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ON THE CAUSAL STRUCTURE OF HOMOGENEOUS MANIFOLDS

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1. Introduction and results.

In a Lorentzian manifold with orientation there is a continuous choice of one half of the double cone determined by the Lorentzian metric in each tangent space. We call the chosen cone at a point $m \in M$ the *forward light cone*. A piecewise differentiable curve γ: $I \to M$, where I is any interval in $\mathbb{R}$, is called *causal* if the derivative $\gamma'(t)$, whenever it exists, is contained in the forward light cone. The manifold M is called *causal* if there are no non-trivial closed causal curves in M and *homogeneous* if there is a transitive orientation preserving Lie group action on M preserving the Lorentzian structure. In that case M is of the form G/H, where G and H are Lie groups, and the Lorentzian metric and the orientation is invariant under the action of G. In particular, the future light cone at $m_0 = H$ is invariant under the action of H. We call any M as described above a *homogeneous Lorentzian manifold.*

There is much literature on causality. Let us first state *our* objectives and explain our new results.

The causal data.

We shall describe new criteria for a homogeneous Lorentzian manifold to be causal. Their basis is the recently developed Lie theory of *semigroups*. The standard reference is [8]. Our methods do not only work for Lorentzian manifolds. Even in that context they free us from the somewhat cumbersome assumption that the Lorentzian metric be invariant; what is really required is that the automorphism group of the manifold preserves future light cones. Throughout this paper we deal with the following situation:

1.1 DEFINITION. We say that a triple (G, H, W) is a set of *causal data* if G is a connected Lie group, H a closed subgroup and W is a wedge (that is, a closed

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convex cone which need not be pointed) in the Lie algebra $\mathfrak{g}$ of G which satisfies the following conditions:

(i) The Lie algebra $\mathfrak{h}$ of H is the largest vector space contained in W.
(ii) The wedge W is invariant under H, that is, $\mathrm{Ad}(h)W = W$ for all $h \in H$.

If we use the right translations to trivialize the tangent bundle of G as usual, then $\mathfrak{g}$ and $T_g(G)$ become identified under the isomorphism $d\rho_g(1)$: $\mathfrak{g} \to T_g(G)$, $\rho_g(x) = xg$, and the vector space automorphism $d\rho_g(1)^{-1}\, d\lambda_g(1)$ is in fact the Lie algebra automorphism $\mathrm{Ad}(g)$. Thus $T(G)$ may be viewed as the semidirect product $\mathfrak{g} \rtimes G$ with G acting on $\mathfrak{g}$ under the adjoint action so that $T_g(G) = \mathfrak{g} \times \{g\}$ and $d\lambda_g(1)(X, 1) = (\mathrm{Ad}(g)(X), g)$, $d\rho_g(1)(X, 1) = (X, g)$. Accordingly, the tangent bundle $T(M)$ may be identified with $(\mathfrak{g}/\mathfrak{h}) \times M$ in such a fashion that the quotient map π: $G \to G/H$, $\pi(g) = gH$ has the differential given by $d\pi(X, g) = (X + \mathfrak{h}, gH)$. If $m = gH$ is an arbitrary point of M, then the left translation λ_g: $G \to G$ induces a diffeomorphism μ_g: $M \to M$ given by $\mu_g(g'H) = gg'H$. Any other left translation mapping $m_0 = H$ to m is given by λ_{gh} with some $h \in H$. The vector space automorphism $d\mu_{gh}(m_0)^{-1}\, d\mu_g(m_0)$ is given by $(X + \mathfrak{h}) \mapsto (\mathrm{Ad}(h)(X) + \mathfrak{h}$.

The left translations allow the transport of the wedge: $d\lambda_g(1)(W \times \{1\}) = \mathrm{Ad}(g)W \times \{g\}$, and $d\lambda_{gh}(1)^{-1} d\lambda_g(1)(W \times \{1\}) = \mathrm{Ad}(h)W \times \{h\} = W \times \{h\}$ in view of Definition 1.1 (ii) for all $\mathfrak{h} \in H$. Therefore the wedge $d\pi(g)\, d\lambda_g(1)(W \times \{1\})$ is independent of the representative g of $m = gH$ and depends only on m; it is therefore justified to call this wedge W_m, and we note $W_m = d\mu_g(m_0)(W_{m_0}) = d\mu_{gh}(m_0)(W_{m_0})$ for all $h \in H$. By Definition 1.1 (i), the wedge W_m is *pointed*, i.e. does not contain non-trivial vector subspaces. It corresponds to the forward light cone at the point m, and we therefore call the assignment $m \mapsto W_m$ *the causal structure of M defined by the data* (G, H, W). A *causal trajectory* is a piecewise smooth map γ: $[0, 1] \to M$ with $\gamma(0) = m_0$ and $\dot{\gamma}(t) \in W_{\gamma(t)}$. We say that M is *causal* if there are no non-constant closed causal trajectories and that M is *totally acausal* if every point of M can be reached by a causal trajectory.

1.2. Definition. A *W-admissible trajectory* or a *W-admissible chain* in G (cf. [8], Definition VI.1.7) is a piecewise differentiable curve γ with $\dot{\gamma}(t) \in d\lambda_{\gamma(t)}(1)(W)$ for all t for which the derivative exists and with $\gamma(0) = 1$. We shall denote the set of all endpoints of W-admissible trajectories by $S(W)$.

Let $J^+_{m_0}$ denote the set of endpoints of all causal trajectories. The crucial concept in our discussion is the pull back

$$S = S_{(G,H,W)} \overset{\mathrm{def}}{=} \pi^{-1}(J^+_{m_0}),$$

The following proposition points out how we determine whether M is causal or not:

1.3. PROPOSITION. (i) *For a set* (G, H, W) *of causal data, the subsets* $S \subset G$ *and* $S(W)$ *are subsemigroups, the latter is invariant under all inner automorphisms by elements of* H *and* $S = HS(W) = S(W)H$.

(ii) *The manifold* M *is causal if and only if* H *is the precise set* $S \cap S^{-1}$ *of invertible elements of* S.

(We shall see the proofs later). Causality of M is now expressed by an algebraic property of the semigroup S; we shall call S *the causal semigroup of the data* (G, H, W). Our program of characterizing causality of M will succeed in the same measure as we are able to deal with S and its relation to the group H. If H is connected, then $S = S(W)$, and the Lie theory of semigroups applies directly to S. If H is not connected we are confronted with a difficult situation and are able to handle only special cases.

It is no loss of generality for our purposes to assume that G is simply connected. Indeed in the contrary case we pass to the simply connected covering and to the full inverse image of H in that covering. Now M is simply connected if and only if H is connected; thus the complications vanish if the manifold M is simply connected.

Reducing the problem.

We have not assumed that W is necessarily very large in $\mathfrak{g}$. Here we would say that W is large in $\mathfrak{g}$ if W is a generating set for the Lie algebra $\mathfrak{g}$. However, the assumption of this sort of thickness is no real restriction for the problem as we see now.

1.4. DEFINITION. Suppose that (G, H, W) are the causal data of a homogeneous manifold $M = G/H$ with a causal structure. Let $\mathfrak{g}_W = \langle\langle W \rangle\rangle$ the Lie subalgebra generated by W, let G_W denote the analytic subgroup $A = \langle \exp \mathfrak{g}_W \rangle$ endowed with its intrinsic Lie group structure so that the inclusion j: $G_W \to G$ is an immersion of Lie groups. We set $H_W = j^{-1}(H)$ and write $M_W = G_W/H_W$.

1.5. PROPOSITION. (The First Reduction Theorem). *(G_W, H_W, W) are the causal data of a homogeneous manifold M_W with causal structure, and there is an immersion of manifolds j_W: $M_W \to M$, given by $j_W(gH_W) = gH$, respecting the causal structure. The manifold M is causal if and only if M_W is causal. The manifold M is totally acausal if and only if $\mathfrak{g}_W = \mathfrak{g}$ and M_W is totally acausal.*

This reduction allows us to assume that W generates the Lie algebra $\mathfrak{g}$. This is certainly the case if W has inner points; Lorentzian manifolds give such examples.

For a set (G, H, W) of causal data we consider the semigroup

$$U = S^{-1} \cap S(W).$$

It is clearly invariant under the inner automorphisms by elements of H and allows us to rephrase causality as follows:

1.6. REMARK. M is causal if and only if U equals H_0, the identity component of H.

We focus on the semigroup U as the source of further reductions. In fact, we shall even use it for the proof of the First Reduction Theorem. We shall see that *U is the set of all points on W-admissible trajectories which end in H*. In particular, U is a path-connected semigroup. Thus the subgroup C generated by U is path-connected, hence analytic by a theorem of Yamabe's. We shall show that *C is the path-component of* 1 *in the semigroup $UH = HU$*. Since C is analytic, it has an intrinsic topology making it into a Lie group Γ with Lie algebra $\mathfrak{c}$. Let Δ denote $H \cap \Gamma$ with the topology induced from Γ.

1.7. THEOREM. *If (G, H, W) is a set of causal data, then $(\Gamma, \Delta, W \cap \mathfrak{c})$ is a set of causal data such that $S(W \cap \mathfrak{c}) = U$. The semigroup $UH = HU$ is a group, namely, the set $S \cap S^{-1}$ of invertible elements of the causality semigroup S.*

The new set $(\Gamma, \Delta, \mathfrak{c} \cap W)$ of causal data is canonically associated to the old one. Usually, the entries $\Gamma, \Delta, \mathfrak{c} \cap W$ are much smaller in dimension than the original data. In fact, they are as small as possible while still having significance for the causality of M. We denote the homogeneous manifold Γ/Δ by $\mathcal{M}$ and call it *the reduction of M*. There is an immersion of $\mathcal{M}$ into M whose image is the submanifold CH/H of M which forms one leaf of a foliation of M.

1.8. THEOREM (The Second Reduction Theorem). *If $\mathcal{M}$ is the reduction of a homogeneous manifold M given by the causal data (G, H, W), then the following conditions are equivalent:*
(1) *M is causal.*
(2) *$\mathcal{M}$ is singleton.*
(3) *$\mathcal{M}$ is causal.*
(4) $\Gamma = \Delta = H_0$.
(5) $U = H_0$.

The difficulty remains that, in general, the reduction $\mathcal{M}$ may be hard to identify. We shall see, however, that in our main results it plays a crucial role. We will encounter a situation where the reduction is either a point or a one-sphere.

The two reductions say in effect that every homogeneous manifold M with a causal structure given by the data (G, H, W) has an immersed submanifold M_W which is "generated by W" (in the sense specified in Proposition 1.6) whose causality determines that of M, and that M_W has in turn a, immersed submanifold $\langle U \cup U^{-1} \rangle H/H$ which is non-singleton precisely when M fails to be causal.

Closing $S(W)$

We let $T = \overline{S(W)}$ denote the closure of the semigroup $S(W)$. *The semigroup $\langle \exp W \rangle$ generated by* $\exp W$, *the semigroup $S(W)$, and the semigroup T have all the same interior, and this interior is dense in all three of them* (see [8]). In particular, $T = \overline{\langle \exp W \rangle}$. It is finally this semigroup T, canonically associated to the data (G, H, W), which we use in order to classify causality. Notice that we are not closing up $S = S(W)H$ and that the semigroup $TH = HT$ may not be closed, unless $H \subset T$. We denote the group $T \cap T^{-1}$ of invertible elements in T by $\mathscr{H}(T)$. Then $H_0 \subset \mathscr{H}(T)$. The set $\mathsf{L}(T)$ of subtangent vectors to T at the origin is the *Lie wedge* of T, i.e. a wedge V satisfying $e^{\operatorname{ad} x}V = V$ for all $x, -x \in V$, and its *edge* $V \cap -V$ is exactly the Lie algebra $\mathfrak{h}_T$ of $\mathscr{H}(T)$. We note $\mathfrak{h} \subset \mathfrak{h}_T$ and $W \subset \mathsf{L}(T)$. Once again, we see a set of causal data ($\mathscr{H}(T)$, $\mathscr{H}(T) \cap H$, $\mathfrak{h}_T \cap W$) and an associated homogeneous manifold $M_T = \mathscr{H}(T)/(\mathscr{H}(T) \cap H)$ with an immersion into M as submanifold $\mathscr{H}(T)H/H$. For these new data we can again form the causality semigroup and, in particular, the semigroup $S(\mathfrak{h}_T \cap W) \subset \mathscr{H}(T)$ of all points on $\mathfrak{h}_T \cap W$-admissible trajectories in G. Recall $U = S^{-1} \cap S(W)$.

1.9. THEOREM. *For any set of causal data (G, H, W) with $H \subset T$, the following relations hold:*

(1) $$S(W \cap \mathfrak{h}_T) = S(W) \cap \mathscr{H}(T).$$

(2) $$U \subset S(W \cap \mathfrak{h}_T).$$

Conclusion (2) says that an element of G is in U if and only if it is the endpoint of a $W \cap \mathfrak{h}_T$-admissible trajectory from 1 to some point of K. It follows quickly that for the groups of invertible elements in the semigroups $S(\mathfrak{h}_T \cap W))$ and $S(W)$ we have

(3) $$H(S(W \cap \mathfrak{h}_T)) = H(S(W)).$$

Since U is entirely contained in $\mathscr{H}(T)$, and since we know that M is causal if and only if $U = H_0$, the issue of causality of M is again decided by the causal data ($\mathscr{H}(T)$, $\mathscr{H}(T) \cap H$, $\mathfrak{h}_T \cap W$) and we have

1.10. THEOREM (The Third Reduction Theorem). *The manifold M associated with the causal data (G, H, W) is causal if and only if the manifold M_T associated with the causal data ($\mathscr{H}(T)$, $\mathscr{H}(T) \cap H$, $\mathfrak{h}_T \cap W$) is causal.*

This reduction is pragmatic in the sense that $T = \overline{\langle \exp W \rangle}$, hence $\mathscr{H}(T) = T \cap T^{-1}$ is immediately given if G and W are given. Given a choice we vastly prefer to deal with closed semigroups than with non-closed ones. Also, if W has inner points in $\mathfrak{g}$ (such as in the Lorentzian case), then $S(W)$ and hence T have inner points in G and, unless $S(W) = G$, the closed subgroup $\mathscr{H}(T)$ is definitely of smaller dimension than G and $\dim M_T < \dim M$.

Strictly causal manifolds.

The Lie theory of semigroups now allows us to consider a refinement of the causality concept which is at the core of our classification. If we are given a Lie group G and a Lie wedge W in its Lie algebra $\mathfrak{g}$, we say that W is *global in* G if there is a subsemigroup Σ of G whose tangent wedge $\mathsf{L}(\Sigma)$ is exactly W. This is tantamount to saying that the semigroups $\langle \exp W \rangle$ and $S(W)$ have W as tangent wedge. If W is a Lie subalgebra, then this is always the case by the Fundamental Theorems of Global Lie Group Theory; for Lie wedges W which are not vector spaces this fails often, and there is an elaborate theory attached to the issue of globality (see [8]). If (G, H, W) is a set of causal data, then *W is global in G if and only if $W = \mathsf{L}(T)$.*

1.11. Definition. A homogeneous manifold M given by the set (G, H, W) of causal data is said to be *strictly causal* if it is causal and W is global in G.

If we were free to speak about the subtangent wedge $\mathsf{L}_{m_0}(J^+_{m_0})$ in $T_{m_0}(M)$ of the set of all points which can be reached from m_0 by a causal trajectory, which is possible if one is willing to face the technicalities involving immersed nonclosed submanifolds, then strict causality adds to causality the information that $\mathsf{L}_{m_0}(J^+_{m_0}) = d\pi(1)(W) \cong W/\mathfrak{h}$.
For the reduction afforded by the Third Reduction Theorem it is instructive to understand the role of the added hypothesis that H is contained in T.

1.12. Proposition. *For the causal data (G, H, W) consider the following statements:*
(i) *H is connected, i.e. $H = H_0$.*
(ii) *$H \subset T$, i.e. $H \subset \mathscr{H}(T)$.*

Then (i) *implies* (ii), *and if* $\mathsf{L}(T) = W$, *then they are both equivalent, and* $H = \mathscr{H}(T)$.

This applies, in particular, when M is strictly causal.
Let us observe that *the hypotheses $H \subset T$ and $\mathsf{L}(T) = W$ imply that M is causal.* We adopt the view point that strict causality is a particularly strong kind of causality, and that we must better understand the situation that *M is not strictly causal.* It is perhaps a bit surprising that the Lie theory of semigroups allows us to describe this situation. *Thus we proceed with our classification according to the following mutually exclusive cases:*
(I) $\mathsf{L}(T) = \mathfrak{g}$.
(II) $\mathsf{L}(T) = W$.
(III) $W \neq \mathsf{L}(T) \neq \mathfrak{g}$.

Classification.

1.13. THEOREM (Classification Theorem, Part A). *Let (G, H, W) be a set of causal data for the homogeneous manifold M. Then M is totally acausal if Case* (I) *holds. If Case* (II) *holds and $H \subset T$, then M is strictly causal.*

Unfortunately, we know little when $H \not\subset T$. By Proposition 1.12 above this occurs if and only if H is disconnected, i.e. if M is not simply connected. It is *not* to be expected that one can prove very general theorems without the condition $H \subset T$. In fact, if $H_0 = \{1\}$ and $W = \mathsf{L}(T)$, we may take any (closed) cyclic subgroup of in the normalizer $\{g \in G \mid \mathrm{Ad}(g)W = W\}$ of W as H and thereby produce causal or totally acausal manifolds depending on the position of the generator of H. Our classification has a gap in this case which may not easily be overcome in the general situation. However, aside from this shortcoming (which instantaneously vanishes if G and M are simply connected) we are now left with the investigation of Case (III).

We state clearly at this point that we will now invoke special hypotheses on the geometry of the cone $W/\mathfrak{h}$, certainly satisfied by all Lorentzian cones. We shall assume that $W/\mathfrak{h}$ is "sufficiently round".

1.14. DEFINITION. We shall say that a wedge W in a vector space $\mathfrak{g}$ with edge $\mathfrak{h} = W \cap -W$ is *sufficiently round* if W has interior points and the following two conditions are satisfied

(i) Every boundary point w of $W \backslash \mathfrak{h}$ is a C^1-point (i.e. has only *one* support hyperplane to W through w).

(ii) Every nonzero boundary point w of W is an E^1-point (i.e. there is at least one support hyperplane E to W through w with $E \cap W = \mathfrak{h}$).

Condition (i) says that the surface of $W/\mathfrak{h}$ has no sharp edges, and (ii) says it has no flat portions.

1.15. THEOREM. *Suppose that (G, H, W) is a set of causal data such that W is sufficiently round and that Case* (III) *prevails. Then either $H \subset T$, or M is totally acausal. In the first case, $\mathsf{L}(T)$ is a half-space bounded by a hyperplane subalgebra $\mathfrak{h}_T$; the closed subgroup $\mathscr{H}(T)$ is a hypersurface.*

We inspect the situation more closely in the spirit of our reduction theorems. By Theorem 1.15 we know that $(\mathscr{H}(T), H, \mathfrak{h}_T \cap W)$ is a set of causal data on a hypersurface.

1.16. THEOREM (Classification Theorem, Part B). *Let (G, H, W) be a set of causal data for the homogeneous manifold M with a sufficiently round cone W and suppose that Case* (III) *holds. If $H \not\subset T$ then M is totally acausal. If $H \subset T$, then the wedge $\mathfrak{h}_T \cap W$ is a half-space in $\mathfrak{a} \overset{\text{def}}{=} (h_T \cap W) - (h_T \cap W)$ and $\mathfrak{h}$ bounds this half*

space in $\mathfrak{a}$. *The Lie algebra* $\mathfrak{c}$ *of* UH (see Theorem 1.7) *satisfies* $\mathfrak{h} \subset \mathfrak{c} \subset \mathfrak{a}$ *leaving exactly two cases:*
(i) $\mathfrak{h} = \mathfrak{c}$,
(ii) $\mathfrak{c} = \mathfrak{a}$.

The causal data $(\Gamma, \Delta, \mathfrak{c} \cap W)$ *yield the reduction* $\mathscr{M}$ *which in Case* (i) *is singleton and in Case* (ii) *is diffeomorphic to* S^1. *In the first case, M is causal, in the second case it is not and has a foliation by timelike circles.*

Before we go on let us summarize the results of the classification for sufficiently round wedges W in a table:

THE CLASSIFICATION TABLE

$\mathsf{L}(T) = \mathfrak{g}$ implies that M is totally acausal, the other cases are represented in the following table

	$H \subset T$		$H \not\subset T$
	$\lvert\mathscr{M}\rvert = 1$	$\mathscr{M} \cong S^1$	
$\mathsf{L}(T) = W$	M is strictly causal	This case is impossible	? impossible if M is simply connected
$W \neq \mathsf{L}(T)$ $\mathsf{L}(T) \neq \mathfrak{g}$	M is causal (not strictly)	M is not causal	M is totally acausal

The Classification Theorem, Part B shows that Case (III) is impossible if W is sufficiently round, $\mathfrak{g}$ contains no hyperplane subalgebras, and M is not totally acausal. This gives immediately the following consequence.

1.17. COROLLARY. *Let M be a homogeneous manifold with a causal structure, which is not totally acausal, given by the data* (G, H, W), *and suppose that W is sufficiently round. If* $\mathfrak{g}$ *contains no subalgebras of of codimension one then M is strictly causal or* $H \not\subset T$.

In particular, *if under the circumstances of* Corollary 1.17, *the manifold M is simply connected, then it is either totally acausal or strictly causal.* Since we have a complete theory for detecting hyperplane subalgebras in a Lie algebra (see [9]), this criterion is rather effective.

The Classification Theorem, Part B shows that in Case (III) the distinguishing feature is the reduction $\mathscr{M}$. We shall give a Lie group theoretical criterion which allows us to distinguish the two cases. Indeed *under the hypotheses of* Theorem

1.16, *the manifold* M *is causal if and only if for all* $x \in (\mathfrak{h}_T \cap W) \backslash \mathfrak{h}$ *the relation* $\exp t \cdot x \in H$ *implies* $t = 0$. (See Proposition 4.2.)

Algorithm.

Suppose we are given a homogeneous manifold M through the causal data (G, H, W) satisfying the hypotheses of the Classification Theorem 1.13 and 1.16. Our results suggest the following procedure for the determination of the causal properties of M.

Firstly, one tests whether W is global or not. In general, this may be a delicate task as the general Lie theory of semigroups shows (see [8]). In practice, one frequently knows the maximal subsemigroups of G; a comparison of W with their tangent wedges may give an affirmative test result. Then we are in Case (II). This procedure is particularly effective for the class of Lie groups, for which maximal subsemigroups are half-space semigroups (cf. [8]). In this case one inspects the (known) list of hyperplane subalgebras of $\mathfrak{g}$, for these bound the tangent wedges of maximal subsemigroups of G [9].

Secondly, if W is global (Case (II)!), we know that M is strictly causal as soon as $H \subset T$. This is automatic if M is simply connected, for which case we may assume that G is simply connected. If $H \not\subset T$, we need a detailed knowledge of $S(W)$ and H/H_0 to determine whether M is causal or not. This is the case marked "?" in the Classification Table.

Thirdly, suppose that W is not global (Cases (I) or (III)). If $S(W) = G$ we are in Case (I). If $S(W) \neq G$ (this implies $T \neq G$, hence Case (III)), we have to check all hyperplane subalgebras $\mathfrak{g}$ (the complete list is known [9]) and check which are support hyperplanes of W. The subalgebra $\mathfrak{h}_T$ must be one of these. We consider the analytic subgroup $\mathscr{H}(T)$ with Lie algebra $\mathfrak{h}_T$ (which, being the group of units of T, has to be closed and to contain H_0.) We inspect H: If $H \not\subset \mathscr{H}(T)$ then M is totally acausal. In the opposite case we have to study the algebras $\mathfrak{a} = (\mathfrak{h}_T \cap W) - (\mathfrak{h}_T \cap W)$ and the topology of the corresponding analytic subgroup of G in order to determine whether the reduction $\mathscr{M}$ is singleton or a 1-sphere.

This test procedure yields the following result valid for a large class of Lie groups. For a convenient formulation let us denote with $\mathscr{E}(\mathfrak{g})$ the set of hyperplane algebras in the Lie algebra $\mathfrak{g}$ (see [9]) which bound a global half-space Lie-wedge. Also recall that a *ray semigroup* in a Lie group is a subsemigroup generated by its one-parameter subsemigroups [8]. If G is a simply connected Lie group such that each simple factor of $\mathfrak{g}/\text{rad}\,\mathfrak{g}$ is either compact or isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ then all maximal closed ray semigroups are half-space semigroups (cf. [11]). This may very well be true for a much larger class of Lie groups, but this is still unknown.

1.18. THEOREM. *Let M be a homogeneous manifold with a causal structure given by the data (G, H, W) such that G is simply connected and H is connected. Suppose that W is sufficiently round and that every maximal closed ray subsemigroup of G is a half-space semigroup. Then the following conclusions hold:*

(i) *If there exists $\mathfrak{e} \in \mathscr{E}(\mathfrak{g})$ such that $\mathfrak{e} \cap W \subset \mathfrak{h}$, then M is strictly causal.*

(ii) *If $\mathfrak{e} \cap \operatorname{int} W \neq \varnothing$ for all $\mathfrak{e} \in \mathscr{E}(\mathfrak{g})$, then M is totally acausal.*

(iii) *If there exists an $\mathfrak{e} \in \mathscr{E}(\mathfrak{g})$ such that $\mathfrak{e} \cap \operatorname{int} W = \varnothing$, and $\mathfrak{e} \cap W \not\subset \mathfrak{h}$, then either M is strictly causal (if W is global) or else M is causal if and only if $H \cap \{\exp tx\colon t > 0\} = \varnothing$ for all $x \in (W \cap \mathfrak{e}) \backslash \mathfrak{h}$.*

We shall illustrate effectiveness of the algorithm by detailing it for some low dimensional examples including those of Levichev [14].

Methods.

The approach to the causality of homogeneous manifolds which we have outlined requires a good deal of technical information on the Lie theory of semigroups which is not yet well known. Much of the current status of this theory is contained in [8]. However, even given that source, we still have to provide many additional details for the proofs of the results specified above. This we will have to do in the main body of the paper. A systematic theory of partial orders on smooth manifolds endowed with cone fields in the frame work we use is recent. For further background see J. D. Lawson's and K. H. Neeb's articles [12] and [16].

2. Proofs and details.

Our first task is to prove Proposition 1.3.

First we show that S is a semigroup. Indeed, suppose that $g_1, g_2 \in S$ and that γ_1, γ_2 are causal curves with $\gamma_i(0) = m_0$ and $\gamma_i(1) = \pi(g_i)$. Then we define a curve $\eta\colon [0,1] \to M$ by $\eta(t) = g_2 \cdot \gamma_1(t)$. Since by assumption the group action on M preserves metric and orientation, i.e., the field of future cones, it follows from the chain rule that η is a causal curve from $\pi(g_1) = \gamma_2(1)$ to $g_2 \cdot \gamma_1(1) = g_2 \cdot \pi(g_1) = \pi(g_2 g_1)$. Thus the concatenation of γ_2 and η is a causal curve from m_0 to $\pi(g_2 g_1)$ which by a change of parameters, shows that $g_2 g_1 \in S$. Since the concatenation of admissible trajectories is again admissible, the set $S(W)$ is a subsemigroup of G. By assumption, W is $\operatorname{Ad}(H)$-invariant; therefore the set of W-admissible trajectories and hence $S(W)$ is invariant under $\operatorname{Ad}(H)$. In particular, this implies $HS(W) = S(W)H$ is a semigroup.

In order to show $S = HS(W)$ we need to relate admissible curves in G and M.

2.1. DEFINITION. (i) A *lifted causal trajectory* is a piecewise smooth map $\gamma\colon [0,1] \to G$ such that $p \circ \gamma\colon [0,1] \to M = G/H$ is a causal trajectory.

(ii) We write S_M for the set of all points $g \in G$ such that there is a lifted causal trajectory γ with $p(\gamma(0)) = m_0 = H$ and with $\gamma(1) = g$.

2.2. LEMMA (The Lifting Lemma). *Let p: $E \to B$ be a locally trivial fibration of C^∞ manifolds. Then every differentiable curve γ: $[0,1] \to B$ lifts differentiably. That is, for a given point $x_0 \in E$ with $p(x_0) = \gamma(0)$ there is a differentiable curve Γ: $[0,1] \to E$ with $\Gamma(0) = x_0$ such that $\gamma = p \circ \Gamma$.*

PROOF. We first prove a sublemma.

SUBLEMMA. Suppose that φ: $I \to \mathsf{R}^n$ is a C^∞ function on an interval $I \subset [0,1]$ with $0 \in I \neq [0,1]$ and with $\varphi(0) = 0$. Given an $\varepsilon \in [0, \sup I]$ we find a C^∞-function Φ: $[0,1] \to \mathsf{R}^n$ with $\varphi(t) = \Phi(t)$ for $0 \leqq t \leqq \sup I - \varepsilon$.

PROOF. Let α: $[0,1] \to [0,1]$ be a C^∞-function such that $\alpha(t) = 1$ for $0 \leqq t \leqq \sup I - \varepsilon$ and $\alpha(t) = 0$ for $\sup T - \dfrac{\varepsilon}{2} \leqq t \leqq 1$. Define

$$\Phi(t) = \begin{cases} \alpha(t) \cdot \varphi(t) & \text{for } 0 \leqq t < \sup I; \\ 0 & \text{for } \sup\ I \leqq t \leqq 1. \end{cases}$$

This function does what we want.

For a proof of the Lifting Lemma we let J denote the set of $r \in [0,1]$ *such that there is a smooth Δ: $[0,r] \to E$ with $\Delta(0) = x_0$ and $p(\Delta(t)) = \gamma(t)$ for $t \in [0,r]$.* Obviously, J is an interval containing 0. We claim that J is open and closed in $[0,1]$. In order to prove the claim let $s = \sup J$. There is an open neighborhood U of $\gamma(s)$ and a diffeomorphism ψ: $F \times U \to p^{-1}(U)$ such that $p(\psi(x,u)) = u$. We choose $0 \leqq s' < s < s'' \leqq 1$ if $s < 1$, else $s'' = s$ so that $\gamma([s', s'']) \subset U$. Set $m = \frac{1}{2}(s' + s)$. Then $m \in J$ and there is a partial lifting Δ: $[0,m] \to E$ by the definition of J. Now there is an open euclidean n-cell C^n such that $C^n \times U$ is mapped homeomorphically onto an open neighborhood W of $\Delta(m)$. Let $I = [r,m]$ with $s' \leqq r < m$ be such that $\Delta([r,m]) \subset W$. Then there is a smooth function φ: $I \to C^n$ such that $\psi(\varphi(t), \gamma(t)) = \Delta(t)$ for $t \in [r,m]$. If $r < r' < m$ then, by the Sublemma in part (a), we find a smooth function Φ: $[m, s''] \to C^n$ which agrees with φ on $[m, r']$. Thus, if we set

$$\Delta'(t) = \begin{cases} \Delta(t) & \text{for } 0 \leqq t \leqq r; \\ \Psi(\Phi(t), \gamma(t)) & \text{for } m < t \leqq s'', \end{cases}$$

then Δ' is a smooth partial lifting of γ over $[0, s'']$ This proves $s'' \in J$. Firstly, this shows that $s \in J$ which guarantees that J is closed. Secondly if $s = 1$, the $J = [0,1]$ and J is open trivially. If $s < 1$, then $s < s''$ means that J is a neighborhood of s and so J is open in this case, too. As an open closed interval of $[0,1]$, the set J must be equal to $[0,1]$ and the Lemma is proved.

Now we can conclude the proof of Proposition 1.3 (i):

By the definition of $S(W)$ and the left invariance of the wedge field $g \mapsto W(g)$, the semigroup $S(W)H$ is exactly the set of endpoints of W-admissible trajectories starting from a point of H. These project onto causal trajectories of M starting from $m_0 = H$. However, by the Lifting Lemma, every causal trajectory in M starting from m_0 lifts to a W-admissible trajectory of G starting from some point of H. Hence, by Definition 2.1 (ii) we conclude $S = S(W)H$.

Next we finish the proof of Proposition 1.3 (ii):

We show first that $S \cap S^{-1} = H$ implies that M is causal. Since clearly H is contained in $S \cap S^{-1}$ it suffices to show that $\pi^{-1}(\gamma([0,1])) \subset S \cap S^{-1}$ for any closed causal curve γ: $[0,1] \to M$. Let $t_0 \in [0,1]$ and $g_0 \in \pi^{-1}(\gamma(t_0))$. Now consider the curve η: $[t_0, 1] \to M$ defined by $\eta(t) = g_0^{-1} \cdot \gamma(t)$, then η is a causal curve from m_0 to $g_0^{-1} \cdot \gamma(1) = \pi(g_0^{-1})$. Hence we have $g_0^{-1} \in S$ and thus $g_0 \in S \cap S^{-1}$.

Conversely, if $g \in S \cap S^{-1}$ we have to show that there is a closed causal curve in M which passes through m_0 and $\pi(g)$. But we find causal curves γ_1 and γ_2 from m_0 to $\pi(g)$ and $\pi(g^{-1})$ respectively, so that translating γ_1 by g^{-1} and concatenating the resulting curves yields the desired causal loop.

The next task is the proof of the three reduction theorems.

The first lemma in this line is of a purely algebraic nature.

2.3. Lemma. *Let S be an arbitrary semigroup in a group G containing* 1 *and H any subgroup of G normalizing S. For an element $s \in S$, the following statements are equivalent:*

(i) *There is an element $h \in H$ such that $sh \in SH$ is a unit in SH.*

(ii) *s is a unit in SH.*

(iii) *$sS \cap H \neq \emptyset$, that is, there is a $t \in S$ such that $st \in H$.*

If $U = S \cap (SH)^{-1}$, then $SH \cap (SH)^{-1} = UH$.

Proof. All of these equivalences are elementary. Since H is contained in $SH \cap (SH)^{-1}$, right and left multiplication with elements of H transforms units of SH into units. Hence (i) and (ii) are equivalent. If (ii) holds, then $s^{-1} \in SH$ and thus $sS \cap H \neq \emptyset$, hence (iii), and vice versa.

The last assertion follows from the preceding.

We now return to the causal data (G, H, W) for a homogeneous manifold $M = G/H$ with a causal structure.

2.4. Lemma. (i) *The semigroup $U = S^{-1} \cap S(W)$ is the set $U(H; W)$ of all points on W-admissible trajectories starting in* 1 *and ending in some point of H.*

(ii) $S \cap S^{-1} = UH$.

(iii) *If $\mathfrak{h} = \mathsf{L}(H) \subset W$, then $\langle U \cup U^{-1} \rangle$, the analytic subgroup generated by all W-admissible trajectories from* 1 *to a point of H, is the arc component of* 1 *in UH.*

(iv) $\mathfrak{h} \subset \mathsf{L}(U) \subset \mathsf{L}(S \cap S^{-1}) \cap \mathsf{L}(S(W))$ *and* $\langle\langle \mathsf{L}(U) \rangle\rangle \subset \mathsf{L}(UH)$, *where $\langle\langle \mathsf{L}(U) \rangle\rangle$ is the Lie algebra generated by $\mathsf{L}(U)$.*

Proof. It follows from the definitions of $U(H;W)$ and $S(W)$ that $U(H;W) = \{s \in S(W) \mid sS(W) \cap H_\lambda\} \neq \emptyset$. In fact, if γ is W admissible with $\gamma(t_0) = s$ and $\gamma(1) \in H$ then $s^{-1}\gamma(t_0 + t) \in S(W)$ for all non-negative t. In particular we have $s(s^{-1}\gamma(1)) \in sS(W) \cap H$. Now we know that $U(H;W)$ is the set of all elements of $S(W)$ which are invertible in $S(W)H$, i.e. we have shown (i).

Further, (ii) is a consequence of Lemma 2.3. For a proof of (iii) we observe first that U is arcwise connected by definition, whence $\langle U \cup U^{-1} \rangle$ is contained in the arc component C of 1 in UH. On the other hand, $\langle U \cup U^{-1} \rangle$ contains H_0, the arc component of 1 in H, because of $\mathfrak{h} \subset W$. We note that C is also the arc component of 1 in $\langle U \cup U^{-1} \rangle H$. But $(\langle U \cup U^{-1} \rangle H)/\langle U \cup U^{-1} \rangle \cong H/(H \cap \langle U \cup U^{-1} \rangle)$, and this last group is a homomorphic image of the group H/H_0 which is countable. Thus the analytic subgroup $\langle U \cup U^{-1} \rangle$ has countable index in the analytic group C. Hence the two agree.

(iv) The preanalytic (cf. [8]) semigroup $U = UH \cap S(W)$ has the Lie wedge $\mathsf{L}(U) \subset \mathsf{L}(UH) \cap \mathsf{L}(S(W))$ as tangent object. Since C contains the subgroup generated by all one parameter semigroups of U it follows that $\langle\langle \mathsf{L}(U) \rangle\rangle \subset \mathsf{L}(C) = \mathsf{L}(UH)$.

As in the introduction, we let Γ denote the analytic subgroup C generated by the arcwise connected subsemigroup U, given its Lie group topology. After Lemma 2.4, the group C is the arc component of 1 in $S \cap S^{-1}$. We set $\mathfrak{c} = \mathsf{L}(\Gamma)$ and $\Delta = \Gamma \cap H$.

Then

$$\mathfrak{c} = \{X \in \mathfrak{g} \mid \exp \mathbb{R} \cdot X \subset HU\}.$$

2.5. Lemma. *Suppose that V is some Lie wedge in $\mathfrak{c} \subset \mathfrak{g}$. The two semigroups $S_\Gamma(V)$ and $S_G(V)$ are well defined subsemigroups of Γ and G, respectively, according to Definition* 1.2. *In particular, we have $S_\Gamma(V) \subset C \subset G$. Then*

$$S_\Gamma(V) = S_G(V). \tag{4}$$

Proof. Suppose that $g \in \Gamma$. The inclusion map $j\colon \Gamma \to G$ induces an inclusion map $T_g(\Gamma) \to T_g(G)$ (which is, in fact, the inclusion $\mathfrak{c} \to \mathfrak{g}$ transported via $d\rho_g(1)$ where $\rho_g(x) = xg$). Under this inclusion $V_\Gamma(g) = d\lambda_g(1)V$ gives exactly $V_G(g) = d\lambda_g(1)(V)$. Clearly, every V-admissible trajectory $\gamma\colon [0,1] \to \Gamma$ in Γ is mapped under j to a V-admissible trajectory $\gamma\colon [0,1] \to G$ of G. But conversely, if $\gamma\colon [0,1] \to G$ is a V-admissible trajectory in G, then $\dot\gamma(t) \in V_G(\gamma(t)) \subset T_{\gamma(t)}(\Gamma)$, and as a consequence $\gamma(t) \in C$ for all $t \in [0,1]$, and the corestriction of γ to C lifts to a V-admissible trajectory in Γ which maps onto γ under j. Thus, the V-admissible trajectories of Γ and G are in a natural bijective correspondence which, on the level of the graphs of functions, becomes equality. In particular, (4) follows.

Relation (4) allows us to speak of the semigroup $S(V)$ unambiguously as subsemigroup of Γ as well as of G. A similar thing can be said of the semigroup U as we shall see now:

2.6. PROPOSITION. (i) $U \subset C$ *and*

$$U = U_\Gamma \overset{\text{def}}{=} S_\Gamma^{-1} \cap S(W \cap \mathfrak{c}) \quad \textit{where} \quad S_\Gamma = S_{(\Gamma, \Delta, W \cap \mathfrak{c})}. \tag{5}$$

(ii) $\Gamma = U_\Gamma \Delta$.
(iii) $U_\Gamma = S(W \cap \mathfrak{c})$.

PROOF. (i) Since Γ is the underlying Lie group of the analytic group generated by U, clearly $U \subset \Gamma$, and U is the set of all elements on W-admissible trajectories from 1 to some point in H. Since these trajectories are contained in Γ they are in fact $W \cap \mathfrak{c}$-admissible, and by Lemma 2.5, equation (5) follows.

(ii) The right side is clearly in the left one. However, C is the arc component of UH, and if $c = uh \in C$ with $u \in U$ and $h \in H$, then $h = u^{-1}c \in C \cap H$. Since the underlying groups of C and Γ, as well as those of $C \cap H$ and Δ agree, the left hand side is in the right one.

(iii) We have $U = U_\Gamma$. By (i) and (ii) above we note $\Gamma = U\Delta \subset S(W \cap \mathfrak{c})\Delta = S_\Gamma \Gamma$, whence $S_\Gamma = \Gamma$ and thus $U = U_\Gamma = S(W \cap \mathfrak{c})$.

Note that Proposition 2.6 (iii) together with Lemma 2.4 (ii) completes the proof of Theorem 1.7.

2.7. LEMMA. *Suppose that* $\mathfrak{h} \subset W$ *and that* $G = S(W)H$. *Then* $G = \langle S(W) \rangle$.

PROOF. We now observe, that it is no loss of generality to assume G simply connected: We consider the universal covering $p\colon \tilde{G} \to G$, set $\tilde{H} = p^{-1}H$ and identify $\mathfrak{g}$ with $\mathsf{L}(\tilde{G})$ in such a fashion that $p \circ \exp_{\tilde{G}} = \exp_G$. Then $p(S_{\tilde{G}}(W)) \subset S_G(W)$. But conversely, every W-admissible trajectory in G starting from 1 has a unique lifting to a W-admissible trajectory in $\tilde{G}$ starting from 1 (cf. Lemma 2.2). Hence also $S_G(W) \subset p(S_{\tilde{G}}(W))$. Thus $\tilde{G} = S_{\tilde{G}}(W)\tilde{H}$.

If now the assertion is true for simply connected groups, then $\langle S_{\tilde{G}}(W) \rangle = \tilde{G}$ and thus $\langle S(W) \rangle = p(\langle S_{\tilde{G}}(W) \rangle) = p(\tilde{G}) = G$.

Hence we assume now that G is simply connected. We observe that H normalizes $S(W)$, hence $A = \langle S(W) \rangle$. From $G = S(W)H \subset AH \subset G$ we now conclude that the analytic subgroup A is normal in G. Since G is simply connected, A is closed. But $H_0 \subset S(W) \subset A$, and H/H_0 is countable. Hence G/A is countable, and this shows $G = A$.

2.8. THEOREM. *Suppose that G is a connected Lie group and H a closed subgroup, W a Lie wedge in $\mathfrak{g}$ containing $\mathfrak{h}$. Further we assume that W invariant under* Ad H.

Then

(i) $U = S(W \cap \mathfrak{c})$,

(ii) $C = S(W \cap \mathfrak{c})(H \cap \Gamma)$, *and*

(iii) Γ *is generated by the semigroup* $S(W \cap \mathfrak{c})$.

(iv) *The set of invertible elements of* $S(W)H$ *is* $S(W \cap \mathfrak{c})H$, *and*

(v) $\langle \exp \mathfrak{c} \rangle$ *is the path component of* 1 *in* $S \cap S^{-1}$.

PROOF. By Proposition 2.6 we know that $U = S(W \cap \mathfrak{c})$ and $\Gamma = S(W \cap \mathfrak{c})(H \cap \Gamma)$. It follows from this and the definition of the group Γ that it is generated by $S(W \cap \mathfrak{c})$ (see also Lemma 2.7). By Lemma 2.4 (i) and Lemma 2.3, the set of invertible elements of S is $UH = S(W \cap \mathfrak{c})H$.

2.9. COROLLARY. *Under the circumstances of Theorem* 2.8, *the following conditions are equivalent:*

(1) $U = H_0$.

(2) $\Gamma = H_0$.

(3) $\mathfrak{c} = \mathfrak{h}$.

(4) $S(W \cap \mathfrak{c}) \subset H_0$.

(5) $U_\Gamma = H_0$.

PROOF. (1) $\Rightarrow$ (2): If (1), then $H_0 \subset \langle U \cup U^{-1} \rangle \subset H_0$. Hence (2).

(2) and (3) are clearly equivalent.

(3) $\Rightarrow$ (4): If (3), then $S(W \cap \mathfrak{c}) = S(W \cap \mathfrak{h}) \subset S(\mathfrak{h}) = H_0$.

(4) $\Rightarrow$ (1): If (4), then by Theorem 2.8 we have $\Gamma = S(W \cap \mathfrak{c})(H \cap \Gamma) \subset H_0 H = H$ and thus $\Gamma \subset H_0$. This implies $U \subset H_0$. The reverse inclusion follows directly from Lemma 2.4 (i) since $\mathfrak{h} \subset W$.

(5) is equivalent to (1) in view of Proposition 2.6.

2.10. COROLLARY. *Under the circumstances of Theorem* 2.8 *and the assumption that* S *generates* G, *the following conditions are also equivalent:*

(1) $G = S(W)H$, *that is,* $G = S$.

(2) $U = S(W)$, *that is,* $S^{-1} \subset S(W)$.

(3) $\Gamma = G$, *that is,* U *algebraically generates* G.

(4) $\mathfrak{c} = \mathfrak{g}$.

PROOF. We recall the Definition of U as $S^{-1} \cap S(W)$ and the equation $S = S(W)H$ (Proposition 1.3 (i)). Thus (1) implies (2). As $S(W) \subset S$, (2) implies that S is a group. Since S generates G, this implies (3). Clearly, (3) and (4) are equivalent. By Theorem 2.8 (v), (4) implies (1).

2.11. PROPOSITION. *Let* (G, H, W) *denote the data of a homogeneous manifold* $M = G/H$ *with a causal structure. Then*

(i) M *is causal if and only if* $U = H_0$ *if and only if* $S \cap S^{-1} = H$.

(ii) M *is totally acausal if and only if* $G = S(W)H = S$.

(iii) *The group* $\mathscr{H}(S(W)) = S(W) \cap S(W)^{-1}$ *is the set of all points on closed W-admissible trajectories. In particular,* $\mathscr{H}(S(W)) \subset U$.
(iv) *If* M *is causal, then* $\mathscr{H}(S(W)) = H_0$.

Proof. (i) By the homogeneity of M, the manifold M is causal if and only if every closed trajectory starting from $m_0 = H$ is constant. This is the case if and only if every W-admissible trajectory γ: $[0,1] \to G$ with $\gamma(0) = 1$ and $\gamma(1) \in H$ satisfies $\gamma([0,1]) \in H$. By Lemma 2.4 (i), this means exactly $U \subset H$. Since U is path connected, this inclusion is equivalent to $U \subset H_0$. The reverse containment, however, is always true. To prove the last equivalence we recall from Lemma 2.4 (ii) that $S \cap S^{-1} = UH$. Therefore, $U = H_0$ implies $S \cap S^{-1} = H_0H = H$. Conversely, $S \cap S^{-1} = H$ implies $U \subset H$ and then $U \subset H_0$, since U is pathconnected.

(ii) By the homogeneity of M we know that M is totally acausal if for every $m \in M$ there is a causal trajectory from m_0 to m. This is equivalent to saying that for every $g \in G$, there exists a W-admissible trajectory starting from 1 and ending in gH. This is equivalent to $G = S(W)H$.

(ii) If x is on a closed W-positive trajectory γ: $[0,1] \to S(W)$ starting at 1, say $x = \gamma(r)$, then $t \mapsto x^{-1}\gamma(t+r)$ is a W-positive trajectory $[0, 1-r] \to S(W)$ leading from 1 to x^{-1}. Thus $x^{-1} \in S(W)$, i.e., $x \in \mathscr{H}(S(W))$. Conversely, if $x \in \mathscr{H}(S(W))$, then there are W-positive trajectories ρ: $[0,1] \to S(W)$ from 1 to x and σ: $[0,1] \to S(W)$ from 1 to x^{-1}. Then the concatenation of ρ and $t \mapsto x\sigma(t)$ is a closed W-positive trajectory containing x.

(iv) Suppose now that $H_0 = U$. It follows from $H_0 \subset S(W)$ that $H_0 \subset \mathscr{H}(S(W))$. By (iii) above we have $\mathscr{H}(S(W)) \subset H_0$. Hence (iv) is proved.

Note that Proposition 2.11 (i) proves Remark 1.6.

We are now ready to prove the reduction theorems. We begin with the First Reduction Theorem.

Recall that we are in the situation of Definition 1.4. By Lemma 2.5, the semigroup $S(W)$ is unambiguously contained in G as well as in G_W. Accordingly, $H_0 \subset U(H; W) = U(H \cap A; W)$. Now by Proposition 2.11, the manifold M is causal if and only if $H_0 = U$, and M_W is causal if and only if $U(H_W; W) = H_0$. Since the semigroups $U(H \cap A; W)$ and $U = U(H; W)$ agree, the first assertion follows.

Now M is totally acausal if and only if $G = S(W)H$ and M_W is totally acausal if and only if $G_W = S(W)H_W$. But $S(W)H \cap A = S(W)(H \cap A)$ and the underlying groups of G_W and A on the one hand and H_W and $H \cap A$ on the other agree. Hence $G = S(W)H$ implies $G_W = S(W)H_W$. Also $G = S(W)H$ implies $G = \langle S(W) \cup S(W)^{-1} \rangle$ by Corollary 2.10. Since $S(W) \subset \langle \exp \mathfrak{g}_W \rangle$ as $W \subset \mathfrak{g}_W$, we conclude $\mathfrak{g}_W = \mathfrak{g}$. Conversely, suppose that this condition is satisfied together with $G_W = S(W)H_W$. Then $\mathfrak{g}_W = \mathfrak{g}$ implies $G = G_W$, consequently $H_W = H$, and $G = S(W)H$ follows.

Next we prove the Second Reduction Theorem 1.8.

From Proposition 2.6 (i) and Theorem 2.8 (i) we know $U_\Gamma = U = S(W \cap \mathfrak{c})$, and the group Γ is algebraically generated by U by its very definition.

We shall now prove the equivalence of conditions (1) through (5) in Theorem 1.8.

(5) $\Leftrightarrow$ (1) $\Rightarrow$ (2): By Proposition 2.11, M is causal iff $U = H_0$. By Corollary 2.9, this is equivalent to $U_\Gamma = H_0 = \Delta_0$. But Γ is the analytic subgroup of G generated by $U = H_0$ hence is also equal to H_0. This implies that $\mathcal{M}$ is singleton.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (4): If $\mathcal{M}$ is causal, then by Proposition 2.11 (i), $U_\Gamma = \Delta_0$. Then (5) in Proposition 2.6 implies $H_0 \subset U = U_\Gamma = \Delta_0 \subset H_0$. Thus $\Gamma = H_0$ and then also $\Delta = \mathrm{H}_0$.

(4) $\Rightarrow$ (5): The relation $\Gamma = H_0$, in view of Lemma 2.4 (i), means $U = H_0$.

We now turn to the proof of Theorem 1.9.

2.12. THEOREM. *For a piecewise smooth curve* γ: $[0,1] \to G$ *with* $\gamma(0) = 1$ *the following conditions are equivalent:*

(1) $\dot\gamma(t) \in (W \cap \mathfrak{h}_T)(\gamma(t))$ *for all* $t \in [0,1]$ *for which the derivative exists.*

(2) $\dot\gamma(t) \in W(\gamma(t))$ *and* $\gamma(t) \in H(T)$ *for all* $t \in [0,1]$ *for which the derivative exists.*

(3) $\dot\gamma(t) \in W(\gamma(t))$ *and* $\gamma(1) \in H(T)$*for all* $t \in [0,1]$ *for which the derivative exists.*

PROOF. The following implications are simple: (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (1), and (2) $\Rightarrow$ (3). The hard implication is (3) $\Rightarrow$ (2) which we prove now:

We suppose (3) and assume that (2) is false in order to derive a contradiction. Then there is an $s \in [0,1]$ such that $\gamma(s) \notin H(T)$. Let U be a compact neighborhood of 1 such that $\gamma(s) \notin UH(T)$. By [8], Theorem V.2.7 there exists a *closed proper* right ideal I of T with

$$\text{(i)} \qquad T \subset UH(T) \cup I.$$

Notice that $UH(T)$ is closed in G. We choose a neighborhood N of 1 in such a fashion that

$$\text{(ii)} \qquad \gamma(1)N \cap I = \varnothing,$$

and

$$\text{(iii)} \qquad \gamma(s)N \cap UH(T) = \varnothing.$$

By [8], Proposition VI.1.13, there is a trajectory f: $[0,1] \to T \subset G$ such that there are elements $X_1, \ldots, X_n \in W$ and numbers $0 = r_0 < r_1 < \ldots < r_{n-1} < 1 = r_n$ for which $f(1) = \exp X_1 \ldots \exp X_n$ and

$$\text{(iv)} \qquad f(t) = \exp X_1 \ldots \exp X_{k-1} \exp(t - r_{k-1})(r_k - r_{k-1})^{-1} \cdot X_k \quad \text{for} \quad r_{k-1} \leqq t < r_k,$$

and that, moreover

(v) $$f(t) \in \gamma(t)N \quad \text{for all} \quad t \in [0,1].$$

From (v) we know $f(s) \in \gamma(s)N$, hence $f(s) \notin UH(T)$ by (iii). Therefore $f(s) \in I$ from (i). Since I is a closed right ideal, (iv) implies then that $f(t) \in I$ for all $t \in [s, 1]$, in particular $f(1) \in I$. Hence $f(1) \in \gamma(1)N \cap I$, and this is a contradiction to (ii). This contradiction proves the claim $\gamma([0, 1]) \subset H(T)$.

Now we prove assertions (1), (2) of Theorem 1.9 and the subsequent condition (3):

In fact, (1) is an immediate consequence of Theorem 2.12 in view of the definition of the semigroups $S(W)$ and $S(W \cap \mathfrak{h}_T)$.

For a proof of (2) we note that (1) implies

(1′) $$S(W \cap \mathfrak{h}_T)^{-1} = S(W)^{-1} \cap H(T).$$

The intersection of the respective sides in (1) and (1′) yields

$$H(S(W \cap \mathfrak{h}_T)) = H(S(W)) \cap H(T),$$

and since $H(S(W)) \subset H(T)$, equation (3) follows.

By Lemma 2.4(i) we have $U \subset S(W) \cap H \subset S(W) \cap \mathscr{H}(T)$, whence $U \subset S(W \cap \mathfrak{h}_T)$ by (1). This completes the proof of (1), (2) and (3).

We proceed with the proof of Proposition 1.12.

The implication (i) $\Rightarrow$ (ii) is trivial. Conversely, under the assumption $\mathsf{L}(T) = W$, if $H \subset T$ then H is contained in the group of units of T which by [8], Theorem V.2.8 and Lemma V.2.2 is a connected subgroup with Lie algebra $\mathfrak{h}_T = \mathsf{L}(T) \cap -\mathsf{L}(T) = \mathfrak{h}$. Therefore $H = H_0$.

We now proceed to prove Theorem 1.13.

We first remark that $\langle \exp W \rangle \subset S(W) \subset T = \overline{S(W)}$. By the First Reduction Theorem 1.5 we may assume that the Lie algebra $\mathfrak{g}$ is generated by W. From [8] V.1.16 we then know that $\operatorname{int} T = \operatorname{int}\langle \exp W \rangle$ and $T = \overline{\operatorname{int} T}$. Hence $\operatorname{int} S(W) = \operatorname{int} T$ is dense in $S(W)$. Also, by Proposition 1.3 (i) we have $S = HS(W)$. Hence the open semigroup $P = H \operatorname{int} T$ is dense in S. We claim that $G = \bar{P}$ implies $G = P$ and thus $G = S$. Indeed let $g \in G$. Then gP^{-1} is open and hence intersects the dense set P. Thus there is a $p \in P$ with $p \in gP^{-1}$ and so $g \in pP \subset P$.

Now if $\mathsf{L}(T) = \mathfrak{g}$, then $T = G$ and thus $\operatorname{int} T$ is dense in G. Then P is dense in G, whence $G = S$ by the preceding. This means that M is totally acausal. Thus the first part of Theorem 1.13 is proved.

Next assume that $\mathsf{L}(T) = W$ and $H \subset T$. Then W is global and by Proposition 1.12 we have $H = \mathscr{H}(T) = H_0 \subset S(W)$. Hence $S = S(W)$ and $H \subset S \cap S^{-1} \subset$

$T \cap T^{-1} = H$. Thus M is causal by Proposition 1.3 (ii). Hence M is strictly causal as W is global.

Before we proceed with the classification we need some special results in the Lie theory of subsemigroups of Lie groups which we collect in the following section.

3. Some results on Lie wedges.

3.1. Definition. Let $W_1 \subset W_2$ Lie wedges in a Lie algebra $\mathfrak{g}$, whose edges we denote with $\mathfrak{h}_1$ and $\mathfrak{h}_2$, respectively. We shall say that W_1 *fits well into* W_2 if $\mathfrak{h}_2 \cap W_1 = \mathfrak{h}_1$.

3.2. Theorem. *Consider two Lie wedges $W_1 \subset W_2$ in the Lie algebra $\mathfrak{g}$. Suppose the following hypotheses:*

(0) *W_1 does not fit well into W_2.*

(C) *Every nonzero boundary point of W_1 is a C^1-point: $\partial W \subset C^1(W) \cup \{0\}$.*

Then either $W_2 = \mathfrak{h}_2 = \mathfrak{g}$ or W_2 is a half-space semialgebra whose boundary $\mathfrak{h}_2$ is a tangent hyperplane of W_1 and a hyperplane subalgebra.

Proof. By condition (0) we find an $x \in W_1 \backslash \mathfrak{h}_1$ with $x \in \mathfrak{h}_2$. Since $x \in H(W_2)$ it follows that x is in the boundary of W_2 and then, being in W_1, in the boundary of W_1. Next, a subtangent vector to W_1 at x is, a fortiori, a subtangent vector to W_2 at x. Thus $T_x(W_1) \subset T_x(W_2) = H(W_2)$. Now (C) applies and shows that $T_x(W_1)$ is a hyperplane. This implies that $H(W_2)$ contains a hyperplane. Thus either $H(W_2) = \mathfrak{g}$, that is, $W_2 = \mathfrak{g}$, or else $H(W_2)$ is a hyperplane itself. In this case, W_2 is a half-space semialgebra with $H(W_2) = T_x(W_1)$.

We notice that hypothesis (C) is satisfied if and only if the pointed cone $W_1/\mathfrak{h}_1$ satisfies (C) in $\mathfrak{g}/\mathfrak{h}_1$, and that is the case in particular if the latter is Lorentzian.

We recall that $E_x = T_x \cap W$ is the exposed face generated by x ([8], Definition I.2.6 and Proposition I.2.7). If W is a Lie wedge with edge $\mathfrak{h} = W \cap -W$, then

$$[\mathfrak{h}, W] \subset W - W. \tag{6}$$

(See [8], Theorem I.2.12.(1).) We recall that $x \in E^1(W)$ if and only if

$$E_x(W) = \mathbb{R}^+ \cdot x + \mathfrak{h}. \tag{7}$$

(See [8], Definition I.2.1).

The following is an application of the observation that the intersection of two Lie wedges is a Lie wedge:

3.3. Proposition. *If W is a Lie algebra $\mathfrak{g}$ and the tangent space $T_x = T_x(W)$ of x is a subalgebra, then E_x is a Lie wedge with edge $\mathfrak{h} = W \cap -W$. If x is an E^1-*

point, then $[x,\mathfrak{h}] \subset \mathsf{R}\cdot x + \mathfrak{h}$, *that is,* $\mathsf{R}\cdot x + \mathfrak{h}$ *is a Lie algebra, and if* $x \notin \mathfrak{h}$, *then* $\mathfrak{h}$ *is a hyperplane subalgebra in it.*

PROOF. Since W and T_x are Lie wedges, and since $\mathfrak{h} \subset T_x$ we know that $E_x = T_x \cap W$ is a Lie wedge with edge $\mathfrak{h}$. In particular, by (6), we have $[x,\mathfrak{h}] \subset E_x - E_x$. If $x \in E^1(W)$, then (7) implies $[x,\mathfrak{h}] \subset \mathsf{R}\cdot x + \mathfrak{h}$ and since $\mathfrak{h}$ is a subalgebra (see [8], Corollary II.1.8), the proposition is proved.

3.4. COROLLARY. *Under the circumstances of Theorem* 3.2, *suppose, in addition that* W_1 *satisfies the following condition:*

(*E*) *Every nonzero boundary point of* W *is an* E^1*-point:* $\partial W \subset E^1(W) \cup \{0\}$.

Then either $\mathfrak{h}_2 = \mathfrak{g}$ *or* $W_1 \cap \mathfrak{h}_2$ *is a half-space semialgebra* $\mathsf{R}^+\cdot x + \mathfrak{h}_1$ *with boundary algebra* $\mathfrak{h}_1$ *and some* $x \in C^1(W_1)$.

PROOF. By Theorem 3.2 W_2 is either $\mathfrak{g}$ or else a half-space semialgebra with $H(W_2) = T_x(W_1)$ for some $x \in C^1(W_1)$. From Hypothesis (E) we know that x is an E^1-point. Now Proposition 3.3 applies and shows that $W_1 \cap \mathfrak{h}_2 = W_1 \cap T_x(W_1) = E_x = \mathsf{R}^+x + \mathfrak{h}_1$.

We recall, that, as a consequence

$$(W_1 \cap \mathfrak{h}_2) - (W_1 \cap \mathfrak{h}_2) = \mathsf{R}\cdot x + \mathfrak{h}_1$$

is a Lie algebra. (Cf. also [8], Proposition II.2.13.)

Observe again that Condition (E) is satisfied if it is satisfied by $W/\mathfrak{h}$ in $\mathfrak{g}/\mathfrak{h}$. It therefore holds, in particular, if $W/\mathfrak{h}$ is Lorentzian.

4. The classification.

We are now ready to prove Theorem 1.15.

Since W is not global by the assumption of Case (III), we conclude from [8] VI.5.2 that W does not fit well into $L(T)$. Since W is sufficiently round. Theorem 3.2 applies and shows that $\mathsf{L}(T) = \mathfrak{g}$ or $\mathsf{L}(T)$ is a half-space semialgebra bounded by $\mathfrak{h}_T$. But $\mathsf{L}(T) = \mathfrak{g}$ is ruled out in Case (III).

Thus T is a half-space semigroup. Hence T is, in particular, a maximal subsemigroup of G. Now HT is a subsemigroup containing T. Hence either $H \subset T$, or $G = HT$. Now H/H_0 is countable and $H_0 \subset T$. Thus $G = HT$ implies that G is a countable union of translates of T. Thus T has inner points by the Baire Category Theorem and thus the interior int T of T is dense in T. As in the proof of Theorem 1.13 we then prove $G = HS(W)$. That is, M is totally acausal.

Our next project is the proof of Theorem 1.16.

Since W is sufficiently round, Corollary 3.4 applies and shows that $W \cap \mathfrak{h}_T$ is a half-space semialgebra in $\mathfrak{a} = (W \cap \mathfrak{h}_T) - (W \cap \mathfrak{h}_T)$ with $\mathfrak{h}$ as boundary hyperplane algebra.

We have $H_0 \subset U \subset C$, whence $\mathfrak{h} \subset \mathfrak{c}$. Now Theorem 1.9 and $W \cap \mathfrak{h}_T \subset \mathfrak{a}$ show

$$U = U(H; W) = U(H; W \cap \mathfrak{h}_T) \subset S(W \cap \mathfrak{h}_T) \subset S(\mathfrak{a}) = \langle \exp \mathfrak{a} \rangle,$$

and thus $\mathfrak{c} = \mathsf{L}(\langle U \cup U^{-1} \rangle) \subset \mathsf{L}(\langle \exp \mathfrak{a} \rangle) = \mathfrak{a}$.

We recall $\mathcal{M} = \Gamma/(\Gamma \cap H)$ and note $H_0 \subset \Gamma \cap H$ and $\mathsf{L}(\Gamma \cap H) = \mathfrak{c} \cap \mathfrak{h}$. Hence $\dim \mathcal{M} = \dim \mathfrak{c}/(\mathfrak{c} \cap \mathfrak{h}) = \dim \mathfrak{c}/\mathfrak{h} \leqq \dim \mathfrak{a}/\mathfrak{h} = 1$ by what we have just shown. Suppose $\dim \mathcal{M} = 1$. A one-dimensional homogeneous space of a Lie group is homeomorphic to R or to S^1. If the manifold $\mathcal{M}$ were homeomorphic to R then it would be causal, whence by the Second Reduction Theorem 1.8 it would have to be singleton, a contradiction. Hence $\mathcal{M}$ is a circle.

The situation of Theorem 1.16 can be described in greater detail.

4.1. Corollary. *If, under the circumstances of Theorem* 1.16, *the reduction* $\mathcal{M}$ *is a circle, in which case then M is not causal, then either* $\Delta = \Gamma \cap H$ *contains the commutator group of* Γ *and* $\mathcal{M}$ *is the circle group, or else there is a closed connected normal subgroup N of* Γ *contained in* Δ *such that* Γ/N *is a covering group of* $\mathrm{PSL}(2; \mathsf{R})$ *and* Δ/N *is a planar subgroup of* Γ/N. *If* Γ *is simply connected, then H is not connected.*

Proof. If the reduction of M is a circle, then M cannot be causal by the Second Reduction Theorem 1.8. We can factor the largest normal connected subgroup N of Γ contained in Δ without changing $\mathcal{M}$ or the causal structure. Then $\Gamma \cong \mathsf{R}/n\mathsf{Z}$ for some $n \in \mathsf{Z}$ or Γ is a covering group of $\mathrm{PSL}(2, \mathsf{R})$. The third case which could occur as a one-dimensional homogeneous space, namely, that Γ is isomorphic to the 2-dimensional solvable group of all

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \qquad a > 0, b \in \mathsf{R},$$

can be excluded since in this case H_0 is conjugate to the subgroup of diagonal matrices, and this group is its own normalizer so that $\Delta = \Delta_0$ and $\mathcal{M}$ couldn't be compact. In the first case, since $\mathcal{M}$ is diffeomorphic to S_1, we have $\Delta = m\mathsf{Z}$ with $m \mid n$ and either m or n non-zero. In the second case, Δ_0 is a planar two dimensional subgroup, and the normalizer of Δ_0 is $\Delta_0 Z$ with the discrete center Z of Γ. Hence Δ is a subgroup of $\Delta_0 Z$.

Suppose that Γ is simply connected and H is connected. Then Δ is connected since $H_0 = \Delta_0 \subset \Delta = H \cap \Gamma \subset H_0$. But then Γ/Δ cannot be homeomorphic to S^1.

It is an exercise to pursue the semigroup $S(W \cap \mathfrak{c})$ in all of these cases (cf. [8], Section V.4, notably Proposition V.4.17 ff., Section V.5).

Through the following result we look at the situation of Theorem 1.16 in a different fashion.

4.2. PROPOSITION. *Let M be a homogeneous manifold with a causal structure given by (G, H, W) such that $H \subset T$ and that $\mathsf{L}(T)$ is a halfspace. We assume that W is sufficiently round. Then M is causal if and only if $H \cap \{\exp tx\colon\ t > 0\} = \emptyset$ for all $x \in (W \cap \mathfrak{h}_T)\backslash\mathfrak{h}$.*

PROOF. Suppose that M is causal and let x be any element of $W \cap \mathfrak{h}_T$ such that $H \cap \{\exp t\cdot x\colon\ t > 0\} \neq \emptyset$. We shall show that $x \in \mathfrak{h}$. Recall that the curve $\exp tx$ is W-admissible (cf. [8], VI.1.16). Therefore $\exp t_0 x \in H \cap \{\exp tx\colon\ t > 0\}$ implies $A_{t_0} \overset{\text{def}}{=} \{\exp tx\colon\ 0 \leqq t \leqq t_0\} \subset U$. Then A_{t_0} is contained in H_0 by Proposition 2.11 (i) and Lemma 2.4 (i). Hence $x \in \mathfrak{h}$.

Conversely, suppose that M is not causal. We have to find an $x \in (W \cap \mathfrak{h}_T)\backslash\mathfrak{h}$ such that $H \cap \{\exp tx\colon\ t > 0\} \neq \emptyset$. Since $\mathsf{L}(T)$ is a half-space and every boundary point of W is an E^1-point, $W \neq \mathsf{L}(T)$ and we are in Case (III). Since M is not causal, $\mathscr{M}$ is not singleton by the Second Reduction Theorem 1.8. Hence we know from Theorem 1.16 that $\mathscr{M}$ is a circle. Now Corollary 4.1 applies. Factoring the largest connected subgroup N of $H_0 = \Delta_0$ which is normal in Γ pass to the quotient Γ/N and temporarily assume that either $\Delta_0 = \{1\}$ and Γ/Δ is a circle group, or else Γ is a covering group of PSL(2) and Δ a two dimensional subgroup (such that Δ is not connected if Γ is simply connected). In both cases, there is a one-parameter semigroup $\exp \mathsf{R}^+ x$ with $x \in (W \cap \mathfrak{h}_T)\backslash\mathfrak{h}$ *which hits* H nontrivially. Lifting this one parameter semigroup in Γ/N *to* Γ brings us back to the general case and the assertion is proved.

Our results show that the situation is considerably simpler if H is connected. In fact, it suffices that $H \subset T$, which is certainly the case if H is connected. If we are willing to consider covering manifolds of our original manifold M we can of course restrict ourselves to connected H. In fact, the canonical map

$$p\colon\ \tilde{M} = G/(H \cap T) \to G/H = M,$$

is a covering. If the causal structure on M is given by the data (G, H, W) then the manifold $\tilde{M}$ carries a causal structure given by the data $(G, H \cap T, W)$. It is clear that M can only be causal if $\tilde{M}$ is causal, since then p maps causal loops to causal loops. The converse is false, as can be easily seen in the abelian case.

4.3. PROPOSITION. *Let M and $\tilde{M}$ be as above, then the causality semigroup S of the data $(G, H \cap T, W)$ equals $S(W)$ and the question mark "?" in the Classification Table does not occur for M.*

PROOF. This follows immediately from $(H \cap T)S(W) = \tilde{S}$ (see Proposition 1.3 applied to $\tilde{M}$).

We still owe an explicit proof of Theorem 1.18 which we present now:

Suppose that the general hypotheses of Theorem 1.18 are satisfied. Since H is connected we have $H \subset T$ by Proposition 1.12 and thus the rightmost column in the classification table is missing.

(i) If $\mathfrak{e} \in \mathscr{E}(G)$ then $\mathfrak{e}$ bounds a global half-space Lie-wedge by definition, and $\mathfrak{e} \cap W \subset \mathfrak{h} \subset \mathfrak{h}_T$ then implies that $\mathsf{L}(T) = \mathsf{L}(\langle \exp W \rangle)$ and thus that W is global [8]. A glance at the Classification Table shows that M is strictly causal.

(ii) Under the assumptions of (ii), the ray semigroup $\langle \exp W \rangle$ is not contained in any maximal ray semigroup. Hence it must equal G and thus M is totally acausal.

(iii) Under the hypotheses it is possible that $\mathsf{L}(T) = W$. Then M is strictly causal according to the Classification Table. Assume the opposite. Then we are in the situation of Proposition 4.2 and the final assertion of Theorem 1.18 follows.

5. Examples.

It is clear from Section 4 that any proposition on the existence or non-existence of semigroups with appropriate given tangent wedges will yield examples of causal or non-causal manifolds. For an account of propositions of this type we refer to [8]. In this paper we restrict our attention to examples to which the classification table and the algorithm applies. The hyperplane subalgebras in any Lie algebra are completely classified [9]. For semisimple algebras this classification is simple and indeed common knowledge. For solvable algebras the situation is more complicated. The general case is composed of these two opposite cases.

We illustrate the situation by first discussing the three dimensional solvable group manifolds. This discussion produces a class of examples including those of Levichev [14].

5.1. Example (cf. [8], Section II.3). The following list contains all three dimensional solvable Lie algebras, described in terms of a basis (e_1, e_2, e_3) and the corresponding commutator relations.

(i)
$$[e_1, e_3] = \lambda e_1 + \omega e_2$$
$$[e_2, e_3] = -\omega e_1 + \lambda e_2,$$

where $\lambda, \omega \in \mathsf{R}$ and $\omega \neq 0$.

(ii)
$$[e_1, e_3] = \lambda e_1$$
$$[e_2, e_3] = e_1 + \lambda e_2,$$

where $\lambda \in \mathsf{R}$.

(iii)
$$[e_1, e_3] = \lambda_1 e_1$$
$$[e_2, e_3] = \lambda_2 e_2,$$

where $\lambda_1, \lambda_2 \in \mathsf{R}$.

The hyperplane subalgebras are given by

(i) the plane $\mathsf{R}e_1 + \mathsf{R}e_2$,

(ii) all planes that contain $\mathsf{R}e_1$,

(iiia) all planes, if $\lambda_1 = \lambda_2$,

(iiib) all planes which contain either $\mathsf{R}e_1$ or $\mathsf{R}e_2$, if $\lambda_1 \neq \lambda_2$,

respectively.

The corresponding groups G are simply connected with the possible exception of those corresponding to (i) or (ii) with $\lambda = 0$. In the second case, $\mathscr{E}(G)$ still contains all hyperplane subalgebras as will be observed in Example 5.2 below. In the first case one verifies directly that the analytic subgroup H corresponding to the hyperplane $\mathsf{R}e_1 + \mathsf{R}e_2$ is closed whether the group G is simply connected or not. However, only for simply connected G does H bound a half-space semigroup. Hence $\mathscr{E}(G) = \varnothing$ for G not simply connected.

5.2. Example. If M is a nilmanifold, i.e., the homogeneous space of a nilpotent group G, then the hyperplane subalgebras of $\mathfrak{g}$ are exactly the hyperplanes containing the commutator algebra (cf. also [8], Theorem II.7.5). If G is simply connected then $\mathscr{E}(G)$ contains all of these hyperplanes. If G is not simply connected and the center of the Lie algebra is contained in the commutator algebra then the center is still contained in all codimension one analytic subgroups and the same conclusion holds.

Using a result analogous to [8], Theorem II.7.5 we find

5.3. Example. If $M = G/H$ with G a complex Lie group then the only hyperplanes in $\mathfrak{g}$ which are subalgebras are the ones which contain the commutator algebra (cf. [8], Theorem II.7.6).

Let us give an example which is no longer a group.

5.4. Example. Let $G = \tilde{M}$ be the (simply connected) oscillator group and W any of the invariant cones in $\mathfrak{g}$ (cf. [3]). The results of [3] and the Classification Table show that $\tilde{M}$ is strictly causal. Since the only hyperplane subalgebra of $\mathfrak{g}$ is the commutator algebra, which is a support hyperplane of W, the causality may also be seen from Theorem 1.18. If now $H = \{\exp(nx)\colon n \in \mathsf{Z}, x \in \mathfrak{g}\backslash[\mathfrak{g}, \mathfrak{g}]\}$ then it follows from the explicit description of $\tilde{S}$ in [3] that $S = H\tilde{S} = G$.

5.5. Example. If $M = G = \mathrm{SO}(3) \times \mathsf{R}$ then $\mathscr{E}(G)$ is $\{\mathrm{so}(3) \times \{0\}\}$.

Note that any 4-dimensional Lie algebra has the property that maximal ray semigroups are half-space semigroups, so that the methods condensed in Theorem 1.18 apply. In particular, one can calculate all hyperplane subalgebras. As an example we consider the 4-dimensional solvable Lie group which occurs as a transitive subgroup of motions for the Einstein space of maximal motion $T^*_{2,5}$ (cf. [1], [17] and also [8], Section II.5).

5.6. Example. Let G be the semidirect product of $\mathbb{R}^n$ and $\mathbb{R}$ where $\mathbb{R}$ acts on $\mathbb{R}^n$ as the group of linear isomorphisms $e^{\mathbb{R}C}$ with an endomorphism C of $\mathbb{R}^n$. Then the Lie algebra $\mathfrak{g}$ is the corresponding semidirect sum of $\mathbb{R}^n \times \mathbb{R}$ with $\mathbb{R}$ acting on $\mathbb{R}^n$ via C. We write an element of G as (v, r) and an element of $\mathfrak{g}$ as (X, T). Then the multiplication is

$$(v, r)(w, s) = (v + e^{rC}w, r + s)$$

and taking the derivative of the inner automorphisms at the identity $(0, 0)$ we find that the adjoint action is given by

$$\mathrm{Ad}(v, r)(X, T) = (e^{rC}X - TCv, T).$$

The intersection Δ of all hyperplane algebras is a characteristic ideal [9]. In the present case it is of the form $J \times \{0\}$ with a C-submodule J of $\mathbb{R}^n$ containing all real Jordan factors for non-real eigenvalues and containing, for each real eigenvalue λ and each generalized eigenspace V^λ, the smallest submodule V_0^λ such that the induced action of C on V^λ/V_0^λ is semisimple, that is, by scalar multiplication. For a classification of the hyperplane subalgebras of $\mathfrak{g}$, we may factor Δ and thus assume that $J = \{0\}$. Now $\mathbb{R}^n$ is a direct sum of eigenspaces V_λ of C and C is diagonalizable. The hyperplane subalgebras are $\mathbb{R}^n \times \{0\}$ and, for each λ and each hyperplane E in V_λ, the hyperplanes $((E + \sum_{\mu \neq \lambda} V_\mu) \times \{0\}) \oplus F$ with any one dimensional vector subspace F of $\mathfrak{g}$ not contained in $\mathbb{R}^n \times \{0\}$.

For $n = 3$ we have the possibilities $\dim J = 0, 2$. If $\dim J = 2$, then $\mathfrak{g}/\Delta$ is the 2-dimensional non-abelian algebra, all of whose 1-dimensional subspaces are hyperplane subalgebras yielding precisely the hyperplane subalgebras of $\mathfrak{g}$. The case $\dim J = 0$ is the case that J is diagonalizable. The spectrum of J has cardinality 3, 2, or 1. The classification of hyperplane subalgebras in the preceding paragraph then provides a complete list in each case. If J has only one eigenvalue, then $\mathfrak{g}$ is an almost abelian Lie algebra and every hyperplane is a hyperplane subalgebra (cf. [8]).

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