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A HOMOLOGICAL PROOF OF A THEOREM BY DAVIS, GERAMITA, ORECCHIA GIVING THE CAYLEY-BACHARACH THEOREM

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Let A be a graded Gorenstein k-algebra. There is a connection between two Hilbert series related characters for a Cohen-Macaulay (C-M) ideal and its algebraic link in A. This was shown by Davis, Geramita, Orecchia [2]. A corollary is the classical Cayley-Bacharach theorem. We will give a homological proof of the D-G-O theorem, using ideas from Peskine-Szpiro [6].

Throughout this paper, let $R = k[X_1, \ldots, X_n]$ (k a field). We call an ideal Gorenstein (C-M) if R/I is Gorenstein (C-M).

We begin by recalling some facts about *algebraicaly linked* ideals (from e.g. [6]).

DEFINITION. *Let $G \subset R$ be a Gorenstein ideal, and I, $J \subset R$ ideals with $G \subset I \cap J$. We call I and J algebraically linked by G if*

$$I = G:J \quad \text{and} \quad J = G:I.$$

i) If this is the case, then $\operatorname{ht} I = \operatorname{ht} J = \operatorname{ht} G$.

ii) Suppose $G = I \cap J$, where I, J are parts of the primary decomposition of G with no common associated prime. Then I and J are algebraically linked.

Next, we recall the definition of the *cone* of a morphism of complexes, and a long exact sequence involving it (from e.g. [1])

Let $u:(C', d') \to (C, d)$ be a morphism of complexes.

DEFINITION. (Con(u), D) *is the complex* $(\mathrm{Con}(u))_n = C'_{n-1} \oplus C_n$ *and*

$$D: C'_n \oplus C'_{n+1} \to C'_{n-1} \oplus C_n$$

$$(y', x) \mapsto (-d'y', dx - uy')$$

There is a long exact sequence

$$\ldots \to H_n(C') \to H_n(C) \to H_n(\mathrm{Con}(u)) \to H_{n-1}(C') \to \ldots.$$

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It will prove helpful to establish some additional facts (which can be found in [3], [4]).

Let A be a graded k-algebra (i.e. $A = R/J$, for some homogeneous ideal J) with Krull-dimension d and projective dimension c.

1) A has Hilbert-series $H_A(z) = p(z)/(1-z)^d$.

DEFINITION. $\sigma(A) = \deg p(z)$.

If now $I \subset A$ is a homogeneous ideal, we define the following two characters

DEFINITION. $\sigma(I) = \sigma(A/I)$.

DEFINITION. $\alpha(I) = \min\{t \mid I_t \neq 0\}$.

2) A has a graded R-resolution

$$0 \to \bigoplus_{i=1}^{b_c} R[-d_{i,c}] \to \ldots \to \bigoplus_{i=1}^{b_1} R[-d_{i,1}] \to R \to A \to 0$$

where all maps are of degree zero ($R[d]_n = R_{n+d}$) and are given by forms of positive degree on the components where they are non-zero (it's called a minimal resolution). Conventionally,

$$d_{1,j} \leqq d_{2,j} \leqq \ldots \leqq d_{b_j,j}, \quad \text{all } j.$$

3) $H_A(z) = (1 - \sum_1^{b_1} z^{d_{i,1}} + \ldots + (-1)^c \sum_1^{b_c} z^{d_{i,c}})/(1-z)^n$.

4) The sequence of lowest degrees is strictly increasing:

$$d_{1,1} < d_{1,2} < \ldots < d_{1,c}.$$

5) If A is C-M, the same for highest degrees:

$$d_{b_1,1} < d_{b_2,2} < \ldots < d_{b_c,c}.$$

6) If A is Gorenstein, then $b_c = 1$, and furthermore, the resolution is symmetric, in the sense that $b_j = b_{c-j}$, for all j, and if $p(z) = 1 - \sum z^{d_{i,1}} + \ldots + (-1)^c z^{d_{1,c}}$ then $p(z) = z^{d_{1,c}} p(1/z)(-1)^c$.

7) Here I is any ideal in any noetherian ring R.

a) $\mathrm{gr}(I)$ = length of a maximal R-sequence in $I = \min\{i \mid \mathrm{Ext}_R^i(R/I, R) \neq 0\}$.

b) $\mathrm{gr}(I) \leqq \mathrm{pd}_R(R/I)$

c) R/I C-M $\Leftrightarrow \mathrm{gr}(I) = \mathrm{pd}_R(R/I)$.

We are now appropriately equipped to prove the following

THEOREM. *Let $G \subset I$ be homogeneous ideals, G Gorenstein and I C-M with the same dimension in $R = k[X_1, \ldots, X_n]$. Let $J = G : I$. Then*

$$\alpha(\bar{J}) + \sigma(\bar{I}) = \sigma(\bar{R}) \quad (\bar{X} = X/G).$$

REMARK. *Then, also, $\alpha(\bar{I}) + \sigma(\bar{J}) = \sigma(\bar{R})$ since also I satisfies the hypothesis and I, J are algebraically linked by G, according to* [6].

PROOF. Put $\dim(R/I) = \dim(R/G) = n - k$ (so both I and G have height k). Then $\mathrm{pd}(R/G) = \mathrm{pd}(R/I) = k$ (use (7)). Take minimal graded R-resolutions $\mathscr{F}_0$, $\mathscr{F}_1$ of R/G and R/I, respectively.

$$\begin{array}{lccccccc}
\mathscr{F}_0 & 0 \to R[-d_k] & \to \dots \to & \bigoplus_{i=1}^{b_1} R[-d_{i,1}] & \to R & \to R/G & \to 0 \\
\downarrow u & \downarrow & & \downarrow & \downarrow & \downarrow & \\
\mathscr{F}_1 & 0 \to \bigoplus_{i=1}^{c_k} R[e_{i,k}] & \to \dots \to & \bigoplus_{i=1}^{c_1} R[e_{i,1}] & \to R & \to R/I & \to 0
\end{array}$$

The lifting u exists (well-known).

NOTE. The resolutions enable us to deduce the Hilbert-series (3), e.g. $H_{R/G}(z) = p(z)/(1 - z)^n$ with $p(z)$ as in (6). This allows us to compute the σ's; $\sigma(R/G) = d_k - k, \sigma(I) = e_{c_k,k} - k$.

Dualize, that is, apply the functor $\mathrm{Hom}_R(.,R)$ (remember $\mathrm{Hom}(R[-a],R) = R[a]$).

$$\begin{array}{lcccc}
\check{\mathscr{F}}_0 & 0 \leftarrow R[d_k] & \leftarrow \dots \leftarrow & \bigoplus_{i=1}^{b_1} R[d_{i,1}] & \leftarrow R \leftarrow 0 \\
\uparrow \check{u} & \uparrow & & \uparrow & \\
\check{\mathscr{F}}_1 & 0 \leftarrow \bigoplus_{i=1}^{c_k} R[e_{i,k}] & \leftarrow \dots \leftarrow & \bigoplus_{i=1}^{c_1} R[e_{i,1}] & \leftarrow R \leftarrow 0
\end{array}$$

Here, let the homological dimension decrease from k (to the right) to 0 (left). Now, take the cone.

$$\mathrm{Con}(\check{u}) \quad 0 \leftarrow R[d_k] \leftarrow \bigoplus_{i=1}^{c_k} R[e_{i,k}] \bigoplus_{i=1}^{b_{k-1}} R[d_{i,k-1}] \leftarrow \dots \leftarrow \bigoplus_{i=1}^{c_1} R[e_{i,1}] \oplus R \leftarrow R \leftarrow 0$$

Shift degrees.

$$\mathrm{Con}(\check{u})[-d_k] \quad 0 \leftarrow R \leftarrow \bigoplus_{i=1}^{c_k} R[e_{i,k} - d_k] \bigoplus_{i=1}^{b_{k-1}} R[d_{i,k-1} - d_k] \leftarrow \dots$$

$$\leftarrow \bigoplus_{i=1}^{c_1} R[e_{i,1} - d_k] \oplus R[-d_k] \leftarrow R[-d_k] \leftarrow 0.$$

CLAIM. $\mathrm{Con}(\check{u})[-d_k] \to R/J \to 0$ is a resolution.

Before proving the claim, let's wrap up.

$$(1 - z)^n H_{R/J}(z) =$$

$$= 1 - \sum z^{d_k - d_{i,k-1}} + \dots + (-1)^{k-1} \sum z^{d_k - d_{i,1}} + (-1)^k z^{d_k}$$
$$- \sum z^{d_k - e_{i,k}} + \dots + (-1)^k \sum z^{d_k - e_{i,1}} + (-1)^{k+1} z^{d_k}.$$

The first row on the right side is $z^{d_k} p(1/z)(-1)^k = p(z)$, so

$$H_{R/J}(z) = H_{R/G}(z) - \frac{\sum z^{d_k - e_{i,k}} - \ldots + (-1)^{k-1} \sum z^{d_k - e_{i,1}} + (-1)^k z_k^d}{(1-z)^n}.$$

Consider this exact sequence of graded R-modules:

$$0 \to J/G \to R/G \to R/J \to 0.$$

It follows that

$$H_{J/G}(z) = H_{R/G}(z) - H_{R/J}(z) = \frac{\sum z^{d_k - e_{i,k}} - \ldots + (-1)^{k-1} \sum z^{d_k - e_{i,1}} + (-1)^k z_k^d}{(1-z)^n}.$$

Now,

$$\alpha(J/G) = \text{the least occuring degree in the numerator} =$$
$$= d_k - e_{c_k,k} \qquad (R/I \text{ C-M, use (5)}).$$

But, as pointed out in the note above,

$$\begin{cases} \sigma(\bar{R}) = d_k - k \\ \sigma(\bar{I}) = \sigma(I) = e_{c_k,k} - k \end{cases}$$

so

$$\alpha(J/G) = \sigma(\bar{R}) - \sigma(\bar{I}).$$

The proof is completed, but for the proof of the claim. Before proceeding with that, we recall from [5, Lemma 2], that there is a functorial isomorphism

$$(*) \qquad \operatorname{Ext}_R^k(M, R) \simeq \operatorname{Hom}_R(M, R/G)$$

for modules M with $\operatorname{Ann}(M) \supset G$.

Proof of claim. We have to show

$$\begin{cases} H_0(\text{Con}) = R/J \\ H_i(\text{Con}) = 0, \quad i > 0 \end{cases}$$

Use the characterization of grade by Ext (7a) to get

$$\begin{cases} \operatorname{Ext}_R^i(R/G, R) = 0, & i < k \\ \operatorname{Ext}_R^i(R/I, R) = 0, & i < k' \end{cases} \text{whence} \begin{cases} H_i(\check{\mathscr{F}}_0) = 0, & i > 0 \\ H_i(\check{\mathscr{F}}_1) = 0, & i > 0 \end{cases}$$

Consider the long exact sequence mentioned in context with the definition of the cone

$$H_i(\check{\mathscr{F}}_0) \to H_i(\text{Con}) \to H_{i-1}(\check{\mathscr{F}}_1).$$

From this, we immediately get

$$i > 1 \Rightarrow H_i(\mathrm{Con}) = 0.$$

We also have, further down in the exact sequence,

$$\begin{array}{ccccccc} 0 \rightarrow H_1(\mathrm{Con}) \rightarrow & H_0(\check{\mathscr{F}}_1) & \overset{\varphi}{\rightarrow} & H_0(\check{\mathscr{F}}_0) & \rightarrow H_0(\mathrm{Con}) \rightarrow 0 \\ & \| & & \| & \\ & \mathrm{Ext}^k_R(R/I, R) & & \mathrm{Ext}^k_R(R/G, R) & \end{array}$$

But there is a commutative diagram

$$\begin{array}{lllll} {}^1\mathrm{xt}^k_R(R/I,R) & \simeq \mathrm{Hom}_R(R/I, R/G) & = \mathrm{Hom}_{R/G}(R/I, R/G) & \simeq \mathrm{Hom}_{R/G}\left(\dfrac{R/G}{I/G}, R/G\right) & \simeq \dfrac{G:I}{G} \\ \downarrow \varphi & \downarrow \beta & & & \\ \mathrm{Ext}^k_R(R/G,R) & \simeq \mathrm{Hom}_R(R/G, R/G) & = \mathrm{Hom}_{R/G}(R/G, R/G) & \simeq \qquad R/G & \end{array}$$

Let's take a moment to explain this. One gets the first quadrant by applying (*) above. The equalities are true, since $G \subset \mathrm{Ann}(R/I) \cap \mathrm{Ann}(R/G)$, and the last isomorphism in the top row is due to the fact that $\mathrm{Hom}_S(S/J, S) \simeq 0:J$ for any ring S. Furthermore, β is mono, since the functor $\mathrm{Hom}(\cdot, R/G)$ is left exact. Now, we have

$$\beta \text{ mono} \Rightarrow \varphi \text{ mono} \Rightarrow H_1(\mathrm{Con}) = 0$$

$$H_0(\mathrm{Con}) = \mathrm{Coker}\left(\frac{G:I}{G} \rightarrow R/G\right) = \frac{R}{G:I} = R/J.$$

This ends proof of claim.

A rather immediate consequence of this theorem is the

CAYLEY-BACHARACH THEOREM. *Let* $(f_1, \ldots, f_r)$ *be a complete intersection in* $k[X_0, X_1, \ldots, X_r]$ *(k alg. closed field),* $\deg f_i = d_i$ *and* $V(f_1, \ldots, f_r) = \{P \in \mathscr{P}^r \mid f_i(P) = 0\, \forall i\} = \{P_1, \ldots, P_m\}$. *Suppose* $m = d_1 \cdot \ldots \cdot d_r$. *Then every hypersurface of degree* $\leqq \sum d_i - \gamma$ $(\gamma \geqq r + 1)$ *which passes through* $m - \binom{\gamma - 1}{r}$ *of the points in V, passes through them all, if the remaining* $\binom{\gamma - 1}{r}$ *don't lie on a hypersurface of degree* $\leqq \gamma - r - 1$.

This is proved in [2]. Of course, the first proof, for $r = 2$, was done in the 1880's, with completely different methods.

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