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THE WEIGHTED POINCARÉ INEQUALITIES

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1. Introduction.

We consider the weighted Poincaré inequalities:

$$\left(\int_D |u(x) - u_{D,\alpha}|^p \, d(x, \partial D)^\alpha \, dx\right)^{1/p} \leqq c \left(\int_D |\nabla u(x)|^p \, d(x, \partial D)^\alpha \, dx\right)^{1/p} \tag{1.1}$$

and

$$\left(\int_D |u(x) - u_{D,\alpha}|^p \, d(x, \partial D)^\alpha \, dx\right)^{1/p} \leqq c \left(\int_D |\nabla u(x)|^p \, dx\right)^{1/p} \tag{1.2}$$

where the number $u_{D,\alpha}$ is the weighted average of u over D:

$$u_{D,\alpha} = \left(\int_D d(x, \partial D)^\alpha \, dx\right)^{-1} \int_D u(x) \, d(x, \partial D)^\alpha \, dx.$$

In particular we are interested in the metric properties of domains D where (1.1) and (1.2) hold in appropriate Sobolev classes.

If $\alpha = 0$, inequality (1.1), as well as (1.2), reduces to the ordinary Poincaré inequality.

We write $\mathscr{P}^1_{p,\alpha}$ (respectively $\mathscr{P}^2_{p,\alpha}$) for the class of bounded domains satisfying inequality (1.1) (respectively (1.2)). We give sufficient conditions of combinatoric nature for $D \in \mathscr{P}^1_{p,\alpha}$ and for $D \in \mathscr{P}^2_{p,\alpha}$, see Theorems 3.2 and 3.4. In particular, domains satisfying both a quasihyperbolic boundary condition and a Whitney cube #-condition belong to $\mathscr{P}^1_{p,\alpha}$ and $\mathscr{P}^2_{p',\alpha'}$ for some α and α', see Theorems 4.1 and 4.2. John domains are examples of such domains, see Section 5.

Weighted inequalities have been studied earlier by T. Horiuchi [Ho], T.

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Iwaniec and C. A. Nolder [IN], A. Kufner and B. Opic [KO], and V. G. Maz'ya [Maz], for example.

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2. Preliminaries.

2.1. NOTATION. Throughout this paper we let D be a domain of euclidean n-space R^n, $n \geqq 2$, with finite measure. We suppose that $\alpha \in \mathsf{R}$, unless otherwise stated, and $p \in [1, \infty)$.

In this paper a Whitney decomposition of D into non-overlapping dyadic closed cubes is denoted as W. For the construction of a Whitney decomposition see [S, VI].

The space $L^p(D, \alpha)$ is a set of functions u on D such that

$$\|u\|_{L^p(D,\alpha)} = \left(\int_D |u(x)|^p \, d(x, \partial D)^\alpha \, dx \right)^{1/p} < \infty.$$

The weighted Sobolev space $W_p^1(D, \alpha)$ is the space of functions $u \in L^p(D, \alpha)$ whose first distributional partial derivatives belong to $L^p(D, \alpha)$. In $W_p^1(D, \alpha)$ we use the norm

$$\|u\|_{W_p^1(D,\alpha)} = \|u\|_{L^p(D,\alpha)} + \|\nabla u\|_{L^p(D,\alpha)}.$$

We set $L^p(D) = L^p(D, 0)$ and $W_p^1(D) = W_p^1(D, 0)$; these are the ordinary Lebesgue and Sobolev spaces, respectively.

The weighted average of a function u over D is

$$u_{D,\alpha} = \left(\int_D d(x, \partial D)^\alpha \, dx \right)^{-1} \int_D u(x) \, d(x, \partial D)^\alpha \, dx,$$

where we suppose

$$\int_D d(x, \partial D)^\alpha \, dx < \infty.$$

If a bounded domain D satisfies a Whitney cube $\sharp$-condition, then this integral is finite also for some $\alpha < 0$, see Section 4. We write $u_D = u_{D,0}$.

We let $c(*, \ldots, *)$ denote a constant which depends only on the quantities appearing in the parentheses.

The following lemma will be used frequently.

2.2. LEMMA. *If* $u \in L^p(D,\alpha)$, $\alpha \in \mathbb{R}$, *then*

$$\|u - u_{D,\alpha}\|_{L^p(D,\alpha)} \leqq 2\,\|u - c\|_{L^p(D,\alpha)}$$

for each $c \in \mathbb{R}$.

PROOF. By the Minkowski inequality

$$\|u - u_{D,\alpha}\|_{L^p(D,\alpha)} \leqq \|u - c\|_{L^p(D,\alpha)} + \|c - u_{D,\alpha}\|_{L^p(D,\alpha)}.$$

The Hölder inequality yields

$$\|u_{D,\alpha} - c\|_{L^p(D,\alpha)} = \left(\int_D \left| \frac{1}{\int_D d(y,\partial D)^\alpha\,dy} \int_D u(y)\,d(y,\partial D)^\alpha\,dy - c \right|^p d(x,\partial D)^\alpha\,dx\right)^{1/p}$$

$$= \frac{1}{\int_D d(y,\partial D)^\alpha\,dy} \left(\int_D \left| \int_D (u(y) - c)\,d(y,\partial D)^\alpha\,dy \right|^p d(x,\partial D)^\alpha\,dx\right)^{1/p}$$

$$\leqq \left(\int_D d(y,\partial D)^\alpha\,dy\right)^{1/p-1} \left(\int_D 1 \cdot |u(y) - c|\,d(y,\partial D)^\alpha\,dy\right)$$

$$\leqq \left(\int_D d(y,\partial D)^\alpha\,dy\right)^{1/p-1+1-1/p} \left(\int_D |u(y) - c|^p\,d(y,\partial D)^\alpha\,dy\right)^{1/p} = \|u - c\|_{L^p(D,\alpha)}.$$

3. Sufficient conditions.

We apply some methods used in [Hu, Sections 4 and 6]:

Let D be a domain and W its Whitney decomposition. Write tQ for the cube with the same center as Q and expanded by a factor $t > 1$. Fix $Q_0 \in W$ and $x_0 \in Q_0$. Join Q_0 to $Q \in W$ with a chain of expanded Whitney cubes $\frac{9}{8}Q_j$, $j = 0, 1, \ldots, k$, $Q_k = Q$, such that

$$Q_i \cap Q_j \neq \emptyset \quad \text{if and only if} \quad |i - j| \leqq 1,$$

see [Hu, the proof for Proposition 6.1]. This construction of expanded Whitney cubes is called a chain, abbreviated $C(Q_k) = (Q_0, Q_1, \ldots, Q_k)$. We let $\ell(C(Q_k)) = k$ denote the length of the chain $C(Q_k)$.

For each $Q \in W$ we fix a chain $C(Q)$. For a fixed cube $A \in W$ we write

$$A(W) = \{Q \in W \mid A \in C(Q)\}.$$

3.1. LEMMA. *For each $Q \in W$ and $u \in W_p^1(D,\alpha)$*

$$\int_{\frac{9}{8}Q} |u(y) - u_{\frac{9}{8}Q}|^p d(y,\partial D)^\alpha \, dy \leqq c(n,p)\,\mathrm{dia}(Q)^p \int_{\frac{9}{8}Q} |\nabla u(y)|^p d(y,\partial D)^\alpha \, dy$$

and for $u \in L^p(D,\alpha)$ such that $|\nabla u| \in L^p(D)$

$$\int_{\frac{9}{8}Q} |u(y) - u_{\frac{9}{8}Q}|^p d(y,\partial D)^\alpha \, dy \leqq c(n,p)\,\mathrm{dia}(Q)^{\alpha+p} \int_{\frac{9}{8}Q} |\nabla u(y)|^p \, dy.$$

PROOF. For each $y \in \frac{9}{8}Q$

$$\tfrac{1}{4} \leqq \frac{d(y,\partial D)}{\mathrm{dia}(Q)} \leqq 20.$$

Thus $u \in W_p^1$ (int $\frac{9}{8}Q$) (in both cases) and the Poincaré inequality without weights in a cube yields the claims.

The quasihyperbolic distance between points x_1 and x_2 in D is given by

$$k_D(x_1,x_2) = \inf_\gamma \int_\gamma \frac{ds}{d(x,\partial D)}$$

where the infimum is taken over all rectifiable curves γ joining x_1 and x_2 in D. For the properties of k_D see [GP] and [GO].

3.2. THEOREM. *Suppose that D is a domain in R^n, $x_0 \in D$, and let $p \in [1,\infty)$. Suppose that*

$$\int_D k_D(x_0,x)^{p-1} d(x,\partial D)^\alpha \, dx < \infty. \tag{3.3}$$

If $p \geqq n + \alpha$, then $D \in \mathscr{P}^1_{p,\alpha}$.
If $p \geqq \max\{-\alpha, n\}$, then $D \in \mathscr{P}^2_{p,\alpha}$.

3.4. THEOREM. *Suppose that D is a domain in R^n, $x_0 \in D$.*
(i) *Let $p \in [1,\infty)$. If for some constant c*

$$\sum_{Q \in A(W)} \int_{\frac{9}{8}Q} (k_D(x_0,x) + 1)^{p-1} d(x,\partial D)^\alpha \, dx \leqq c\,\mathrm{dia}(A)^{n+\alpha-p} \tag{3.5}$$

whenever $A \in W$, then $D \in \mathscr{P}^1_{p,\alpha}$.
(ii) *Let $p \in [\max\{-\alpha, 1\},\infty)$. If for some constant c*

$$\sum_{Q\in A(W)} \int\limits_{\frac{9}{8}Q} (k_D(x_0, x) + 1)^{p-1} d(x, \partial D)^\alpha \, dx \leqq c \operatorname{dia}(A)^{n-p} \tag{3.6}$$

whenever $A \in W$, *then* $D \in \mathscr{P}^2_{p,\alpha}$.

We note that for $\alpha = 0$ Theorems 3.2 and 3.4 reduce to Theorem 6.7 (P_1), (P_3) in [Hu].

Proof of Theorems 3.2 and 3.4. Let $Q_0 \in W$ be such that $x_0 \in Q_0$. By Lemma 2.2 it suffices to estimate

$$\int\limits_D |u(y) - u_{\frac{9}{8}Q_0}|^p \, d(y, \partial D)^\alpha \, dy.$$

We shall employ properties of Whitney cubes.

First

$$\int\limits_D |u(y) - u_{\frac{9}{8}Q_0}|^p \, d(y, \partial D)^\alpha \, dy = \sum_{Q\in W} \int\limits_Q |u(y) - u_{\frac{9}{8}Q_0}|^p \, d(y, \partial D)^\alpha \, dy \tag{3.7}$$

$$\leqq 2^{p-1}\Bigg(\sum_{Q\in W} \int\limits_{\frac{9}{8}Q} |u(y) - u_{\frac{9}{8}Q}|^p \, d(y, \partial D)^\alpha \, dy + \sum_{Q\in W} \int\limits_{\frac{9}{8}Q} |u_{\frac{9}{8}Q} - u_{\frac{9}{8}Q_0}|^p \, d(y, \partial D)^\alpha \, dy\Bigg).$$

By Lemma 3.1

$$\begin{aligned} &\sum_{Q\in W} \int\limits_{\frac{9}{8}Q} |u(y) - u_{\frac{9}{8}Q}|^p \, d(y, \partial D)^\alpha \, dy \\ &\leqq c_1(n,p,\alpha) \sum_{Q\in W} \operatorname{dia}(Q)^p \int\limits_{\frac{9}{8}Q} |\nabla u(y)|^p \, d(y, \partial D)^\alpha \, dy \\ &\leqq c_2(n,p,\alpha) \operatorname{dia}(D)^p \int\limits_D |\nabla u(y)|^p \, d(y, \partial D)^\alpha \, dy \end{aligned} \tag{3.8}$$

and

$$\sum_{Q\in W} \int\limits_{\frac{9}{8}Q} |u(y) - u_{\frac{9}{8}Q}|^p \, d(y, \partial D)^\alpha \, dy \tag{3.9}$$

$$\leqq c_3(n,p,\alpha)\sum_{Q\in W}\operatorname{dia}(Q)^{p+\alpha}\int_{\frac{9}{8}Q}|\nabla u(y)|^p\,dy$$

$$\leqq c_4(n,p,\alpha)\operatorname{dia}(D)^{p+\alpha}\int_D|\nabla u(y)|^p\,dy,$$

if $p+\alpha\geqq 0$.

To estimate the last sum in (3.7) we fix $Q\in W$ and join Q_0 to Q with the chain $C(Q_0)=(Q_0,Q_1,\dots,Q_k)$, $Q_k=Q$. Write

$$u_j=u_{\frac{9}{8}Q_j}=\fint_{\frac{9}{8}Q_j}u(y)\,dy.$$

Now the ordinary Poincaré inequality in a cube yields

$$|u_{\frac{9}{8}Q}-u_{\frac{9}{8}Q_0}|^p\leqq\left(\sum_{j=1}^k|u_j-u_{j-1}|\right)^p\leqq k^{p-1}\sum_{j=1}^k|u_j-u_{j-1}|^p$$

$$=k^{p-1}\sum_{j=1}^k\ \fint_{\frac{9}{8}Q_{j-1}\cap\frac{9}{8}Q_j}|u_j-u_{j-1}|^p\,dy$$

$$=(2k)^{p-1}\sum_{j=1}^k\frac{1}{|\frac{9}{8}Q_{j-1}\cap\frac{9}{8}Q_j|}\left(\int_{\frac{9}{8}Q_{j-1}}|u_{j-1}-u(y)|^p\,dy+\int_{\frac{9}{8}Q_j}|u_j-u(y)|^p\,dy\right)$$

$$\leqq c_5(n,p)k^{p-1}\sum_{j=0}^k\operatorname{dia}(Q_j)^{p-n}\int_{\frac{9}{8}Q_j}|\nabla u(y)|^p\,dy$$

Write $k=\ell(C(Q))$; now

$$\text{(3.10)}\quad\sum_{Q\in W}\int_{\frac{9}{8}Q}|u_{\frac{9}{8}Q}-u_{\frac{9}{8}Q_0}|^p\,d(y,\partial D)^\alpha\,dy$$

$$=c_5(n,p)\sum_{Q\in W}\int_{\frac{9}{8}Q}\ell(C(Q))^{p-1}\,d(y,\partial D)^\alpha\,dy\sum_{A\in C(Q)}\operatorname{dia}(A)^{p-n}\int_{\frac{9}{8}A}|\nabla u(x)|^p\,dx;$$

and changing the order of summation we obtain

$$(3.11)\quad \sum_{Q\in W}\int_{\frac{9}{8}Q}|u_{\frac{9}{8}Q}-u_{\frac{9}{8}Q_0}|^p\,d(y,\partial D)^\alpha\,dy$$
$$\leqq c_5(n,p)\sum_{A\in W}\sum_{Q\in A(W)}\int_{\frac{9}{8}Q}\ell(C(Q))^{p-1}\,d(y,\partial D)^\alpha\,dy\,\mathrm{dia}(A)^{p-n}\int_{\frac{9}{8}A}|\nabla u(x)|^p\,dx.$$

Next we shall employ the inequality ([Hu, Proposition 6.1])

$$\ell(C(Q))\leqq c(n)(k_D(x_0,x)+1)\quad \text{for each}\quad x\in Q.$$

If $p\geqq n+\alpha$, we obtain from (3.10)

$$\sum_{Q\in W}\int_{\frac{9}{8}Q}\ell(C(Q))^{p-1}\,d(y,\partial D)^\alpha\,dy\sum_{A\in C(Q)}\mathrm{dia}(A)^{p-n-\alpha}\int_{\frac{9}{8}A}|\nabla u(x)|^p\,d(x,\partial D)^\alpha\,dx$$
$$\leqq c_6\,\mathrm{dia}(D)^{p-n-\alpha}\sum_{Q\in W}\int_{\frac{9}{8}Q}(k_D(x_0,x)+1)^{p-1}\,d(x,\partial D)^\alpha\,dx\int_D|\nabla u(x)|^p\,d(x,\partial D)^\alpha\,dx,$$

where $c_6=c_6(\alpha,n,p)$. This together with (3.3), (3.7) and (3.8) yields $D\in\mathscr{P}^1_{p,\alpha}$, if $p\geqq n+\alpha$. If $p\geq\max\{-\alpha,n\}$, then (3.3), (3.7), (3.9) and (3.10) imply $D\in\mathscr{P}^2_{p,\alpha}$. Hence Theorem 3.2 is proved.

From (3.11) we obtain

$$\sum_{Q\in W}\int_{\frac{9}{8}Q}|u_{\frac{9}{8}Q}-u_{\frac{9}{8}Q_0}|^p\,d(y,\partial D)^\alpha\,dy$$
$$\leqq c_7\sum_{A\in W}\sum_{Q\in A(W)}\int_{\frac{9}{8}Q}(k_D(x_0,x)+1)^{p-1}\,d(x,\partial D)^\alpha\,dx\,\mathrm{dia}(A)^{p-n-\alpha}\int_{\frac{9}{8}A}|\nabla u(x)|^p\,d(x,\partial D)^\alpha\,dx.$$

where $c_7=c_7(\alpha,n,p)$. This together with (3.5), (3.7) and (3.8) implies $D\in\mathscr{P}^1_{p,\alpha}$. If $p+\alpha\geqq 0$, then (3.6), (3.7), (3.9) and (3.11) yield $D\in\mathscr{P}^2_{p,\alpha}$, Thus Theorem 3.4 is proved.

4. Domains satisfying (3.3), (3.5) or (3.6).

Here we give examples of domains which satisfy the conditions in Theorems 3.2 and 3.4. These examples show that the Poincaré domains can be quite non-smooth.

John domains. [MS] A domain D is called an (α,β)-*John domain*, $0<\alpha\leqq\beta<\infty$, if there is $x_0\in D$ such that each $x\in D$ can be joined to x_0 by a curve $\gamma\colon [0,\ell] \to D$ parametrized by arc length with $\ell \leqq \beta$ and

$$d(\gamma(t), \partial D) \geqq \frac{\alpha}{\ell} t, \quad t \in [0,\ell].$$

Domains satisfying a quasihyperbolic boundary condition. A domain D satisfies a quasihyperbolic boundary condition with a constant $a > 0$, if there exists a point x_0 such that

$$k_D(x_0, x) \leqq a \log\left(1 + \frac{|x_0 - x|}{\min\{d(x,\partial D), d(x_0, \partial D)\}}\right)$$

for all $x \in D$, see [GM, 3.6], [HV, Section 2] and [Hu, 7.2].

John domains form a proper subclass of domains satisfying a quasihyperbolic boundary condition.

Plumpness. Following O. Martio and J. Väisälä [MaVä, 2.1] we say that a domain D is *α-plump*, $0 < \alpha \leqq 1$, if there is $\sigma > 0$ such that for every $y \in \partial D$ and for all $t \in (0, \sigma]$ there is $x \in D \cap \overline{B^n(y,t)}$ with $d(x, \partial D) > \alpha t$.

We consider only bounded domains satisfying a quasihyperbolic boundary condition.

Whitney cube #-condition ([MaVu, 2.1]). Suppose that for a bounded domain D

$$D = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{N_k} Q_j^k$$

where the Whitney decomposition W of D is arranged in such a way that the Whitney cubes Q_j^k satisfy

$$\operatorname{dia}(Q_j^k) = \operatorname{dia}(D) 2^{-k}, \quad j = 1, \ldots, N_k.$$

A domain D is said to satisfy a *Whitney cube #-condition* with $\lambda < n$, if there are constants $M < \infty$ and $\lambda \in (0, n)$ such that

$$N_k \leqq M 2^{\lambda k} \quad \text{for each} \quad k.$$

A John domain satisfies a Whitney cube #-condition, [MaVu, Lemmas 6.3 and 2.8], and, more generally, if a domain D is plump, then it satisfies a Whitney cube #-condition, see [MaVu, 2.7 and Lemma 2.8].

4.1. THEOREM. *Let D be a domain in $\mathbf{R}^n$ satisfying a quasihyperbolic boundary condition with a constant a and a Whitney cube #-condition with $\lambda < n$. Let $\alpha > \lambda - n$. Now* (i) *$D \in \mathscr{P}^1_{p,\alpha}$ for each $p \geqq \alpha + n$ and* (ii) *$D \in \mathscr{P}^2_{p,\alpha}$ for each $p \geqq n$.*

PROOF FOR THE CLAIM (i). Write $c_0 = \sup_{x \in D} d(x, \partial D)/d(x_0, \partial D)$.

The quasihyperbolic boundary condition yields

$$\int_D k_D(x_0,x)^{p-1}\, d(x,\partial D)^\alpha\, dx = \sum_{Q\in W} \int_Q k_D(x_0,x)^{p-1}\, d(x,\partial D)^\alpha\, dx$$

$$\leqq a^{p-1} \sum_{Q\in W} \int_Q \left(\log\left(1 + \frac{c_0|x_0 - x|}{d(x,\partial D)}\right)\right)^{p-1} d(x,\partial D)^\alpha\, dx$$

and the Whitney cube #-condition gives

$$\sum_{Q\in W} \int_Q k_D(x_0,x)^{p-1}\, d(x,\partial D)^\alpha\, dx$$

$$\leqq a^{p-1} \sum_{k=1}^{\infty} \sum_{j=1}^{N_k} \left(\log \frac{2c_0 \operatorname{dia}(D)}{\operatorname{dia}(Q_j^k)}\right)^{p-1} |Q_j^k| \operatorname{dia}(Q_j^k)^\alpha$$

$$\leqq c \sum_{k=1}^{\infty} k^{p-1} 2^{k(\lambda - n - \alpha)} < \infty,$$

where the constant c depends on $n, p, a, \lambda, d(x_0, \partial D)$, and $dia(D)$. Thus Theorem 3.2 yields $D \in \mathscr{P}^1_{p,\alpha}$.

A similar estimate as above also yields the claim (ii).

If instead of the Whitney cube #-condition plumpness is used, then we obtain better estimates than in Theorem 4.1 for the exponents α and p.

4.2. THEOREM. *Let D be a domain in R^n satisfying a quasihyperbolic boundary condition with a constant a, let D be β-plump, and let* $\varepsilon = (\log(1 + (\beta/24)^n))/\log(120/\beta)$ *and* $p \in [1, \infty)$.

(i) *If $\alpha > -\varepsilon$ and $p > (\alpha + n)\left(1 - \dfrac{1}{2a}\right) - \dfrac{\varepsilon}{2a}$, then $D \in \mathscr{P}^1_{p,\alpha}$.*

(ii) *If $\alpha > \max\{-\varepsilon, -p\}$ and $p > n - (\alpha + \varepsilon + n)/2a$, then $D \in \mathscr{P}^2_{p,\alpha}$.*

For the proof we decompose D into Whitney cubes and construct chains as explained at the beginning of Section 3. We need the following lemma.

4.3. LEMMA ([Hu, Lemma 7.27]). *Suppose that D satisfies a quasihyperbolic boundary condition and D is β-plump. Then for each $A \in W$*

$$\sum_{Q\in B_j} |Q| \leqq c 2^{-j\varepsilon} \operatorname{dia}(A)^{(n+\varepsilon)/2a}$$

where

$$B_j = \left\{ Q \in A(W) \,\middle|\, 2^{-j} \leqq \frac{\operatorname{dia}(Q)}{c_1 \operatorname{dia}(A)^{1/a}} \leqq 2^{-j+1} \right\},$$

$j = 1, 2, \ldots$; constants c and c_1 depend at most on $n, p, \beta, d(x_0, \partial D)$ and $\operatorname{dia}(D)$.

PROOF OF (i) IN THEOREM 4.2. Fix $A \in W$. Write $c_0 = \sup_{x \in D} d(x, \partial D)/d(x_0, \partial D)$. Constants c_i, $i = 1, 2, 3$, below depend at most on $n, p, \alpha, \beta, a, d(x_0, \partial D)$, and $\operatorname{dia}(D)$. The quasihyperbolic boundary condition and Lemma 4.3 imply

$$\begin{aligned}
&\sum_{Q \in B_j} \int_{\frac{9}{8}Q} (k_D(x_0, x) + 1)^{p-1} d(x, \partial D)^\alpha \, dx \\
&\leqq c_1 \sum_{Q \in B_j} \left(\log \frac{2ec_0 \operatorname{dia}(D)}{\operatorname{dia}(Q)} \right)^{p-1} \operatorname{dia}(Q)^\alpha |Q| \\
&\leqq \frac{c_2}{\delta} j^{p-1} 2^{-j(\alpha+\varepsilon)} \operatorname{dia}(A)^{(n+\varepsilon)/2a + \alpha/a - \delta}
\end{aligned}$$

where $0 < \delta < (n + \varepsilon)/2a + \alpha/a$. Summing over $j = 1, 2, \ldots$ we obtain

$$\sum_{Q \in A(W)} \int_{\frac{9}{8}Q} (k_D(x_0, x) + 1)^{p-1} d(x, \partial D)^\alpha \, dx \leqq \frac{c_3}{\delta} \operatorname{dia}(A)^{(n+\varepsilon)/2a + \alpha/a - \delta},$$

see [Hu, Lemma 7.13]. Theorem 3.4 yields the claim (i).

The proof for the claim (ii) in Theorem 4.2 is analogous.

4.4. REMARK. If $D \subset \mathsf{R}^n$ is a John domain with a Whitney cube #-constant $\lambda < n$ and if $\delta \in (\lambda - n, \infty)$, then $D \in \mathscr{P}^1_{p,\delta}$ for each $p \geqq 1$, see Theorem 5.2. The following example shows that the lower bound for p in Theorems 4.1 (i) and 4.2 (i) is essentially sharp for a non-John domain.

4.5. EXAMPLE. Let G_0 be the open rectangle bounded by the lines

$$x_1 = 0, x_2 = 0, x_1 = 1, x_2 = -1$$

and for $j = 1, 2, \ldots$ let G_j be the open triangle bounded by

$$x_1 = 2^{-2j}, x_2 = 2^{-2j} - 2^{-2bj}, x_1 + x_2 = 2^{-2j} - 2^{-2bj},$$

where $b \geqq 2$ is a constant; cf. [GM, Example 2.2b]. Denote by G^* the reflection of the domain $\bigcup_{j=0}^{\infty} G_j$ with respect to the line $x_2 = -\frac{1}{2}$. Set

$$G = \bigcup_{j=0}^{\infty} G_j \cup G^*.$$

Let $T: \mathsf{R}^2 \to \mathsf{R}^2$ be a translation such that $T(x_1, x_2) = (x_1, x_2 + \frac{1}{2})$. Set $D = T(G)$.

The domain D satisfies a quasihyperbolic boundary condition with $a = 36b$ and also a Whitney cube #-condition for some $\lambda_0 \in [1, 2)$. Thus by Theorem 4.1 (i)

$D \in \mathscr{P}^1_{p,\delta}$ at least for each $p \geqq 2 + \delta$, where $\delta \in (2 - \lambda, 0]$, $\lambda \in [\lambda_0, 2)$ will be fixed later. We show that $D \notin \mathscr{P}^1_{p,\delta}$, if $p < 2 + \delta - \dfrac{1}{2b}(4 + \delta)$.

Let G_j^1 be the open set bounded by the lines

$$x_1 = 2^{-2j},\ x_2 = 2^{-2j} - 2^{-2bj},\ x_2 = 2^{-2bj},\ x_1 + x_2 = 2^{-2j} - 2^{-2bj}.$$

Let G_j^{1*} be the image of G_j^1 under reflection across the line $x_2 = -\frac{1}{2}$. Set $T(G_j^1) = D_j^1$ and $T(G_j \backslash (G_j^1 \cap G_0)) = D_j^2$ and $T(G_j^{1*}) = D_j^{1*}$ and $T\ (G_j^* \backslash (G_j^{1*} \cap G_0)) = D_j^{2*}$,see Figure 4.1.

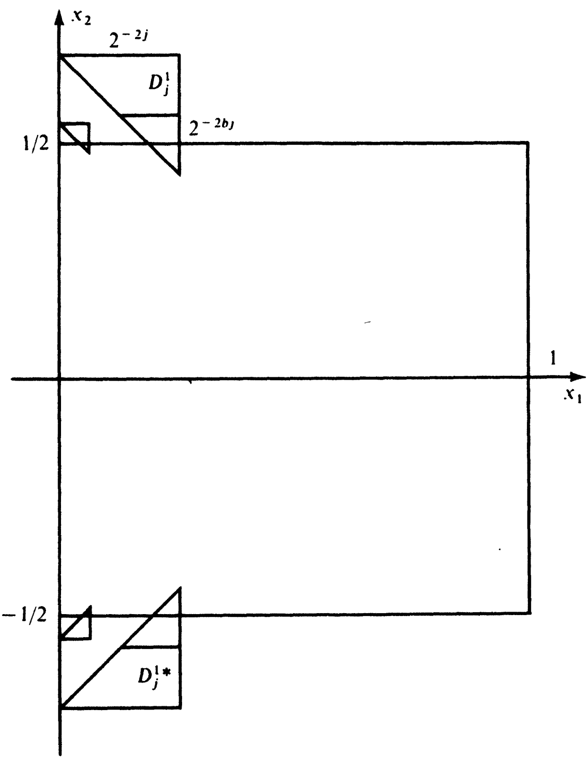

Figure 4.1.

Fix $D_j^2, j = 1, \ldots$. Now D_j^2 is an (α_j, β_j)-John domain with $\alpha_j = \dfrac{1}{2} 2^{-2bj}$ and $\beta_j = 3 \cdot 2^{-2bj}$. Let $W(D_j^2)$ be a Whitney decomposition of D_j^2. The proofs of [MaVu, Lemmas 6.3 and 2.8] yield that

$$\#\{Q \in W(D_j^2) \mid \ \operatorname{dia}(Q) = \operatorname{dia}(D_j^2) 2^{-k}\} \leqq A 2^{\lambda_1 k},$$

$k = 1, 2, \ldots$, and $\lambda_1 < 2$. Constants c_i, $i = 1, \ldots, 5$, depend at most on δ, p, and G. Set $\lambda = \max\{\lambda_0, \lambda_1\}$. Now

$$\int_{D_j^2} d(x, \partial D_j^2)^\delta \, dx = \sum_{Q \in W(D_j^2)} \int_Q d(x, \partial D_j^2)^\delta \, dx \tag{4.6}$$

$$= \sum_{k=1}^{\infty} \sum_{\substack{Q \in W(D_j^2) \\ \operatorname{dia}(Q) = \operatorname{dia}(D_j^2) 2^{-k}}} \int_Q d(x, \partial D_j^2)^\delta \, dx \leqq c_1 \operatorname{dia}(D_j^2)^{2+\delta} \sum_{k=1}^{\infty} 2^{-(2+\delta-\lambda)k}$$

$$= c_2 \operatorname{dia}(D_j^2)^{2+\delta} = c_3 2^{-2b(2+\delta)j},$$

if $\delta > \lambda - 2$.

Fix $\delta \in (\lambda - 2, 0]$. Choose a piecewise linear continuous function $u: D \to R$ such that

$$u(x) = \begin{array}{rll} 2^{(4+2\delta)j} & \text{in} & D_j^1, j = 1, 2, \ldots \\ 0 & \text{in} & \{(x_1, x_2) \mid x_1 \in (0,1),\ x_2 \in (-\frac{1}{2}, \frac{1}{2})\} \\ -2^{(4+2\delta)j} & \text{in} & D_j^{1*}, j = 1, 2, \ldots \end{array}$$

Now $u_D = 0$ and

$$\int_D |u(x)|^p d(x, \partial D)^\delta \, dx \geqq \sum_{j=1}^{\infty} \int_{D_j^1} 2^{(4+2\delta)j} d(x, \partial D)^\delta \, dx$$

$$\geqq \sum_{j=1}^{\infty} 2^{(4+2\delta)j} (2 \cdot 2^{-2j})^\delta |D_j^1| = c_4 \sum_{j=1}^{\infty} 2^{2(2+\delta)j} 2^{-2(2+\delta)j} = \infty.$$

On the other hand,

$$\int_D |\nabla u(x)|^p d(x, \partial D)^\delta \, dx = 2 \sum_{j=1}^{\infty} \int_{D_j^2} 2^{(4+\delta+2bp)j} d(x, \partial D)^\delta \, dx$$

$$\leqq 2 \sum_{j=1}^{\infty} 2^{(4+\delta+2bp)j} \int_{D_j^2} d(x, \partial D_j^2)^\delta \, dx$$

and by (4.6)

$$\int_D |\nabla u(x)|^p d(x, \partial D)^\delta \, dx \leqq c_5 \sum_{j=1}^{\infty} 2^{(4+\delta+2bp-4b-2b\delta)j} < \infty,$$

if $p < 2 + \delta - \dfrac{1}{2b}(4 + \delta)$. Hence $D \notin \mathscr{P}^1_{p,\delta}$, if $p < 2 + \delta - \dfrac{1}{2b}(4 + \delta)$.

5. The weighted Poincaré inequality in John domains.

We will show that an (α, β)-John domain $D \in \mathscr{P}^1_{p,\gamma}$ for each $p \in [1, \infty)$ whenever $\gamma \in (c(n, \beta/\alpha), \infty)$ where $c(n, \beta/\alpha) < 0$ is a constant. Our method is based on a potential estimate and the method of Martio [M], and it differs from that used in Section 3.

5.1. THEOREM. *Let D be a bounded domain in R^n and let W be its Whitney decomposition. If D satisfies a Whitney cube #-condition, with constants $M < \infty$ and $\lambda < n$ such that*

$$\#\{Q \in W \mid \operatorname{dia}(Q) = \operatorname{dia}(D)2^{-j}\} \leqq M 2^{\lambda j}$$

for each $j = 1, 2, \ldots$, then for all $y \in D$

$$\int_D |x - y|^{1-n} d(x, \partial D)^\gamma \, dx \leqq c(n, \gamma, \lambda) M^{1/(2n-1)} \operatorname{dia}(D)\, d(y, \partial D)^\gamma \tag{5.2}$$

where $(\lambda - n)/2n < \gamma \leqq 0$.

PROOF. Let $x, y \in D$. Since D is bounded,

$$d(y, \partial D) \leqq \operatorname{dia}(D)^{2n/(2n-1)} d(x, \partial D)^{1/(1-2n)}$$

for all $x, y \in D$. Thus

$$\int_D |x - y|^{1-n} d(x, \partial D)^\gamma \, dx = \int_D |x - y|^{1-n} d(x, \partial D)^{\gamma/(1-2n)} d(x, \partial D)^{\gamma(1 - 1/(1-2n))} \, dx$$

$$\leqq \operatorname{dia}(D)^{2n\gamma/(1-2n)} d(y, \partial D)^\gamma \int_D |x - y|^{1-n} d(x, \partial D)^{2n\gamma/(2n-1)} \, dx.$$

By the Hölder inequality with exponents $(2n - 1)/2(n - 1)$ and $2n - 1$

$$\int_D |x - y|^{1-n} d(x, \partial D)^{2n\gamma/(2n-1)} \, dx$$

$$\leqq \left(\int_D |x - y|^{(1-2n)/2} \, dx \right)^{2(n-1)/(2n-1)} \left(\int_D d(x, \partial D)^{2n\gamma} \, dx \right)^{1/(2n-1)}$$

where

$$\int_D |x-y|^{(1-2n)/2}\,dx \leqq \int_{B^n(y,(|D|/\Omega_n)^{1/n})} |x-y|^{(1-2n)/2}\,dx$$

$$= c_1(n) \int_0^{(|D|/\Omega_n)^{1/n}} \rho^{n-1+\frac{1}{2}-n}\,d\rho = c_2(n)\,|D|^{\frac{1}{2n}}.$$

Using a Whitney decomposition W of D we obtain

$$\int_D d(x,\partial D)^{2n\gamma}\,dx = \sum_{j=1}^{\infty} \sum_{\substack{Q\in W\\ \operatorname{dia}(Q)\\ =\operatorname{dia}(D)2^{-j}}} \int_Q d(x,\partial D)^{2n\gamma}\,dx$$

$$\leqq M\,\operatorname{dia}(D)^{n(1+2\gamma)} \sum_{j=1}^{\infty} 2^{-j(n+2n\gamma-\lambda)} < \infty,$$

if $n + 2n\gamma - \lambda > 0$.

The above inequalities yield (5.2) and the theorem is proved.

5.3. THEOREM. *Let $p\in[1,\infty)$. An (α,β)-John domain D belongs to $\mathscr{P}^1_{p,\gamma}$ where $(\lambda-n)/2n < \gamma \leqq 0$ and $\lambda = \lambda(n,\beta/\alpha) < n$ is the Whitney cube #-constant.*

PROOF. Let x_0 be a John center and let $x\in D$. Now [M, Theorem 2.2] implies that there is an L-bilipschitz mapping T_x of $B^n(0,\alpha)$ into D such that $T_x(0) = x_0$, $x\in T_x(B^n(0,\alpha))$, and $L = c(n)\left(\dfrac{\beta}{\alpha}\right)^4$. Write $A = T_x(B^n(0,\alpha))$ and $E = B^n\left(x_0, c(n)\dfrac{\alpha^5}{\beta^4}\right)$.

Let $u\in W_p^1(D,d(x,\partial D)^\gamma)$. The proofs of [M, Lemmas 2.3, 2.4 and 2.5] yield that $\bar{A}\subset D$. Hence $u\in W_p^1(A,d(x,\partial D)^\gamma)$ and the norms $\|u\|_{W_p^1(A,d(x,\partial D)^\gamma)}$ and $\|u\|_{W_p^1(A)}$ are equivalent. Thus $W_p^1(A,d(x,\partial D)^\gamma) = \overline{C^\infty(A)}$, where the closure is taken with respect to the norm $\|\cdot\|_{W_p^1(A,d(x,\partial D)^\gamma)}$. Hence we obtain from [M, Lemma 3.3]

$$|u(x)-u_E| \leqq c_1(n,\alpha,\beta)\int_A |x-y|^{1-n}|\nabla u(y)|\,dy \leqq c_2(n,\alpha,\beta)\int_D |x-y|^{1-n}|\nabla u(y)|\,dy$$

for x. Since $x\in D$ was an arbitrary point,

$$|u(x)-u_E| \leqq c_2(n,\alpha,\beta)\int_D |x-y|^{1-n}|\nabla u(y)|\,dy$$

for each $x\in D$.

The Hölder inequality yields

$$|u(x) - u_E|^p \leqq c_2(n,\alpha,\beta)\left(\int_D (|x-y|)^{1-1/p}(|x-y|^{1-n})^{1/p}|\nabla u(y)|\,dy\right)^p$$

$$\leqq c_2(n,\alpha,\beta)\left(\int_D |x-y|^{1-n}\,dy\right)^{p-1}\int_D |x-y|^{1-n}|\nabla u(y)|^p\,dy,$$

where

$$\int_D |x-y|^{1-n}\,dy \leqq \int_{B^n(x,(|D|/\Omega_n)^{1/n})} |x-y|^{1-n}\,dy \leqq n\Omega_n(|D|/\Omega_n)^{1/n}.$$

Multiplying with $d(x,\partial D)^\gamma$ on both sides of the inequality

$$|u(x) - u_E|^p \leqq c_3(n,p,\alpha,\beta)\,|D|^{(p-1)/n}\int_D |x-y|^{1-n}|\nabla u(y)|^p\,dy$$

and integrating over D with respect to the variable x and using Fubini's theorem we obtain

$$\int_D |u(x) - u_E|^p\,d(x,\partial D)^\gamma\,dx$$

$$\leqq c_3(n,p,\alpha,\beta)\,|D|^{(p-1)/n}\int_D\left(\int_D |x-y|^{1-n}|\nabla u(y)|^p\,dy\right)d(x,\partial D)^\gamma\,dx$$

$$= c_3(n,p,\alpha,\beta)\,|D|^{(p-1)/n}\int_D |\nabla u(y)|^p\left(\int_D |x-y|^{1-n}\,d(x,\partial D)^\gamma\,dx\right)dy$$

An (α,β)-John domain satisfies a Whitney cube $\sharp$-condition with $\lambda = \lambda(n,\beta/\alpha) < n$ and $M = M(n,\alpha,\beta)$ [MaVu, Lemmas 6.3 and 2.8]. Thus Theorem 5.1 implies

$$\int_D |x-y|^{1-n}\,d(x,\partial D)^\gamma\,dx \leqq c_4(n,\alpha,\beta,\gamma)\,\mathrm{dia}(D)\,d(y,\partial D)^\gamma$$

where $(\lambda - n)/2n < \gamma \leqq 0$. Thus combining the above inequalities we obtain

$$\int_D |u(x) - u_E|^p\,d(x,\partial D)^\gamma\,dx$$

$$\leqq c_5(n,p,\alpha,\beta,\gamma)\,|D|^{(p-1)/n}\,\mathrm{dia}(D)\int_D |\nabla u(y)|^p\, d(y,\partial D)^\gamma\, dy$$

$$\leqq c_6(n,p,\alpha,\beta,\gamma)\,\mathrm{dia}(D)^p\int_D |\nabla u(y)|^p\, d(y,\partial D)^\gamma\, dy.$$

Lemma 2.2 completes the proof.

5.4. Remark. Theorem 3.4 (i) and [Hu, Lemmas 8.3 and 8.4] yield that an (α,β)-John domain D belongs to $\mathscr{P}^1_{p,\gamma}$ for all $p \geqq 1$, if $\gamma \geqq 0$.

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