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ON MODULES INDUCED OR COINDUCED FROM HOPF SUBALGEBRAS

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Let A be a Hopf algebra over a commutative ring k, $B \subset A$ a Hopf subalgebra, and W a left A-module. Koppinen and Neuvonen [2] showed that W is induced from B, that is $W \cong A \otimes_B V$ for a left B-module V, if and only if W admits a system of imprimitivity based on B; their result assumes however that the antipode of A is bijective and that A as a right B-module is a finitely generated and projective generator. Essentially the same result holds for coinduced modules, i.e. modules of the form $W \cong \mathrm{Hom}_B(A, V)$. The rather strong assumptions in [2] were made in order to apply the Morita theorems, and Koppinen and Neuvonen asked whether these assumptions can be weakened ([2], Remark). The present paper gives proofs for the above results which do not use Morita theory, and which only assume A to be finitely generated and projective over B. In the induced case a more general result is given which only needs A to be flat over B; this is closely related to [1] and [4].

In the following A denotes a Hopf algebra over a commutative ring k, and $B \subset A$ a Hopf subalgebra. The antipode and counit are denoted by λ and ε, respectively, and the coproduct by δ. "A-module" will mean left A-module.

1. Induced Modules.

Let F denote the k-algebra considered in [2]; as a k-module F consists of all right B-linear maps $f\colon A \to k$, i.e. $f(ab) = f(a)\varepsilon(b)$ for $a \in A$, $b \in B$, and the product is given by

$$(f \cdot f')(a) = \sum f(a_{(2)})f'(a_{(1)}), \qquad f, f' \in F.$$

F is an A-module with $(af)(a') = f(\lambda(a)a')$ for $a, a' \in A$. By definition, an A-module W admits a system of imprimitivity based on B if it is a left F-module satisfying

$$(1) \qquad a(fw) = \sum (a_{(1)}f)(a_{(2)}w), \qquad a \in A, f \in F, w \in W.$$

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In the following let $C = A \otimes_B k$. This is naturally an A-module coalgebra with coproduct $C \to C \otimes C$, $a \otimes 1 \mapsto \sum (a_{(1)} \otimes 1) \otimes (a_{(2)} \otimes 1)$, $a \in A$. (We note that $C \cong A/AB^+$ in the notation of [4], and that C represents the kernel of $\mathrm{Sp}(A) \to \mathrm{Sp}(B)$ if A is commutative, [5], p. 14). Let ${}_A^C\mathfrak{M}$ be the category of A-modules W which are supplied with a left C-coaction $\alpha \colon W \to C \otimes W$ such that

$$\alpha(aw) = \delta(a)\alpha(w), \qquad a \in A,\ w \in W. \tag{2}$$

A morphism in ${}_A^C\mathfrak{M}$ is a map which is both A-linear and C-colinear.

LEMMA 1.1. *Assume A_B is finitely generated and projective. Then an A-module W admits a system of imprimitivity based on B if and only if W is an object of ${}_A^C\mathfrak{M}$.*

PROOF. First note that F is essentially the opposite of the dual algebra C^* of C. For

$$\zeta\colon \mathrm{Hom}_k(C, k) \to F, \qquad \zeta(g)(a) = g(a \otimes 1),$$

is an anti-isomorphism of k-algebras with $\zeta^{-1}(f)(a \otimes 1) = f(a)$; ζ is an algebra antimorphism since the product of F is defined by the transpose coproduct of A. Since A_B is finitely generated and projective, $C = A \otimes_B k$ is so over k. Therefore, W is a left F-module if and only if W is a left C-comodule, the actions being determined by each other through the formula

$$f \cdot w = \sum \langle \zeta^{-1}(f), w_{(-1)} \rangle w_{(0)}, \qquad f \in F,\ w \in W.$$

Since $C \otimes W \cong \mathrm{Hom}_k(F, W)$, one sees that (1) is equivalent to

$$(1 \otimes a)\alpha(w) = \sum (\lambda(a_{(1)}) \otimes 1)\alpha(a_{(2)} w), \qquad a \in A,\ w \in W. \tag{3}$$

The latter is evidently satisfied if (2) holds. Conversely, applying (3) with a replaced by $a_{(2)}$ one obtains $\delta(a)\alpha(w) = \sum (a_{(1)}\lambda(a_{(2)}) \otimes 1)\alpha(a_{(3)}w) = \alpha(aw)$. Hence (2) is equivalent to (1), and this completes the proof.

For any left B-module V, $W = A \otimes_B V$ is naturally an object of ${}_A^C\mathfrak{M}$ with coaction $W \to C \otimes W$, $a \otimes v \mapsto \sum (a_{(1)} \otimes 1) \otimes (a_{(2)} \otimes v)$. Conversely, we want to show that any $W \in {}_A^C\mathfrak{M}$ is induced from B if A_B is flat. This is closely related to [1], Thm. 2.11, and [4], Thm. 2, and essentially the same proof as in [1] can be employed. It is based on the following lemma.

LEMMA 1.2. *For any A-module X the map*

$$\eta_X\colon A \otimes_B X \to C \otimes X, \qquad a \otimes x \mapsto \sum (a_{(1)} \otimes 1) \otimes a_{(2)} x,$$

is an isomorphism.

PROOF. For $b \in B$ we have $\eta_X(ab \otimes x) = \sum (a_{(1)} \otimes \varepsilon(b_{(1)})) \otimes (a_{(2)} b_{(2)} x) = \eta_X(a \otimes bx)$; hence η_X is well-defined, and $(a \otimes 1) \otimes x \mapsto \sum a_{(1)} \otimes \lambda(a_{(2)})x$ gives a (well-defined) inverse.

THEOREM 1.3. *Let A be a Hopf k-algebra, $B \subset A$ a Hopf subalgebra, and $C = A \otimes_B k$. Assume A is flat as a right B-module. Then an A-module W is induced from B if and only if W is an object of ${}_A^C\mathfrak{M}$* (i.e. *a left C-comodule satisfying* (2)).

PROOF. Let α: $W \to C \otimes W$ be the coaction of W. Regard $C \otimes W$ as an A-module by $a \cdot (c \otimes w) = \delta(a)(c \otimes w)$. Then both α and i: $W \to C \otimes W$, $w \mapsto (1 \otimes 1) \otimes w$, are B-linear. Set $W_0 = \{w \in W | \alpha(w) = i(w)\}$. Since A is B-flat, the sequence

$$(4) \qquad A \otimes_B W_0 \to A \otimes_B W \underset{1 \otimes i}{\overset{1 \otimes \alpha}{\rightrightarrows}} A \otimes_B (C \otimes W)$$

is exact. Consider μ_W: $A \otimes_B W_0 \to W$, $a \otimes w \mapsto aw$, which is a morphism in ${}_A^C\mathfrak{M}$. It is easy to see that $(\mu_W, \eta_W, \eta_{C \otimes W})$ transforms (4) into the sequence

$$W \xrightarrow{\alpha} C \otimes W \underset{\delta \otimes 1}{\overset{1 \otimes \alpha}{\rightrightarrows}} C \otimes C \otimes W.$$

But this sequence is exact for any comodule. Hence Lemma 1.2 implies that μ_W is an isomorphism.

Let ${}_B\mathfrak{M}$ denote the category of left B-modules.

COROLLARY 1.4. (cf. [4], Thm. 2). *Assume A_B is faithfully flat. Then the functor ${}_B\mathfrak{M} \to {}_A^C\mathfrak{M}$, $V \mapsto A \otimes_B V$, is an equivalence.*

PROOF. For $V \in {}_B\mathfrak{M}$ consider the B-linear map v: $V \to (A \otimes_B V)_0$, $v \mapsto 1 \otimes v$. Since $\mu(1 \otimes v)$: $A \otimes_B V \to A \otimes_B V$ is the identity, we have that $1 \otimes v$, hence v, is an isomorphism. It follows that $W \mapsto W_0$ is a quasi-inverse for $V \mapsto A \otimes_B V$.

REMARK 1.5. For $B = k$ theorem 1.3 gives the descent theorem for Hopf modules [3], Thm. 4.1.1. It should be noted however that the proof given in [3] works also for A not k-flat (and k not a field).

2. Coinduced Modules.

In the following we work with the k-algebra $E = \mathrm{Hom}_B(A, k)$ of all *left* B-linear maps f: $A \to k$. The product is defined by

$$(f \cdot f')(a) = \sum f(a_{(1)}) f'(a_{(2)}), \qquad f, f' \in E.$$

E is an A-module algebra with $(af)(a') = f(a'a)$. We shall consider the category ${}_A\mathfrak{M}_E$ of left A-modules and right E-modules W satisfying

$$(5) \qquad a(wf) = \sum (a_{(1)} w)(a_{(2)} f), \qquad a \in A,\ w \in W,\ f \in E.$$

EXAMPLE. Let $W = \mathrm{Hom}_B(A, V)$ for a left B-module V. For $g \in W$ and $f \in E$ define $gf \in W$ by $(gf)(a) = \sum g(a_{(1)}) f(a_{(2)})$, $a \in A$. Then W is an object of ${}_A\mathfrak{M}_E$ with natural A-action $(ag)(a') = g(a'a)$.

REMARK 2.1. The condition for an A-module W to be an object of ${}_A\mathfrak{M}_E$ is slightly different from that of admitting a system of imprimitivity. There is no difference if A is cocommutative, for then E is commutative and $f \mapsto f\lambda$ gives an algebra isomorphism $F \xrightarrow{\sim} E$. There appears however to be a gap at the end of the proof in [2] for the coinduced case. The proof provides an action on W by the right B-endomorphisms of A, but for $W \cong \mathrm{Hom}_B(A, V)$ a (right) action by the left B-endomorphisms of A is needed.

Let $W \in {}_A\mathfrak{M}_E$. We define a left B-module W_0 by the exact sequence

$$W \otimes E \underset{m'}{\overset{m}{\rightrightarrows}} W \xrightarrow{p} W_0 \tag{6}$$

where $m(w \otimes f) = wf$, and $m'(w \otimes f) = wf(1)$. Note that m and m' are B-linear if we regard $W \otimes E$ as a B-module by $b \cdot (w \otimes f) = \delta(b)(w \otimes f)$. We want to show that the left A- and right E-linear map

$$\mu_W\colon W \to \mathrm{Hom}_B(A, W_0), \qquad \mu_W(w)(a) = p(aw), \tag{7}$$

is an isomorphism if ${}_BA$ is finitely generated and projective. The proof is in some sense dual to that in section 1, and uses the following lemma (cf. the lemma in [2]).

LEMMA 2.2. *Let X be an A-module, and suppose ${}_BA$ is finitely generated and projective. Then*

$$\vartheta_X\colon X \otimes E \to \mathrm{Hom}_B(A, X), \qquad \vartheta_X(x \otimes f)(a) = \sum a_{(1)} f(a_{(2)}) x,$$

is an isomorphism.

PROOF. Let ${}^{\varepsilon}X = X$ with left B-action $b \cdot x = \varepsilon(b)x$. Then

$$\beta\colon \mathrm{Hom}_B(A, {}^{\varepsilon}X) \to \mathrm{Hom}_B(A, X), \qquad \beta(\varphi)(a) = \sum a_{(1)} \varphi(a_{(2)}),$$

is an isomorphism with $\beta^{-1}(\psi)(a) = \sum \lambda(a_{(1)}) \psi(a_{(2)})$ for $\psi \in \mathrm{Hom}_B(A, X)$. Set ${}^{\varepsilon}\vartheta = \beta^{-1}\vartheta$. Then for $f \in E$, $a \in A$, and $x \in X$

$${}^{\varepsilon}\vartheta(x \otimes f)(a) = \sum \lambda(a_{(1)}) \vartheta(x \otimes f)(a_{(2)}) = \sum \lambda(a_{(1)}) a_{(2)} f(a_{(3)}) x = f(a)x.$$

Now choose a projective coordinate system $f_i \in \mathrm{Hom}_B(A, B)$, $a_i \in A$, $1 \leq i \leq n$; then $\mathrm{Hom}_B(A, {}^{\varepsilon}X) \to X \otimes E$, $\varphi \mapsto \sum \varphi(a_i) \otimes \varepsilon \circ f_i$, is an inverse for ${}^{\varepsilon}\vartheta$ as follows from $\sum f_i(a) a_i = a$ for $a \in A$.

THEOREM 2.3. *Let A be a Hopf k-algebra, $B \subset A$ a Hopf subalgebra, $E = \mathrm{Hom}_B(A, k)$, and assume that A is finitely generated and projective as a left*

B-module. Then an A-module W is coinduced from B if and only if W is an object of ${}_A\mathfrak{M}_E$ (i.e. *a right E-module satisfying* (5)).

PROOF. For any right E-module W there is a canonical exact sequence

$$(8) \qquad W \otimes E \otimes E \rightrightarrows W \otimes E \to W$$

defined by the action of E on W. Suppose $W \in {}_A\mathfrak{M}_E$ and consider $W \otimes E$ as a left A-module by $a \cdot (w \otimes f) = \delta(a)(w \otimes f)$. Then $(\vartheta_{W \otimes E}, \vartheta_W, \mu_W)$, with μ_W defined in (7), transforms (8) into the exact sequence obtained from (6) by applying $\mathrm{Hom}_B(A, -)$. Observe that

$$\vartheta_{W \otimes E}(w \otimes f \otimes f')(a) = \sum a_{(1)} f'(a_{(2)})(w \otimes f) = \sum f'(a_{(3)})(a_{(1)} w \otimes a_{(2)} f),$$

and $p(wf) = p(wf(1))$. In particular, $(\mu_W m)(w \otimes f)(a) = p(a(wf)) = p(\sum (a_{(1)} w)(a_{(2)} f)) = p(\sum a_{(1)} w f(a_{(2)})) = p\vartheta_W(w \otimes f)(a)$. It follows therefore from lemma 2.2 that μ_W is an isomorphism.

COROLLARY 2.4. *Suppose ${}_BA$ is finitely generated and projective. Then the functor ${}_A\mathfrak{M}_E \to {}_B\mathfrak{M}$, $W \mapsto W_0$, is an equivalence if and only if $B \subset A$ is a left B-direct summand of A.*

PROOF. First observe that $V \mapsto \mathrm{Hom}_B(A, V)$ is a right adjoint for $W \mapsto W_0$, the adjunction morphisms being μ, and ν: $\mathrm{Hom}_B(A, V)_0 \to V, p(g) \mapsto g(1)$, for $g \in \mathrm{Hom}_B(A, V)$. Furthermore, the composite

$$\mathrm{Hom}_B(A, V) \xrightarrow{\mu} \mathrm{Hom}_B(A, \mathrm{Hom}_B(A, V)_0) \xrightarrow{\mathrm{Hom}(A, \nu)} \mathrm{Hom}_B(A, V)$$

is the identity, since $p(ag) = (ag)(1) = g(a)$ for $a \in A$. Hence $\mathrm{Hom}(A, \nu)$ is bijective. Now, if $A = B \oplus X$, we may conclude that ν is bijective by decomposing $\mathrm{Hom}(A, \nu) = \mathrm{Hom}(B, \nu) \oplus \mathrm{Hom}(X, \nu)$. Conversely, suppose that ν is bijective for $V = B$. Then there exists $g \in \mathrm{Hom}_B(A, B)$ with $g(1) = 1$, and therefore $A = B \oplus \mathrm{Ker}(g)$. □

REMARK 2.5. $B \subset A$ is a left B-direct summand of A iff A is a left B-generator; for suppose there exists a B-epimorphism $A^{(I)} \to B$. Pick $u \in A^{(I)}$ such that $u \mapsto 1$. Then g: $A \to Au \to B$ is a B-epimorphism with $g(1) = 1$, and hence with $A = B \oplus \mathrm{Ker}(g)$.

REMARK 2.6. For $B = k$ the assumption of thm. 2.3 can not be omitted. This can be seen as follows. Let X be an A-module. Then $W = X \otimes E$ is naturally a right E-module and an object of ${}_A\mathfrak{M}_E$ with A-action defined by δ. It is not difficult to see that

$$W \otimes E \underset{m'}{\overset{m}{\rightrightarrows}} W \xrightarrow{p} X$$

is exact where $p(x \otimes f) = f(1)x$ for $x \in X$, and $f \in E$. Hence in this case $W_0 = X$.

Furthermore, for $\mu_W\colon X \otimes E \to \mathrm{Hom}_B(A, X)$ we obtain

$$\mu_W(x \otimes f)(a) = p(a \cdot (x \otimes f)) = p(\delta(a)(x \otimes f)) = \sum a_{(1)} f(a_{(2)})x.$$

Thus $\mu_W = \vartheta_X$. Suppose now that μ_W is bijective for $X = A$. Then also ${}^{\varepsilon}\vartheta = \beta^{-1}\vartheta$ is an isomorphism, and in case $B = k$ this means that A is finitely generated and projective over k.

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