_z^{s+md-\alpha d-\beta-1/2}(\Gamma,F_{\alpha d+\beta})}$$

for all

$$(F, G) \in H_z^s(\Omega, E) \oplus \bigoplus_{\alpha,\beta} H_z^{s+md-\alpha d-\beta-1/2}(\Gamma, F_{\alpha d+\beta})$$

uniformly in z with some constant $C > 0$.

2.2. In solving the parabolic problem (1.1–3) with homogeneous initial values, we shall make essential use of the Laplace transformation $\mathscr{L}$,

$$(\mathscr{L}v)(z) = \int_0^\infty e^{-zt}\, v(t)\, dt.$$

Let $s \geqq 0$ and $\rho > 0$. Referring to [P] for more details, we recall that $\mathscr{L}$ is an isomorphism from the space

$$H_{[0]}^{(s)}(Q, \rho, E^t) = H^0(\mathsf{R}_+, \rho; H^s(\Omega, E)) \cap H_{(0)}^{s/d}(\mathsf{R}_+, \rho; H^0(\Omega, E))$$

to the space

$$\mathscr{H}^{(s)}(\mathsf{C}_\rho, \Omega, E) = \mathscr{H}^0(\mathsf{C}_\rho; H^s(\Omega, E)) \cap \mathscr{H}^{s/d}(\mathsf{C}_\rho; H^0(\Omega, E)),$$

and an analogous result holds in the case of Γ and S. Here the space $H_{(0)}^r = H_{(0)}^r(\mathsf{R}_+, \rho; H)$ is defined as the closure H_0^r in H^r of the space of C^∞ functions with compact support in R_+, valued in a Hilbert space H, unless $r \equiv 1/2 \bmod 1$, in which case $H_{(0)}^r$ is defined by interpolation. The space $\mathscr{H}^r(\mathsf{C}_\rho; H)$ consists of such analytic functions U from $\mathsf{C}_\rho = \{z \in \mathsf{C}\colon \operatorname{Re} z > \rho\}$ to H that

$$\|U\|_{\mathscr{H}^r(\mathsf{C}_\rho; H)}^2 = \sup_{\sigma > \rho} \int_{-\infty}^{\infty} |\sigma + i\tau|^{2r}\, \|U(\sigma + i\tau)\|_H^2\, d\tau < \infty.$$

In particular we then note that

$$\mathscr{L} A(\partial_t) \begin{bmatrix} u \\ w \end{bmatrix} = A(z)\, \mathscr{L} \begin{bmatrix} u \\ w \end{bmatrix}$$

for all

$$(u, w) \in H_{[0]}^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H_{[0]}^{(s+md+d-\beta-1/2)}(S, \rho, F_{\alpha d+\beta}^t).$$

For the proof of this, as well as for the following result we refer to our considerations in [P, Section 3]; only some modifications are needed.

2.3. THEOREM. *Let $s \geqq 0$ and let $\rho > 0$ be chosen as in 2.1. Assume the operator $A(\partial_t)$ to be parabolic. Then the parameter-elliptic operator $A(z)$ is an isomorphism from*

$$\mathscr{H}^{(s+md)}(\mathsf{C}_\rho, \Omega, E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(s+md+d-\beta-1/2)}(\mathsf{C}_\rho, \Gamma, F_{\alpha d+\beta})$$

onto

$$\mathscr{H}^{(s)}(\mathbf{C}_\rho, \Omega, E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(s+md-\alpha d-\beta-1/2)}(\mathbf{C}_\rho, \Gamma, F_{\alpha d+\beta}).$$

3. Parabolic problems.

3.1. In this main section we suppose that the operator $A(\partial_t)$ is parabolic, and consider the parabolic initial-boundary value problem (1.1–3).

It will turn out important to know how the higher order traces of u and w with respect to t are connected with the data f, g, h, and η. Therefore we shall first state the next theorem giving such a relation. The proof of the result is then technical and will be omitted here. In what follows we write

$$\gamma^j = \gamma_t \partial_t^j \quad \text{for } j \in \mathbf{N} \quad \text{and} \quad \gamma^\nu = \gamma_t I^\nu(\partial_t) \quad \text{for } \nu \in \mathbf{Z}^{md},$$

and, if the identity on $E \oplus F$ is given in the diagonal form $\operatorname{diag}(I_E, I_{F_0}, \ldots, I_{F_{md-1}})$, set

$$I^0 = \operatorname{diag}(I_E, 0, \ldots, 0)$$

and

$$I_k = \operatorname{diag}(0, 0, \ldots, 0, I_{F_k}, 0, \ldots, 0) \quad \text{for } k = 0, \ldots, md-1.$$

3.2. THEOREM. *Suppose that* $s \geqq 0$,

$$(u, w) \in H^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta}),$$

and set

$$\begin{bmatrix} f \\ g \end{bmatrix} = A(\partial_t)\begin{bmatrix} u \\ w \end{bmatrix}.$$

(a) *If* $s > d/2$ *and* $l_0 = \max\{l \in \mathbf{N} : ld < s - d/2\}$, $v^0 = (v^0_{\alpha d+\beta})_{\alpha,\beta} \in \mathbf{N}^{md}$ *with*

$$v^0_{\alpha d+\beta} = \max\{k \in \mathbf{N} : kd < s + md - l_0 d - \alpha d - \beta - 1/2 - d/2\}$$

for $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$, *then we have*

$$\begin{bmatrix} \gamma^{m+l} u \\ \gamma^{\mu+\bar{l}} w \end{bmatrix} = \mathscr{M}^l_0 \begin{bmatrix} f \\ g \end{bmatrix} + \sum_{\kappa=0}^{m-1} \mathscr{N}^{l-\kappa}_0 \sum_{i=0}^{m-1-\kappa} A^{(m-i)} \begin{bmatrix} \gamma^{\kappa+i} u \\ \gamma^{\mu-\overline{(m-\kappa-i)}} w \end{bmatrix}$$

for all v *with* $0 \leqq v \leqq v^0$ i.e., $0 \leqq v_{\alpha d+\beta} \leqq v^0_{\alpha d+\beta}$ *for all* α, β). *Here the operators* $\mathscr{M}^l_\nu$ *and* $\mathscr{N}^l_\nu$ *are defined as follows:*

$$\mathscr{M}^0_0 = \begin{bmatrix} \gamma^0 & 0 \\ 0 & \gamma^0 \end{bmatrix}, \quad \mathscr{M}^{-l}_0 = 0 \quad \text{for } l = 1, 2, \ldots,$$

$$\mathcal{M}_0^l = \begin{bmatrix} \gamma^l & 0 \\ 0 & \gamma^{\bar{l}} \end{bmatrix} - \sum_{j=0}^{m-1} A^{(m-j)} \mathcal{M}_0^{l-m+j} \quad \text{for } l = 1, 2, \ldots,$$

$$\mathcal{M}_\nu^l = I^0 \mathcal{M}_0^l + \sum_{k=0}^{md-1} I_k \mathcal{M}_0^{l+\nu_k} \quad \text{for } l \in \mathbf{Z}, \nu \in \mathbf{N}^{md},$$

and

$$\mathcal{N}_0^0 = -\begin{bmatrix} I_E & 0 \\ 0 & I_F \end{bmatrix}, \ \mathcal{N}_0^{-l} = 0 \quad \text{for } l = 1, 2, \ldots,$$

$$\mathcal{N}_0^l = -\sum_{j=0}^{m-1} A^{(m-j)} \mathcal{N}_0^{l-m+j} \quad \text{for } l = 1, 2, \ldots,$$

$$\mathcal{N}_\nu^l = I^0 \mathcal{N}_0^l + \sum_{k=0}^{md-1} I_k \mathcal{N}_0^{l+\nu_k} \quad \text{for } l \in \mathbf{Z}, \nu \in \mathbf{N}^{md}.$$

(b) *If* $0 \leqq s \leqq d/2$ *and* $0 \leqq \alpha \leqq m-1$, $0 \leqq \beta \leqq d-1$ *(with* $M_{\alpha d+\beta} > 0$*) such that* $s + md - \alpha d - \beta - 1/2 - d/2 > 0$, *then*

$$\gamma^{\alpha+1+k} w_{\alpha d+\beta} = \gamma^k g_{\alpha d+\beta} - \sum_{j=0}^{\alpha} \left(T_{\alpha d+\beta}^{(m-j)} \gamma^{k+j} u + \sum_{\alpha'=m-1-j}^{m-1} \sum_{\beta'=0}^{d-1} R_{\alpha d+\beta, \alpha' d+\beta'}^{(m-j)} \gamma^{k+\alpha'-m+1+j} w_{\alpha' d+\beta'} \right)$$

for all $k = 0, \ldots, k_{\alpha d+\beta}^0$, *where*

$$k_{\alpha d+\beta}^0 = \max \{k \in \mathbf{N} : kd < s + md - \alpha d - \beta - 1/2 - d/2\}.$$

3.3. THEOREM. *Let* $s \geqq 0$ *be given, and let* $\rho > 0$ *be as in* 2.1. *Then the parabolic initial-boundary value problem* (1.1 – 3) *has a solution*

$$(u, w) \in H^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F_{\alpha d+\beta}^t).$$

PROOF. We divide the proof according to the value of s.

3.3.1. First we suppose that $s \not\equiv d/2 \bmod d$ and $s - \beta \not\equiv 1/2 + d/2 \bmod d$ for every β with $M_{\alpha d+\beta} > 0$ for some α.

A. Let $s > d/2$. Let us define

$$h^s = (h_j^s)_j \in \bigoplus_{j=0}^{m+l_0} H^{s+md-jd-d/2}(\Omega, E)$$

and

$$\eta^s = (\eta_{\alpha d+\beta, j}^s)_{\alpha,\beta,j} \in \bigoplus_{\alpha,\beta} \bigoplus_{j=0}^{\alpha+1+l_0+\nu_{\alpha d+\beta}^0} H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma, F_{\alpha d+\beta})$$

by setting

$$h_j^s = h_j \qquad \text{for } j = 0, \ldots, m-1, \tag{3.1}$$

(3.2) $$\eta^s_{\alpha d+\beta,j} = \eta_{\alpha d+\beta,j} \qquad \text{for } j = 0,\dots,\alpha,$$

and further (see Theorem 3.2 for notation and motivation)

(3.3) $$\begin{bmatrix} h^s_{m+l} \\ \eta^s_{\mu+\bar{l}} \end{bmatrix} = \mathscr{M}^l_0 \begin{bmatrix} f \\ g \end{bmatrix} + \sum_{\kappa=0}^{m-1} \mathscr{N}_0^{l-\kappa} \sum_{i=0}^{m-1-\kappa} A^{(m-i)} \begin{bmatrix} h_{\kappa+i} \\ \eta_{\mu-(\overline{m-\kappa-i})} \end{bmatrix}$$

for $l = 0,\dots,l_0$, and

(3.4) $$\begin{bmatrix} h^s_{m+l_0} \\ \eta^s_{\mu+\bar{l}_0+\nu} \end{bmatrix} = \mathscr{M}^{l_0}_\nu \begin{bmatrix} f \\ g \end{bmatrix} + \sum_{\kappa=0}^{m-1} \mathscr{N}_0^{l_0-\kappa} \sum_{i=0}^{m-1-\kappa} A^{(m-i)} \begin{bmatrix} h_{\kappa+i} \\ \eta_{\mu-(\overline{m-\kappa-i})} \end{bmatrix}$$

for $0 \leqq \nu \leqq \nu^0$. Here as well as in what follows we use the notation

$$\eta_\nu = (\eta_{\alpha d+\beta,\nu_{\alpha d+\beta}})_{\alpha,\beta} \quad \text{for } \nu = (\nu_{\alpha d+\beta})_{\alpha,\beta} \in \mathbf{Z}^{md}$$

(also with s), where by definition $\eta_{\alpha d+\beta,\nu_{\alpha d+\beta}} = 0$ for $\nu_{\alpha d+\beta} < 0$. By the trace theorem there exists

$$(u^0, w^0) \in H^{(s+md)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})$$

such that

(3.5) $$\gamma^j u^0 = h^s_j \quad \text{for } j = 0,\dots,m+l_0$$

and, for $\alpha = 0,\dots,m-1$, $\beta = 0,\dots,d-1$,

(3.6) $$\gamma^j w^0_{\alpha d+\beta} = \eta^s_{\alpha d+\beta,j} \quad \text{for } j = 0,\dots,\alpha+1+l_0+\nu^0_{\alpha d+\beta}.$$

Now we set

(3.7) $$\begin{bmatrix} f^0 \\ g^0 \end{bmatrix} = A(\partial_t) \begin{bmatrix} u^0 \\ w^0 \end{bmatrix}.$$

Then, by (3.1)–(3.6) and using Theorem 3.2, we obtain

$$\mathscr{M}^l_0 \begin{bmatrix} f^0 \\ g^0 \end{bmatrix} = \mathscr{M}^l_0 \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{for } l = 0,\dots,l_0$$

and

$$\mathscr{M}^{l_0}_\nu \begin{bmatrix} f^0 \\ g^0 \end{bmatrix} = \mathscr{M}^{l_0}_\nu \begin{bmatrix} f \\ g \end{bmatrix} \quad \text{for } 0 \leqq \nu \leqq \nu^0.$$

Therefore, it follows from the definitions of the operators $\mathscr{M}^l_0$ and $\mathscr{M}^{l_0}_\nu$ (see Theorem 3.2) that

$$(\gamma^l f^0, \gamma^{\bar{l}} g^0) = (\gamma^l f, \gamma^{\bar{l}} g) \quad \text{for } l = 0,\dots,l_0$$

and

$$(\gamma^{l_0} f^0, \gamma^{\bar{l}_0+\nu} g^0) = (\gamma^{l_0} f, \gamma^{\bar{l}_0+\nu} g) \quad \text{for } 0 \leqq \nu \leqq \nu^0.$$

Under the assumptions on s we thus have

$$f - f^0 \in H^{(s)}_{[0]}(Q, \rho, E^t)$$

and

$$g_{\alpha d+\beta} - g^0_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta})$$

for $\alpha = 0, \ldots, m-1, \beta = 0, \ldots, d-1$.

B. If $0 \leqq s < d/2$, then for all $\alpha = 0, \ldots, m-1, \beta = 0, \ldots, d-1$ we make use of the definition (3.2) again, and if

$$\kappa^0_{\alpha d+\beta} = \max\{k \in \mathbf{Z}: kd < s + md - \alpha d - \beta - 1/2 - d/2\} \geqq 0,$$

we define

$$(3.8)\quad \eta^s_{\alpha d+\beta,\alpha+1+k} = \gamma^k g_{\alpha d+\beta} - \sum_{j=0}^{\alpha} \left(T^{(m-j)}_{\alpha d+\beta} h_{k+j} + \sum_{\alpha'=m-1-j}^{m-1} \sum_{\beta'=0}^{d-1} R^{(m-j)}_{\alpha d+\beta,\alpha' d+\beta'} \eta^s_{\alpha' d+\beta',k+\alpha'-m+1+j} \right)$$

for $k = 0, \ldots, \kappa^0_{\alpha d+\beta}$. Hence we have

$$\eta^s = (\eta^s_{\alpha d+\beta,j})_{\alpha,\beta,j} \in \bigoplus_{\alpha,\beta} \bigoplus_{j=0}^{\alpha+1+\kappa^0_{\alpha d+\beta}} H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma, F_{\alpha d+\beta}).$$

Now there exists

$$(u^0, w^0) \in H^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta})$$

such that

$$(3.9)\qquad \gamma^j u^0 = h_j \quad \text{for } j = 0, \ldots, m-1$$

and, for $\alpha = 0, \ldots, m-1, \beta = 0, \ldots, d-1$,

$$(3.10)\qquad \gamma^j w^0_{\alpha d+\beta} = \eta^s_{\alpha d+\beta,j} \quad \text{for } j = 0, \ldots, \alpha + 1 + \kappa^0_{\alpha d+\beta}.$$

Let f^0 and g^0 be defined by (3.7). Then we have immediately

$$f - f^0 \in H^{(s)}_{[0]}(Q, \rho, E^t)$$

and

$$(3.11)\qquad g_{\alpha d+\beta} - g^0_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta})$$

when $\kappa^0_{\alpha d+\beta} = -1$. If $\kappa^0_{\alpha d+\beta} \geqq 0$, then (3.11) follows from (3.8)–(3.10) and Theorem 3.2 (b).

C. Now we continue the consideration jointly. Applying the Laplace transformation $\mathscr{L}$ to $f - f^0$ and $g - g^0$, we have $\mathscr{L}(f - f^0) \in \mathscr{H}^{(s)}(\mathbf{C}_\rho, \Omega, E)$ and $\mathscr{L}(g - g^0) \in \bigoplus_{\alpha,\beta} \mathscr{H}^{(s+md-\alpha d-\beta-1/2)}(\mathbf{C}_\rho, \Gamma, F_{\alpha d+\beta})$. By Theorem 2.3, we can find

$$(U, W) \in \mathscr{H}^{(s+md)}(\mathbf{C}_\rho, \Omega, E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(s+md+d-\beta-1/2)}(\mathbf{C}_\rho, \Gamma, F_{\alpha d+\beta})$$

which satisfies the equation

$$A(z)\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} \mathscr{L}(f - f^0) \\ \mathscr{L}(g - g^0) \end{bmatrix}.$$

By using the inverse transformation $\mathscr{L}^{-1}$ of $\mathscr{L}$, we then have

$$(\mathscr{L}^{-1} U, \mathscr{L}^{-1} W) \in H_{[0]}^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H_{[0]}^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta})$$

such that (see 2.2)

$$A(\partial_t)\begin{bmatrix} \mathscr{L}^{-1} U \\ \mathscr{L}^{-1} W \end{bmatrix} = \begin{bmatrix} f - f^0 \\ g - g^0 \end{bmatrix}.$$

This, however, means that the definition

$$(u, w) = (u^0 + \mathscr{L}^{-1} U, w^0 + \mathscr{L}^{-1} W)$$

gives a solution of the problem.

3.3.2. Next we consider the case $s = k_0 d + d/2$ with $k_0 \in \mathsf{N}$; note that then $s - \beta \not\equiv 1/2 + d/2 \bmod d$ for every $\beta = 0, \dots, d-1$.

We use here the method of [GG–S]: One adds a variable $x_{n+1} \in]0, \infty[= \mathsf{R}_{x_{n+1},+}$ to the space coordinates and considers Ω the boundary of $\tilde{\Omega} = \Omega \times]0, \infty[$, and Q the boundary of $\tilde{Q} = Q \times]0, \infty[$. Let $\tilde{E}$ denote the trivial extension of E over $\tilde{\Omega}$, and $\tilde{E}^t$ the trivial extension of E^t over $\tilde{Q}$. We also use, for $r \geqq 0$, the corresponding spaces

$$H^r(\tilde{\Omega}, \tilde{E}) = H^0(]0, \infty[; H^r(\Omega, E)) \cap H^r(]0, \infty[; H^0(\Omega, E)),$$

$$H^{(r)}(\tilde{Q}; \rho, \tilde{E}^t) = H^0(]0, \infty[; H^{(r)}(Q, \rho, E^t)) \cap H^r(]0, \infty[; H^{(0)}(Q, \rho, E^t)),$$

and

$$H_{[0]}^{(r)}(\tilde{Q}; \rho, \tilde{E}^t) = H^0(]0, \infty[; H_{[0]}^{(r)}(Q, \rho, E^t)) \cap H^r(]0, \infty[; H^{(0)}(Q, \rho, E^t)).$$

Further, define analogously $\tilde{\Gamma}, \tilde{S}, \tilde{F}_{\alpha d+\beta}, \tilde{F}^t_{\alpha d+\beta}$, and the spaces $H^r(\tilde{\Gamma}, \tilde{F}_{\alpha d+\beta})$, $H^{(r)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta})$, $H_{[0]}^{(r)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta})$.

Now, if we let γ_{n+1} denote the usual trace operator with respect to x_{n+1}, then it follows from the trace theorem that there are sections $\tilde{f} \in H^{(s+1/2)}(\tilde{Q}; \rho, \tilde{E}^t)$ and $\tilde{g}_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta})$ such that

$$\gamma_{n+1} \tilde{f} = f \tag{3.12}$$

and

$$\gamma_{n+1} \tilde{g}_{\alpha d+\beta} = g_{\alpha d+\beta} \tag{3.13}$$

for $\alpha = 0, \dots, m-1$, $\beta = 0, \dots, d-1$. Similarly we can find

$$\tilde{h}_j \in H^{s+md-jd-d/2+1/2}(\tilde{\Omega}, \tilde{E}) \quad \text{for } j = 0, \dots, m-1$$

and

$$\tilde{\eta}_{\alpha d+\beta,j} \in H^{s+md+d-\beta-jd-d/2}(\tilde{\Gamma}, \tilde{F}_{\alpha d+\beta}) \quad \text{for } j = 0,\ldots,\alpha$$

such that

$$\gamma_{n+1}\tilde{h}_j = h_j \tag{3.14}$$

and

$$\gamma_{n+1}\tilde{\eta}_{\alpha d+\beta,j} = \eta_{\alpha d+\beta,j} \tag{3.15}$$

for $\alpha = 0,\ldots,m-1, \beta = 0,\ldots,d-1$. We now set (cf. 3.3.1, part A)

$$\tilde{h}^s_j = \tilde{h}_j \qquad \text{for } j = 0,\ldots,m-1, \tag{3.16}$$

$$\tilde{\eta}^s_{\alpha d+\beta,j} = \tilde{\eta}_{\alpha d+\beta,j} \qquad \text{for } j = 0,\ldots,\alpha, \tag{3.17}$$

and define (using the same notation to the natural extensions of the operators)

$$\begin{bmatrix} \tilde{h}^s_{m+l} \\ \tilde{\eta}^s_{\mu+\bar{l}} \end{bmatrix} = \mathcal{M}^l_0 \begin{bmatrix} \tilde{f} \\ \tilde{g} \end{bmatrix} + \sum_{\kappa=0}^{m+1} \mathcal{N}_0^{l-\kappa} \sum_{i=0}^{m-1-\kappa} A^{(m-i)} \begin{bmatrix} \tilde{h}_{\kappa+i} \\ \eta_{\mu-(\overline{m-\kappa-i})} \end{bmatrix} \tag{3.18}$$

for $l = 0,\ldots,k_0$, and further

$$\begin{bmatrix} \tilde{h}^s_{m+k_0} \\ \tilde{\eta}^s_{\mu+\bar{k}_0+\nu} \end{bmatrix} = \mathcal{M}^{k_0}_\nu \begin{bmatrix} \tilde{f} \\ \tilde{g} \end{bmatrix} + \sum_{\kappa=0}^{m-1} \mathcal{N}_\nu^{k_0-\kappa} \sum_{i=0}^{m-1-\kappa} A^{(m-i)} \begin{bmatrix} \tilde{h}_{\kappa+i} \\ \tilde{\eta}_{\mu-(\overline{m-\kappa-i})} \end{bmatrix} \tag{3.19}$$

for $0 \leqq \nu \leqq \mu^0$ with $\mu^0_{\alpha d+\beta} = m-1-\alpha$. Then there exists

$$(\tilde{u}^0, \tilde{w}^0) \in H^{(s+md+1/2)}(\tilde{Q}; \rho, \tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta})$$

such that

$$\gamma^j \tilde{u}^0 = \tilde{h}^s_j \quad \text{for } j = 0,\ldots,m+k_0 \tag{3.20}$$

and, for $\alpha = 0,\ldots,m-1, \beta = 0,\ldots,d-1$,

$$\gamma^j \tilde{w}^0_{\alpha d+\beta} = \tilde{\eta}^s_{\alpha d+\beta,j} \quad \text{for } j = 0,\ldots,m+k_0. \tag{3.21}$$

If we set

$$\begin{bmatrix} \tilde{f}^0 \\ \tilde{g}^0 \end{bmatrix} = A(\partial_t) \begin{bmatrix} \tilde{u}^0 \\ \tilde{w}^0 \end{bmatrix} \in \begin{matrix} H^{(s+1/2)}(\tilde{Q}; \rho, \tilde{E}^t) \\ \oplus \\ \oplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta}) \end{matrix}, \tag{3.22}$$

then we have (cf. Theorem 3.2)

$$\begin{bmatrix} \gamma^{m+l}\tilde{u}^0 \\ \gamma^{\mu+\bar{l}}\tilde{w}^0 \end{bmatrix} = \mathcal{M}^l_0 \begin{bmatrix} \tilde{f}^0 \\ \tilde{g}^0 \end{bmatrix} + \sum_{\kappa=0}^{m-1} \mathcal{N}_0^{l-\kappa} \sum_{i=0}^{m-1-\kappa} A^{(m-i)} \begin{bmatrix} \gamma^{\kappa+i}\tilde{u}^0 \\ \gamma^{\mu-(\overline{m-\kappa-i})}\tilde{w}^0 \end{bmatrix} \tag{3.23}$$

for $l = 0, \ldots, k_0$, and

$$\begin{bmatrix} \gamma^{m+k_0}\tilde{u}^0 \\ \gamma^{\mu+\bar{k}_0+\nu}\tilde{w}^0 \end{bmatrix} = \mathcal{M}_\nu^{k_0}\begin{bmatrix} \tilde{f}^0 \\ \tilde{g}^0 \end{bmatrix} + \sum_{\kappa=0}^{m-1}\mathcal{N}_\nu^{k_0-\kappa}\sum_{i=0}^{m-1-\kappa}A^{(m-i)}\begin{bmatrix} \gamma^{\kappa+i}\tilde{u}^0 \\ \gamma^{\mu-(\overline{m-\kappa-i})}\tilde{w}^0 \end{bmatrix} \tag{3.24}$$

when $0 \leqq \nu \leqq \mu^0$. As in 3.3.1, it now follows from (3.16)–(3.24) that

$$(\gamma^l\tilde{f}^0, \gamma^{\bar{l}}\tilde{g}^0) = (\gamma^l\tilde{f}, \gamma^{\bar{l}}\tilde{g}) \quad \text{for every } l = 0, \ldots, k_0 \tag{3.25}$$

and

$$(\gamma^{k_0}\tilde{f}^0, \gamma^{\bar{k}_0+\nu}\tilde{g}^0) = (\gamma^{k_0}\tilde{f}, \gamma^{\bar{k}_0+\nu}\tilde{g}) \quad \text{for all } \nu, 0 \leqq \nu \leqq \mu^0. \tag{3.26}$$

Therefore, taking into account that $s = k_0 d + d/2$, we have

$$\tilde{f} - \tilde{f}^0 \in H_{[0]}^{(s+1/2)}(\tilde{Q}; \rho, \tilde{E}^t)$$

and

$$\tilde{g}_{\alpha d+\beta} - \tilde{g}_{\alpha d+\beta}^0 \in H_{[0]}^{(s+md-\alpha d-\beta)}(\tilde{S}; \rho, \tilde{F}_{\alpha d+\beta}^t) \quad \text{for } \beta > 0,$$

and hence (cf. [L–M, Chapter 4, p. 10])

$$\gamma_{n+1}(\tilde{f} - \tilde{f}^0) \in H_{[0]}^{(s)}(Q, \rho, E^t) \tag{3.27}$$

and

$$\gamma_{n+1}(\tilde{g}_{\alpha d+\beta} - \tilde{g}_{\alpha d+\beta}^0) \in H_{[0]}^{(s+md-\alpha d-\beta-1/2)}(S, \rho, F_{\alpha d+\beta}^t) \quad \text{for } \beta > 0. \tag{3.28}$$

Now, let us define

$$\begin{bmatrix} u^0 \\ w^0 \end{bmatrix} = \gamma_{n+1}\begin{bmatrix} \tilde{u}^0 \\ \tilde{w}^0 \end{bmatrix} \in \begin{matrix} H^{(s+md)}(Q, \rho, E^t) \\ \oplus \\ \oplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F_{\alpha d+\beta}^t) \end{matrix} \tag{3.29}$$

and

$$\begin{bmatrix} f^0 \\ g^0 \end{bmatrix} = \gamma_{n+1}\begin{bmatrix} \tilde{f}^0 \\ \tilde{g}^0 \end{bmatrix} \in \begin{matrix} H^{(s)}(Q, \rho, E^t) \\ \oplus \\ \oplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta-1/2)}(S, \rho, F_{\alpha d+\beta}^t) \end{matrix}. \tag{3.30}$$

Since γ_{n+1} and $A(\partial_t)$ commute, it follows from (3.22) that

$$A(\partial_t)\begin{bmatrix} u^0 \\ w^0 \end{bmatrix} = \begin{bmatrix} f^0 \\ g^0 \end{bmatrix}.$$

By combining (3.14), (3.16), (3.20), and (3.29), we also get

$$\gamma^j u^0 = h_j \quad \text{for } j = 0, \ldots, m-1,$$

and similarly, for $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$, we have, by (3.15), (3.17), (3.21), and (3.29),

$$\gamma^j w_{\alpha d+\beta}^0 = \eta_{\alpha d+\beta, j} \quad \text{for } j = 0, \ldots, \alpha.$$

Now we deduce from (3.12), (3.27), and (3.30) that

$$f - f^0 \in H_{[0]}^{(s)}(Q, \rho, E^t),$$

and from (3.13), (3.28), and (3.30) that

$$g_{\alpha d+\beta} - g^0_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta}) \quad \text{for } \beta > 0.$$

For $\beta = 0$ it follows from (3.13), (3.25), (3.26), and (3.30) that $\gamma^j(g_{\alpha d} - g^0_{\alpha d}) = 0$ for $j = 0,\dots, m + k_0 - \alpha - 1$. Since $s + md - \alpha d - 1/2 \not\equiv d/2 \bmod d$, this implies that

$$g_{\alpha d} - g^0_{\alpha d} \in H^{(s+md-\alpha d-1/2)}_{[0]}(S, \rho, F^t_{\alpha d}).$$

Therefore, to complete the proof in this case, it suffices to use the same reasoning as in 3.3.1.

3.3.3. Finally we consider the case in which $s - \beta_0 \equiv 1/2 + d/2 \bmod d$ for some β_0 such that $M_{\alpha d+\beta_0} > 0$ for some α. Hence $s = k_0 d + \beta_0 + 1/2 + d/2$ with $k_0 \geqq -1$, which implies $s \not\equiv d/2 \bmod d$.

A. Let us first suppose that $k_0 \geqq 0$. Let $\tilde{f}$, $\tilde{g}_{\alpha d+\beta}$, $\tilde{h}_j$, and $\tilde{\eta}_{\alpha d+\beta,j}$ be given as in 3.3.2.

Then we make use of (3.16)–(3.19) to define $\tilde{h}^s_j$ for $j = 0,\dots, m + k_0$, and $\tilde{\eta}^s_{\alpha d+\beta,j}$ for $\alpha = 0,\dots, m - 1$, $\beta = 0,\dots, d - 1$, $j = 0,\dots,\alpha + 1 + \mu^0_{\alpha d+\beta}$, where this time the component $\mu^0_{\alpha d+\beta}$ of $\mu^0 \in \mathbf{N}^{md}$ is given by

$$\text{(3.31)} \qquad \mu^0_{\alpha d+\beta} = \max\{k \in \mathbf{Z}: kd < (m - \alpha + k_0)d + \beta_0 - \beta + 1/2\}.$$

According to the trace theorem, there exists

$$(\tilde{u}^0, \tilde{w}^0) \in H^{(s+md+1/2)}(\tilde{Q}; \rho, \tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta})$$

such that

$$\gamma^j \tilde{u}^0 = \tilde{h}^s_j \quad \text{for } j = 0,\dots, m + k_0$$

and

$$\gamma^j \tilde{w}^0 = \tilde{\eta}^s_{\alpha d+\beta,j} \quad \text{for } j = 0,\dots,\alpha + 1 + \mu^0_{\alpha d+\beta}.$$

If we set again

$$\begin{bmatrix} \tilde{f}^0 \\ \tilde{g}^0 \end{bmatrix} = A(\partial_t)\begin{bmatrix} \tilde{u}^0 \\ \tilde{w}^0 \end{bmatrix},$$

then the same reasoning as made in 3.3.2 shows that

$$\gamma^j(\tilde{f} - \tilde{f}^0) = 0 \quad \text{for } j = 0,\dots, k_0$$

and, for $\alpha = 0,\dots, m - 1$, $\beta = 0,\dots, d - 1$,

$$\gamma^j(\tilde{g}_{\alpha d+\beta} - \tilde{g}^0_{\alpha d+\beta}) = 0 \quad \text{for } j = 0,\dots, \mu^0_{\alpha d+\beta}.$$

Since now $s \not\equiv d/2 \bmod d$ and

$$\gamma_{n+1}(\tilde{f} - \tilde{f}^0) \in H^{(s)}(Q, \rho, E^t)$$

with $\gamma^j \gamma_{n+1}(\tilde{f} - \tilde{f}^0) = 0$ for $j = 0, \ldots, k_0$, it follows that

$$\gamma_{n+1}(\tilde{f} - \tilde{f}^0) \in H^{(s)}_{[0]}(Q, \rho, E^t).$$

Further, by assumption on s, we obtain

$$\tilde{g}_{\alpha d+\beta} - \tilde{g}^0_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta)}_{[0]}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta}),$$

so that (cf. [L–M, Chapter 4, p. 10])

$$\gamma_{n+1}(\tilde{g}_{\alpha d+\beta} - \tilde{g}^0_{\alpha d+\beta}) \in H^{(s+md-\alpha d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta}).$$

Next we define $(u^0, w^0) = \gamma_{n+1}(\tilde{u}^0, \tilde{w}^0)$ and $(f^0, g^0) = \gamma_{n+1}(\tilde{f}^0, \tilde{g}^0)$. As in 3.3.2, it then follows that we have

$$A(\partial_t)\begin{bmatrix} u^0 \\ w^0 \end{bmatrix} = \begin{bmatrix} f^0 \\ g^0 \end{bmatrix}$$

such that

$$\gamma^j u^0 = h_j \qquad \text{for } j = 0, \ldots, m-1,$$

$$\gamma^j w^0_{\alpha d+\beta} = \eta_{\alpha d+\beta, j} \qquad \text{for } j = 0, \ldots, \alpha,$$

and

$$f - f^0 \in H^{(s)}_{[0]}(Q, \rho, E^t),$$

$$g_{\alpha d+\beta} - g^0_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta})$$

for $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$.

B. In the case $k_0 = -1$ we have $s = \beta_0 + 1/2 - d/2$; note that then $\beta_0 \geqq d/2$. Assume again that $\tilde{f}$, $\tilde{g}_{\alpha d+\beta}$, $\tilde{h}_j$, and $\tilde{\eta}_{\alpha d+\beta, j}$ are given as in 3.3.2.

Now we set, for $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$,

(3.32) $$\tilde{\eta}^s_{\alpha d+\beta, j} = \tilde{\eta}_{\alpha d+\beta, j} \quad \text{for } j = 0, \ldots, \alpha,$$

and then, if $\mu^0_{\alpha d+\beta} \geqq 0$ (see (3.31)), we define

(3.33) $$\tilde{\eta}^s_{\alpha d+\beta, \alpha+1+k} = \gamma^k \tilde{g}_{\alpha d+\beta} - \sum_{j=0}^{\alpha} \left(T^{(m-j)}_{\alpha d+\beta} \tilde{h}_{k+j} + \sum_{\alpha'=m-1-j}^{m-1} \sum_{\beta'=0}^{d-1} R^{(m-j)}_{\alpha d+\beta, \alpha' d+\beta'} \tilde{\eta}_{\alpha' d+\beta', k+\alpha'-m+1+j} \right)$$

for $k = 0, \ldots, \mu^0_{\alpha d+\beta}$. By the trace theorem we find

$$(\tilde{u}^0, \tilde{w}^0) \in H^{(s+md+1/2)}(\tilde{Q}; \rho, \tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta})$$

such that

$$\gamma^j \tilde{u}^0 = \tilde{h}_j \quad \text{for } j = 0, \dots, m-1 \tag{3.34}$$

and

$$\gamma^j \tilde{w}^0_{\alpha d+\beta} = \tilde{\eta}^s_{\alpha d+\beta, j} \quad \text{for } j = 0, \dots, \alpha + 1 + \mu^0_{\alpha d+\beta}. \tag{3.35}$$

Define

$$\begin{bmatrix} \tilde{f}^0 \\ \tilde{g}^0 \end{bmatrix} = A(\partial_t) \begin{bmatrix} \tilde{u}^0 \\ \tilde{w}^0 \end{bmatrix}.$$

When $\mu^0_{\alpha d+\beta} \geqq 0$ it follows from (3.32)–(3.35) that (see 3.3.2 and Theorem 3.2 (b))

$$\gamma^j \tilde{g}^0_{\alpha d+\beta} = \gamma^j \tilde{g}_{\alpha d+\beta} \quad \text{for every } j = 0, \dots, \mu^0_{\alpha d+\beta}.$$

Since $s + md - \alpha d - \beta \not\equiv d/2 \bmod d$, this implies that

$$\tilde{g}_{\alpha d+\beta} - \tilde{g}^0_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta)}_{[0]} (\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta}). \tag{3.36}$$

If $\mu^0_{\alpha d+\beta} = -1$, then (3.36) holds immediately.

As in part A, we now define $(u^0, w^0) = \gamma_{n+1}(\tilde{u}^0, \tilde{w}^0)$ and $(f^0, g^0) = \gamma_{n+1}(\tilde{f}^0, \tilde{g}^0)$, and have then

$$A(\partial_t) \begin{bmatrix} u^0 \\ w^0 \end{bmatrix} = \begin{bmatrix} f^0 \\ g^0 \end{bmatrix}$$

such that

$$\gamma^j u^0 = h_j \quad \text{for } j = 0, \dots, m-1$$

and, for $\alpha = 0, \dots, m-1$, $\beta = 0, \dots, d-1$,

$$\gamma^j w^0_{\alpha d+\beta} = \eta_{\alpha d+\beta, j} \quad \text{for } j = 0, \dots, \alpha.$$

Moreover, since $s < d/2$,

$$f - f^0 \in H^{(s)}_{[0]}(Q, \rho, E^t),$$

and from (3.36) it follows that

$$g_{\alpha d+\beta} - g^0_{\alpha d+\beta} \in H^{(s+md-\alpha d-\beta-1/2)}_{[0]} (S, \rho, F^t_{\alpha d+\beta})$$

for $\alpha = 0, \dots, m-1, \beta = 0, \dots, d-1$.

C. The proof in this case can now be completed with the same reasoning as in 3.3.1.

The theorem is proved.

3.4. We shall next show that the parabolic initial-boundary value problem (1.1–3) can admit at most one solution (u, w). This will be a consequence of the following a priori estimate for a solution in the case $s = 0$.

3.5 THEOREM. *For $s = 0$ let $\rho > 0$ be chosen as in 2.1. Then there is a constant $C > 0$ such that the inequality*

$$\|(u, w)\|_{H^{(md)}(Q,\rho,E^t)\oplus \bigoplus_{\alpha,\beta} H^{(md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})}$$
$$\leqq C\Bigg(\Bigg\|A(\partial_t)\begin{bmatrix} u \\ w \end{bmatrix}\Bigg\|_{H^{(0)}(Q,\rho,E^t)\oplus \bigoplus_{\alpha,\beta} H^{(md-\alpha d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})}$$
$$+ \sum_{j=0}^{m-1} \|\gamma^j u\|_{H^{md-jd-d/2}(\Omega,E)}$$
$$+ \sum_{\alpha,\beta} \sum_{j=0}^{\alpha} \|\gamma^j w_{\alpha d+\beta}\|_{H^{md+d-\beta-jd-1/2-d/2}(\Gamma,F_{\alpha d+\beta})}\Bigg)$$

holds for all

$$(u, w) \in H^{(md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta}).$$

PROOF. For brevity we set

$$\begin{bmatrix} f \\ g \end{bmatrix} = A(\partial_t)\begin{bmatrix} u \\ w \end{bmatrix}, \tag{3.37}$$

$$h_j^0 = \gamma^j u \quad \text{for } j = 0, \ldots, m-1, \tag{3.38}$$

and, for all $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$,

$$\eta^0_{\alpha d+\beta,j} = \gamma^j w_{\alpha d+\beta} \quad \text{for } j = 0, \ldots, \alpha + 1 + \kappa^0_{\alpha d+\beta}, \tag{3.39}$$

where $\kappa^0_{\alpha d+\beta} = \max\{k \in \mathbf{Z} : kd < md - \alpha d - \beta - 1/2 - d/2\} \geqq -1$. When $\kappa^0_{\alpha d+\beta} \geqq 0$, it follows from Theorem 3.2(b) that

$$\begin{aligned} \eta^0_{\alpha d+\beta,\alpha+1+k} &= \gamma^k g_{\alpha d+\beta} \\ &- \sum_{j=0}^{\alpha}\Bigg(T^{(m-j)}_{\alpha d+\beta}\gamma^{k+j}u + \sum_{\alpha'=m-1-j}^{m-1}\sum_{\beta'=0}^{d-1} R^{(m-j)}_{\alpha d+\beta,\alpha' d+\beta'}\gamma^{k+\alpha'-m+1+j}w_{\alpha' d+\beta'}\Bigg) \end{aligned} \tag{3.40}$$

for every $k = 0, \ldots, \kappa^0_{\alpha d+\beta}$. By the trace theorem, we now find $u^0 \in H^{(md)}(Q, \rho, E^t)$ such that $\gamma^j u^0 = h_j^0$ for $j = 0, \ldots, m-1$ in a continuous way, that is, the estimate

$$\|u^0\|_{H^{(md)}(Q,\rho,E^t)} \leqq C \sum_{j=0}^{m-1} \|h_j^0\|_{H^{md-jd-d/2}(\Omega,E)} \tag{3.41}$$

is satisfied with some constant $C > 0$.

Similarly, for any $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$, there exists $w^0_{\alpha d+\beta} \in H^{(md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta})$ such that $\gamma^j w^0_{\alpha d+\beta} = \eta^0_{\alpha d+\beta,j}$ for $j = 0, \ldots, \alpha + 1 + \kappa^0_{\alpha d+\beta}$ and

$$\begin{aligned} &\|w^0_{\alpha d+\beta}\|_{H^{(md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\leqq C \sum_{j=0}^{\alpha+1+\kappa^0_{\alpha d+\beta}} \|\eta^0_{\alpha d+\beta,j}\|_{H^{md+d-\beta-jd-1/2-d/2}(\Gamma,F_{\alpha d+\beta})}. \end{aligned} \tag{3.42}$$

Therefore, it follows that

$$(u - u^0, w - w^0) \in H^{(md)}_{[0]}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(md+d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta})$$

with $w^0 = (w^0_{\alpha d+\beta})_{\alpha,\beta}$, and hence

$$(\mathscr{L}(u - u^0), \mathscr{L}(w - w^0)) \in \mathscr{H}^{(md)}(\mathrm{C}_\rho, \Omega, E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(md+d-\beta-1/2)}(\mathrm{C}_\rho, \Gamma, F_{\alpha d+\beta}).$$

In view of Theorem 2.3, we now have the estimate (note that the symbol C denotes a generic positive constant)

$$\begin{aligned} &\|(\mathscr{L}(u - u^0), \mathscr{L}(w - w^0))\|_{\mathscr{H}^{(md)}(\mathrm{C}_\rho, \Omega, E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(md+d-\beta-1/2)}(\mathrm{C}_\rho, \Gamma, F_{\alpha d+\beta})} \\ &\leqq C \left\| A(z) \begin{bmatrix} \mathscr{L}(u - u^0) \\ \mathscr{L}(w - w^0) \end{bmatrix} \right\|_{\mathscr{H}^{(0)}(\mathrm{C}_\rho, \Omega, E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(md-\alpha d-\beta-1/2)}(\mathrm{C}_\rho, \Gamma, F_{\alpha d+\beta})}, \end{aligned}$$

and hence

$$\begin{aligned} &\|(u - u^0, w - w^0)\|_{H^{(md)}_{[0]}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(md+d-\beta-1/2)}_{[0]}(S,\rho,F^t_{\alpha d+\beta})} \\ &\leqq C \left\| A(\partial_t) \begin{bmatrix} u - u^0 \\ w - w^0 \end{bmatrix} \right\|_{H^{(0)}_{[0]}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(md-\alpha d-\beta-1/2)}_{[0]}(S,\rho,F^t_{\alpha d+\beta})}. \end{aligned}$$

From this we then derive the inequality

$$\begin{aligned} &\|(u, w)\|_{H^{(md)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\leqq C \Big(\|(u^0, w^0)\|_{H^{(md)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\qquad + \|(f, g)\|_{H^{(0)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(md-\alpha d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \Big). \end{aligned}$$

By combining the above inequality with (3.41) and (3.42), and using the equations (3.37)–(3.40), we obtain the desired estimate.

3.6. Remark. Essentially the same reasoning yields the corresponding a priori estimate for all $0 \leqq s < d/2$ with $s - \beta \not\equiv 1/2 + d/2 \bmod d$ for every β (with $M_{\alpha d+\beta} > 0$ for some α).

3.7. Theorem. *Let $s \geqq 0$ be given. Then for all $\rho > 0$ sufficiently large the parabolic initial-boundary value problem* (1.1–3) *has only one solution*

$$(u, w) \in H^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta}).$$

Proof. In view of Theorem 3.3, this is a corollary of Theorem 3.5.

3.8. THEOREM. *Let $s \geqq 0$ be given, and let $\rho > 0$ be as in Theorem 3.7. Then the parabolic operator $A(\partial_t)$ satisfies the a priori estimate*

$$\|(u, w)\|_{H^{(s+md)}(Q,\rho,E^t) \oplus \oplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \tag{3.43}$$

$$\leqq C\left(\left\|A(\partial_t)\begin{bmatrix} u \\ w \end{bmatrix}\right\|_{H^{(s)}(Q,\rho,E^t) \oplus \oplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})}\right.$$

$$+ \sum_{j=0}^{m-1} \|\gamma^j u\|_{H^{s+md-jd-d/2}(\Omega,E)}$$

$$\left. + \sum_{\alpha,\beta} \sum_{j=0}^{\alpha} \|\gamma^j w_{\alpha d+\beta}\|_{H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma,F_{\alpha d+\beta})}\right)$$

for all

$$(u, w) \in H^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta})$$

with a positive constant C.

PROOF. Let

$$(u, w) \in H^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta})$$

be given. To simplify notation, we write

$$\begin{bmatrix} f \\ g \end{bmatrix} = A(\partial_t)\begin{bmatrix} u \\ w \end{bmatrix},$$

$$h_j = \gamma^j u \quad \text{for } j = 0, \ldots, m-1,$$

and, for $\alpha = 0, \ldots, m-1$, $\beta = 0, \ldots, d-1$,

$$\eta_{\alpha d+\beta,j} = \gamma^j w_{\alpha d+\beta} \quad \text{for } j = 0, \ldots, \alpha.$$

Note that in this proof the symbol C is used to denote a generic positive constant.

3.8.1. Suppose first that $s \not\equiv d/2 \bmod d$ and $s - \beta \not\equiv 1/2 + d/2 \bmod d$ for every $\beta = 0, \ldots, d-1$ with $M_{\alpha d+\beta} > 0$ for some α. In view of Remark 3.6, it is then enough to consider the case $s > d/2$.

Now, let the integers l_0 and $v^0_{\alpha d+\beta}$ be given as in Theorem 3.2(a). Then the traces

$$h^s_j = \gamma^j u \in H^{s+md-jd-d/2}(\Omega, E) \quad \text{for } j = 0, \ldots, m + l_0 \tag{3.44}$$

and

$$\eta^s_{\alpha d+\beta,j} = \gamma^j w_{\alpha d+\beta} \in H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma, F_{\alpha d+\beta}) \tag{3.45}$$

$$\text{for } j = 0, \ldots, \alpha + 1 + l_0 + v^0_{\alpha d+\beta}$$

are well-defined. By the trace theorem, there exists

$$(u^0, w^0) \in H^{(s+md)}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S, \rho, F^t_{\alpha d+\beta})$$

such that

$$\gamma^j u^0 = h^s_j \quad \text{for } j = 0, \dots, m + l_0, \tag{3.46}$$

$$\gamma^j w^0_{\alpha d+\beta} = \eta^s_{\alpha d+\beta,j} \quad \text{for } j = 0, \dots, \alpha + 1 + l_0 + v^0_{\alpha d+\beta}, \tag{3.47}$$

and furthermore,

$$\|u^0\|_{H^{(s+md)}(Q,\rho,E^t)} \leqq C \sum_{j=0}^{m+l_0} \|h^s_j\|_{H^{s+md-jd-d/2}(\Omega,E)} \tag{3.48}$$

and

$$\begin{aligned} &\|w^0_{\alpha d+\beta}\|_{H^{(s+md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\leqq C \sum_{j=0}^{\alpha+1+l_0+v^0_{\alpha d+\beta}} \|\eta^s_{\alpha d+\beta,j}\|_{H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma,F_{\alpha d+\beta})} \end{aligned} \tag{3.49}$$

for $\alpha = 0, \dots, m-1, \beta = 0, \dots, d-1$. From (3.44)–(3.47) we now get

$$(u - u^0, w - w^0) \in H^{(s+md)}_{[0]}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta}).$$

Therefore, using the Laplace transformation and Theorem 2.3, we thus obtain the inequality (cf. the proof of Theorem 3.5)

$$\begin{aligned} &\|(u, w)\|_{H^{(s+md)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\leqq C\Big(\|(f, g)\|_{H^{(s)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\qquad + \|(u^0, w^0)\|_{H^{(s+md)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})}\Big). \end{aligned} \tag{3.50}$$

Here it follows from (3.44), (3.45), (3.48), (3.49), and Theorem 3.2 that the latter term on the right-hand side of (3.50) can be estimated by the right-hand side of the required inequality (3.43), which then implies the assertion.

3.8.2. Let us next suppose that $s = k_0 d + d/2$ with $k_0 \in \mathbf{N}$. We first observe (cf. 3.3.2) that there exists

$$(\tilde{f}, \tilde{g}) \in H^{(s+1/2)}(\tilde{Q}; \rho, \tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta)}(\tilde{S}; \rho, \tilde{F}^t_{\alpha d+\beta})$$

which satisfies the equation

$$\gamma_{n+1}(\tilde{f}, \tilde{g}) = (f, g) \tag{3.51}$$

and the inequalities

$$\|\tilde{f}\|_{H^{(s+1/2)}(\tilde{Q};\rho,\tilde{E}^t)} \leqq C\|f\|_{H^{(s)}(Q,\rho,E^t)} \tag{3.52}$$

and

$$\|\tilde{g}_{\alpha d+\beta}\|_{H^{(s+md-\alpha d-\beta)}(\tilde{S};\rho,\tilde{F}^t_{\alpha d+\beta})} \leqq C\|g_{\alpha d+\beta}\|_{H^{(s+md-\alpha d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \tag{3.53}$$

for $\alpha = 0,\ldots,m-1$, $\beta=0,\ldots,d-1$. Analogously we find

$$\tilde{h}_j \in H^{s+md-jd-d/2+1/2}(\tilde{\Omega},\tilde{E}) \quad \text{for } j = 0,\ldots,m-1$$

such that

$$\gamma_{n+1}\tilde{h}_j = h_j \tag{3.54}$$

and

$$\|\tilde{h}_j\|_{H^{s+md-jd-d/2+1/2}(\tilde{\Omega},\tilde{E})} \leqq C\|h_j\|_{H^{s+md-jd-d/2}(\Omega,E)}, \tag{3.55}$$

and further $\tilde{\eta}_{\alpha d+\beta,j} \in H^{s+md+d-\beta-jd-d/2}(\tilde{\Gamma},\tilde{F}_{\alpha d+\beta})$ such that

$$\gamma_{n+1}\tilde{\eta}_{\alpha d+\beta,j} = \eta_{\alpha d+\beta,j} \tag{3.56}$$

and

$$\|\tilde{\eta}_{\alpha d+\beta,j}\|_{H^{s+md+d-\beta-jd-d/2}(\tilde{\Gamma},\tilde{F}_{\alpha d+\beta})} \leqq C\|\eta_{\alpha d+\beta,j}\|_{H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma,F_{\alpha d+\beta})} \tag{3.57}$$

for $\alpha = 0,\ldots,m-1$, $\beta = 0,\ldots,d-1$, $j = 0,\ldots,\alpha$.

For these $\tilde{f}$, $\tilde{g}$, $\tilde{h}_j$, and $\tilde{\eta}_{\alpha d+\beta,j}$ we define

$$\tilde{h}^s_j \in H^{s+md-jd-d/2+1/2}(\tilde{\Omega},\tilde{E}) \quad \text{for } j = 0,\ldots,m+k_0$$

and

$$\tilde{\eta}^s_{\alpha d+\beta,j} \in H^{s+md+d-\beta-jd-d/2}(\tilde{\Gamma},\tilde{F}_{\alpha d+\beta}) \quad \text{for } j = 0,\ldots,m+k_0$$

by the equations (3.16)–(3.19). Then there exists

$$(\tilde{u}^0,\tilde{w}^0) \in H^{(s+md+1/2)}(\tilde{Q};\rho,\tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta)}(\tilde{S};\rho,\tilde{F}^t_{\alpha d+\beta})$$

such that

$$\gamma^j\tilde{u}^0 = \tilde{h}^s_j \quad \text{for } j = 0,\ldots,m+k_0, \tag{3.58}$$

$$\gamma^j\tilde{w}^0_{\alpha d+\beta} = \tilde{\eta}^s_{\alpha d+\beta,j} \quad \text{for } j = 0,\ldots,m+k_0, \tag{3.59}$$

and

$$\|\tilde{u}^0\|_{H^{(s+md+1/2)}(\tilde{Q};\rho,\tilde{E}^t)} \leqq C\sum_{j=0}^{m+k_0}\|\tilde{h}^s_j\|_{H^{s+md-jd-d/2+1/2}(\tilde{\Omega},\tilde{E})}, \tag{3.60}$$

$$\|\tilde{w}^0_{\alpha d+\beta}\|_{H^{(s+md+d-\beta)}(\tilde{S};\rho,F^t_{\alpha d+\beta})} \leqq C\sum_{j=0}^{m+k_0}\|\tilde{\eta}^s_{\alpha d+\beta,j}\|_{H^{s+md+d-\beta-jd-d/2}(\tilde{\Gamma},\tilde{F}_{\alpha d+\beta})}. \tag{3.61}$$

Now we set

$$\begin{bmatrix} \tilde{f} \\ \tilde{g}^0 \end{bmatrix} = A(\partial_t)\begin{bmatrix} \tilde{u}^0 \\ \tilde{w}^0 \end{bmatrix} \tag{3.62}$$

and

$$(u^0, w^0) = \gamma_{n+1}(\tilde{u}^0, \tilde{w}^0), \quad (f^0, g^0) = \gamma_{n+1}(\tilde{f}^0, \tilde{g}^0). \tag{3.63}$$

Then it can be seen as in 3.3.2 that

$$A(\partial_t)\begin{bmatrix} u^0 \\ w^0 \end{bmatrix} = \begin{bmatrix} f^0 \\ g^0 \end{bmatrix}, \tag{3.64}$$

$$\gamma^j u^0 = h_j \quad \text{for } j = 0, \dots, m-1, \tag{3.65}$$

$$\gamma^j w^0_{\alpha d+\beta} = \eta_{\alpha d+\beta, j} \quad \text{for } j = 0, \dots, \alpha, \tag{3.66}$$

and, furthermore, that

$$(f - f^0, g - g^0) \in H^{(s)}_{[0]}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta}).$$

According to Theorem 2.3, there exists now

$$(U, W) \in \mathscr{H}^{(s+md)}(\mathrm{C}_\rho, \Omega, E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(s+md+d-\beta-1/2)}(\mathrm{C}_\rho, \Gamma, F_{\alpha d+\beta})$$

such that

$$A(z)\begin{bmatrix} U \\ W \end{bmatrix} = \begin{bmatrix} \mathscr{L}(f - f^0) \\ \mathscr{L}(g - g^0) \end{bmatrix}$$

and

$$\begin{aligned} &\|(U, W)\|_{\mathscr{H}^{(s+md)}(\mathrm{C}_\rho,\Omega,E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(s+md+d-\beta-1/2)}(\mathrm{C}_\rho,\Gamma,F_{\alpha d+\beta})} \\ &\leqq C\|(\mathscr{L}(f - f^0), \mathscr{L}(g - g^0))\|_{\mathscr{H}^{(s)}(\mathrm{C}_\rho,\Omega,E) \oplus \bigoplus_{\alpha,\beta} \mathscr{H}^{(s+md-\alpha d-\beta-1/2)}(\mathrm{C}_\rho,\Gamma,F_{\alpha d+\beta})}. \end{aligned} \tag{3.67}$$

Thus we have the equation

$$A(\partial_t)\begin{bmatrix} \mathscr{L}^{-1} U \\ \mathscr{L}^{-1} W \end{bmatrix} = \begin{bmatrix} f - f^0 \\ g - g^0 \end{bmatrix}$$

with

$$(\mathscr{L}^{-1} U, \mathscr{L}^{-1} W) \in H^{(s+md)}_{[0]}(Q, \rho, E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}_{[0]}(S, \rho, F^t_{\alpha d+\beta}).$$

But then, by Theorem 3.7, it follows from (3.64)–(3.66) that $(\mathscr{L}^{-1} U, \mathscr{L}^{-1} W) = (u - u^0, w - w^0)$. Therefore, by using (3.63), (3.67), (3.51)–(3.53), and (3.62), we get

$$\begin{aligned} &\|(u, w)\|_{H^{(s+md)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\leqq C\Big(\|(f, g)\|_{H^{(s)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\ &\qquad + \|(\tilde{u}^0, \tilde{w}^0)\|_{H^{(s+md+1/2)}(\tilde{Q};\rho,\tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta)}(\tilde{S};\rho,\tilde{F}^t_{\alpha d+\beta})}\Big), \end{aligned}$$

where it follows from (3.60), (3.61), (3.16)–(3.19), and (3.52), (3.53), (3.55), (3.57) that

$$\begin{aligned}
&\|(\tilde{u}^0, \tilde{w}^0)\|_{H^{(s+md+1/2)}(\tilde{Q};\rho,\tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md+d-\beta)}(\tilde{S};\rho,\tilde{F}^t_{\alpha d+\beta})} \\
&\leqq C\Bigg(\|(\tilde{f},\tilde{g})\|_{H^{(s+1/2)}(\tilde{Q};\rho,\tilde{E}^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta)}(\tilde{S};\rho,\tilde{F}^t_{\alpha d+\beta})} \\
&\qquad + \sum_{j=0}^{m-1} \|\tilde{h}_j\|_{H^{s+md-jd-d/2+1/2}(\tilde{\Omega},\tilde{E})} \\
&\qquad + \sum_{\alpha,\beta} \sum_{j=0}^{\alpha} \|\tilde{\eta}_{\alpha d+\beta,j}\|_{H^{s+md+d-\beta-jd-d/2}(\tilde{\Gamma},\tilde{F}_{\alpha d+\beta})} \Bigg) \\
&\leqq C\Bigg(\|(f,g)\|_{H^{(s)}(Q,\rho,E^t) \oplus \bigoplus_{\alpha,\beta} H^{(s+md-\alpha d-\beta-1/2)}(S,\rho,F^t_{\alpha d+\beta})} \\
&\qquad + \sum_{j=0}^{m-1} \|h_j\|_{H^{s+md-jd-d/2}(\Omega,E)} \\
&\qquad + \sum_{\alpha,\beta} \sum_{j=0}^{\alpha} \|\eta_{\alpha d+\beta,j}\|_{H^{s+md+d-\beta-jd-1/2-d/2}(\Gamma,F_{\alpha d+\beta})} \Bigg).
\end{aligned}$$

This clearly implies the desired inequality.

3.8.3. It remains to consider the case in which $s = k_0 d + \beta_0 + 1/2 + d/2$ with $k_0 \geqq -1$ for some β_0 such that $M_{\alpha d+\beta} > 0$ for some α. However, if we proceed essentially as in 3.8.2, modifying now the reasonings in accordance with 3.3.3, we obtain the required estimate in this case, too.

REFERENCES

[GG] G. Grubb, *Functional Calculus of Pseudo-Differential Boundary Problems*, Progress in Mathematics 65, Birkhäuser, Boston, 1986.

[GG–S] G. Grubb and V. A. Solonnikov, *Solution of parabolic pseudo-differential initial-boundary value problems*, J. Differential Equations (to appear).

[L–M] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications I, II*, Grundlehren der mathematischen Wissenschaften 181, 182, Springer-Verlag, Berlin-Heidelberg, 1972.

[P] V. T. Purmonen, *Parabolic pseudo-differential initial-boundary value problems*, Math. Scand. 65 (1989), 221–244.

UNIVERSITY OF JYVÄSKYLÄ
DEPARTMENT OF MATHEMATICS
SEMINAARINKATU 15
SF–40100 JYVÄSKYLÄ
FINLAND