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STANLEY DECOMPOSITIONS OF THE BRACKET RING

BERND STURMFELS* and NEIL WHITE*

Abstract.

Methods from algebraic combinatorics are applied to give an explicit Stanley decomposition of the commutative ring generated by the $d \times d$-minors of a generic $n \times d$-matrix.

1. Introduction.

Let $\mathsf{K}[x_{ij}]$ denote the polynomial ring freely generated over a field K by the entries of a generic $n \times d$-matrix (x_{ij}) where $n \geqq d$. The *bracket ring* $\mathsf{B}_{n,d}$ is the subring of $\mathsf{K}[x_{ij}]$ generated by the $d \times d$-minors

$$[i_1 i_2 \ldots i_d] := \det\begin{pmatrix} x_{i_1,1} & \cdots & x_{i_1,d} \\ \vdots & \ddots & \vdots \\ x_{i_d,1} & \cdots & x_{i_d,d} \end{pmatrix}$$

of (x_{ij}). Let $\Lambda_{n,d}$ denote the set of brackets $[i_1 i_2 \ldots i_d]$ with $1 \leqq i_1 < i_2 < \ldots < i_d \leqq n$, and let $\mathsf{K}[\Lambda_{n,d}]$ denote the polynomial ring freely generated by the $\binom{n}{d}$-element set $\Lambda_{n,d}$. Products of brackets (i.e. monomials in $\mathsf{K}[\Lambda_{n,d}]$) are called *tableaux*.

We write the bracket ring as $\mathsf{B}_{n,d} = \mathsf{K}[\Lambda_{n,d}]/I_{n,d}$ where $I_{n,d}$ is the ideal of *syzygies* or algebraic dependencies among the $d \times d$-minors. Recall the *First and Second Fundamental Theorems of Invariant Theory* which state respectively that $\mathsf{B}_{n,d}$ is the invariant ring under the canonical action of SL (K^d) on $\mathsf{K}[x_{ij}]$ and that the syzygy ideal $I_{n,d}$ is generated by the quadratic Grassmann-Plücker relations [7], [14].

It is the objective of this note to derive an explicit representation of the bracket ring as finite direct sum of K-vector spaces

$$(1) \qquad \mathsf{B}_{n,d} = \bigotimes_{i=1}^{m_{n,d}} T_i \, \mathsf{K}[B_{i,}{}^0, B_{i,1}, \ldots, B_{i,d(n-d)}],$$

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where the T_i are suitable tableaux and the $B_{i,j}$ are suitable brackets. Such a *Stanley decomposition* for the special case $d = 2$ has been found by Cushman, Sanders and White [6]. We remark that explicit combinatorial decompositions of classical invariant rings such as $\mathsf{B}_{n,d}$ are of considerable importance in the Cushman-Sanders normal form theory for nilpotent vector fields [5].

The reader is referred to our earlier paper [15] for an introduction to Stanley decompositions of affine algebras in general. In [15, Section 2] an algorithm based on Buchberger's Gröbner bases method [4] is given for computing Stanley decompositions. The result of the present note is an application of several interesting techniques from algebraic combinatorics, including the classical straightening algorithm, Stanley-Reisner rings and shellability of distributive lattices. We wish to thank Paul Edelman and Michelle Wachs for helpful ideas regarding this paper.

2. The construction.

We begin by considering Stanley decompositions arising from shellings of simplicial complexes. See [3], [9], [10], [11] for details on shellings. The square-free monomials in the polynomial ring $\mathsf{K}[x_1,\ldots,x_t]$ are identified with the subsets of $\{x_1,\ldots,x_t\}$. The *Stanley-Reisner ring* of a simplicial complex Δ on $\{x_1,x_2,\ldots,x_t\}$ is the quotient ring $\mathsf{K}[\Delta] := \mathsf{K}[x_1,\ldots,x_t]/I_\Delta$ where I_Δ is the ideal generated by all (square-free) monomials not contained in Δ.

LEMMA 1. *Suppose that Δ is shellable and that $(\sigma_1,\sigma_2,\ldots,\sigma_m)$ is an ordering of the facets of Δ arising from a shelling. Let η_i be the unique minimal (with respect to inclusion) face of σ_i which is not a face of the subcomplex $\bigcup_{j=1}^{i-1} \sigma_j$, for $i = 1,2,\ldots,m$. Then*

$$\mathsf{K}[\Delta] = \bigoplus_{i=1}^{m} \eta_i\,\mathsf{K}[\sigma_i]$$

is a Stanley decomposition of the Stanley-Reisner ring $\mathsf{K}[\Delta]$.

Lemma 1 is an easy analogue to Kind & Kleinschmidt's [10] and Garsia's [9] constructive proof for the Cohen-Macaulayness of shellable complexes, and to Stanley's decompositions of diophantine rings in [12, Theorem 5.2]. Every monomial $\mu \in \mathsf{K}[x_1,\ldots,x_t]$ which is non-zero in $\mathsf{K}[\Delta]$ appears in exactly summand $\eta_i\,\mathsf{K}[\sigma_i]$ in Lemma 1. The corresponding index i is the least integer such that the face $\mathrm{supp}(\mu)$ is in the facet σ_i.

In order to apply Lemma 1 to the bracket ring $\mathsf{B}_{n,d}$, we need the following definitions. A tableau $T = [i_{1,1}i_{1,2}\cdots i_{1,d}][i_{2,1}i_{2,2}\cdots i_{2,d}]\cdots[i_{m,1}i_{m,2}\cdots i_{m,d}] \in \mathsf{K}[\Lambda_{n,d}]$ is *standard* if $i_{r,s} < i_{r,s+1}$ and $i_{r,s} \leq i_{r+1,s}$ for all r, s; otherwise T is *non-standard.* If we consider T as a rectangular $m \times d$-tableau, then T is standard if its rows are increasing and its columns are non-decreasing. Let $J_{n,d} \subset \mathsf{K}[\Lambda_{n,d}]$ denote the monomial ideal generated by all non-standard tableaux.

In [14] we used the classical straightening algorithm to see that $J_{n,d}$ is the initial ideal of the syzygy ideal $I_{n,d}$ with respect to a suitable term order on $\mathsf{K}[\Lambda_{n,d}]$. By the results of [15], it suffices to find a Stanley decomposition for the ring $\mathsf{K}[\Lambda_{n,d}]/J_{n,d}$. More precisely, [15, Lemma 2.6] implies that every Stanley decomposition of $\mathsf{K}[\Lambda_{n,d}]/J_{n,d}$ is automatically a Stanley decomposition of $\mathsf{B}_{n,d}$.

Identifying the subsets of $\Lambda_{n,d}$ with the square-free monomials of $\mathsf{K}[\Lambda_{n,d}]$, the set of (square-free) standard tableaux defines a simplicial complex $\Delta_{n,d}$ on the set $\Lambda_{n,d}$. The Stanley-Reisner ring of $\Delta_{n,d}$ equals the above monomial ring: $\mathsf{K}[\Delta_{n,d}] = \mathsf{K}[\Lambda_{n,d}]/J_{n,d}$. Using Lemma 1 it suffices to show that $\Delta_{n,d}$ is shellable.

LEMMA 2. *The simplicial complex $\Delta_{n,d}$ of square-free standard tableaux of shape k by d (where k is necessarily less than or equal to $d(n-d)+1$) is an $(n-d)d$-dimensional shellable complex on the set $\Lambda_{n,d}$ of brackets. The maximal simplices of $\Delta_{n,d}$ are in one-to-one correspondence with the standard tableaux of shape d by $n-d$ on $d(n-d)$ distinct symbols.*

PROOF. We define a partial order "$\prec$" on the set $\Lambda_{n,d}$ by setting

$$[i_1 i_2 \dots i_d] \prec [j_1 j_2 \dots j_d] \Leftrightarrow i_1 \leqq j_1 \text{ and } i_2 \leqq j_2 \text{ and } \dots \text{ and } i_d \leqq j_d.$$

The simplicial complex $\Delta_{n,d}$ is the chain complex of the resulting poset $(\Lambda_{n,d}, \prec)$ and hence it suffices to show that the poset $(\Lambda_{n,}{}^{d}, \prec)$ is shellable.

Now, $\Lambda_{n,d}$ is a distributive lattice because it is the lattice of order ideals of the poset $C_d \times C_{n-d}$ (= the direct product of a d-chain and an $(n-d)$-chain). A result of J. S. Provan [11, Section 3.4.2] states that every distributive lattrice is shellable, and hence $\Lambda_{n,d}$ is shellable.

The maximal chains of $\Lambda_{n,d}$ are in one-to-one correspondence with the linear extensions of $C_d \times C_{n-d}$, that is, with labelings of $C_d \times C_{n-d}$ with the integers $1, 2, 3, \dots, d(n-d)$ which respect the ordering, which are in turn the standard d by $n-d$ tableaux. The dimension of $\Delta_{n,d}$ equals the length of the maximal chains in $\Lambda_{n,d}$, which equals $\#(C_d \times C_{n-d}) = d(n-d)$. The maximal simplices of $\Delta_{n,d}$ are the maximal chains of $\Lambda_{n,d}$, and h, the number of such simplices, equals the number of standard d by $n-d$ tableaux, which may be computed by the well-known hook-length formula of Frame, et. al. [8].

A general method for constructing shellings of distributive lattices (and other classes of posets) has been given by A. Björner [3], namely by listing the chains in lexicographic (or reverse lexicographic) order. We need to be even more explicit and list the T_i's in order to get the Stanley decomposition, and the results of [3] yield also that information. The resulting construction is summarized in the following Theorem.

THEOREM. *Let $\tau_1, \tau_2, \ldots, \tau_h$ be a list of all standard $d \times (n-d)$ tableaux on the symbols $\{1, 2, \ldots, d(n-d)\}$, given in the order τ_i before τ_j if the largest symbol in which they differ is in an earlier row of τ_j than τ_i. The maximal simplex σ_i of $\Delta_{n,d}$ which corresponds to τ_i is given by $\sigma_i = B_{i,0}, B_{i,1}, \ldots, B_{i,d(n-d)}$, where $B_{i,0} = [1, 2, \ldots, d]$, and $B_{i,k}$ is obtained from $B_{i,k-1}$ by increasing the $(d+1-l)$th entry in the bracket $B_{i,k-1}$ by 1 if k occurs in the l-th row of τ_i. Define T_i to be the bracket monomial which is the product of the brackets $B_{i,k}$ such that k occurs in a later row of τ_i than $k+1$, that is, such that k is in the descent set of τ_i. Then* (1) *defines a Stanley decomposition of the bracket ring $\mathsf{B}_{n,d}$.*

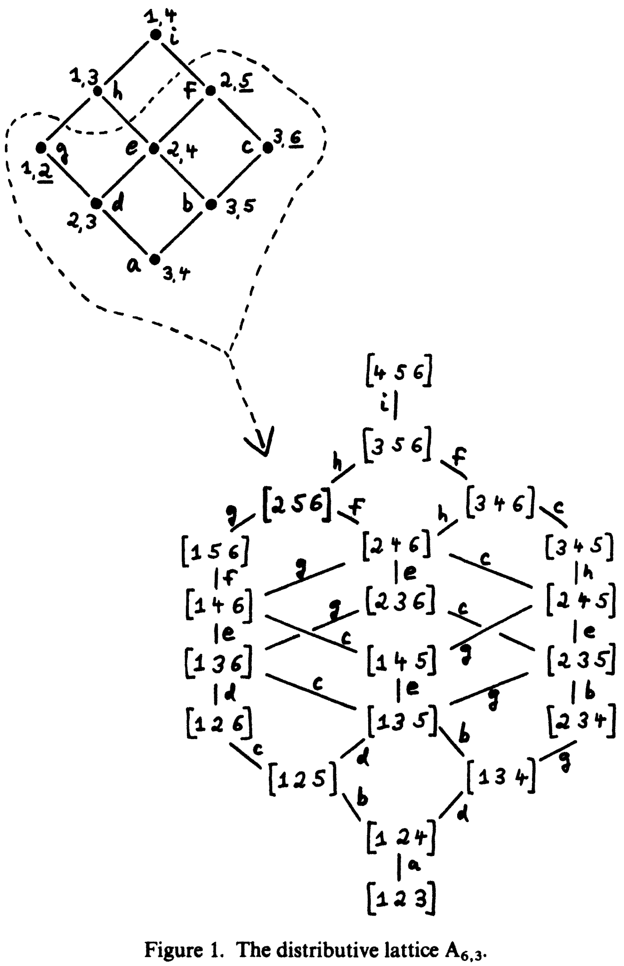

Figure 1. The distributive lattice $A_{6,3}$.

PROOF. We wish to provide an edge labeling of the (Hasse diagram of the) distributive lattice $\Lambda_{n,d}$ which is an L-labeling in the sens of Björner, as follows. If $[j_1, j_2, \ldots, j_d]$ covers $[j_1, j_2, \ldots, j_{l-1}, j_l - 1, j_{l+1}, \ldots, j_d]$ in $\Lambda_{n,d}$, assign the label $(d + 1 - l, j_l - l)$ to the edge between them. We choose lexicographic total ordering on these labels, $(1,1) < (1,2) < \ldots < (1, n-d) < (2,1) < \ldots < (d, n-d)$. Note that what we are really doing is giving a compatible linearly ordered labeling to the underlying poset $C_d \times C_{n-d}$, and using it to induce an edge labeling of the lattice of ideals , $\Lambda_{n,d}$. This always gives an L-labeling of the distributive lattice, and by Theorem 2.3 and Lemma 2.6 of Björner [3], the

Table 1. Stanley Decomposition of $B_{6,3}$

Standard Tableau		Corresponding Summand
123.456.789		K[[123], [124], [125], [126], [136], [146], [156], [256], [356], [456]] ⊕
124.356.789	[135]	K[[123], [124], [125], [135], [136], [146], [156], [256], [356], [456]] ⊕
134.256.789	[134]	K[[123], [124], [134], [135], [136], [146], [156], [256], [356], [456]] ⊕
125.346.789	[145]	K[[123], [124], [125], [135], [145], [146], [156], [256], [356], [456]] ⊕
135.246.789	[145] [134]	K[[123], [124], [134], [135], [145], [146], [156], [256], [356], [456]] ⊕
123.457.689	[246]	K[[123], [124], [125], [126], [136], [146], [246], [256], [356], [456]] ⊕
124.357.689	[246] [135]	K[[123], [124], [125], [135], [136], [146], [246], [256], [356], [456]] ⊕
134.257.689	[246] [134]	K[[123], [124], [134], [135], [136], [146], [246], [256], [356], [456]] ⊕
125.347.689	[246] [145]	K[[123], [124], [125], [135], [145], [146], [246], [256], [356], [456]] ⊕
135.247.689	[246] [145] [134]	K[[123], [124], [134], [135], [145], [146], [246], [256], [356], [456]] ⊕
123.467.589	[236]	K[[123], [124], [125], [126], [136], [236], [246], [256], [356], [456]] ⊕
124.367.589	[236] [135]	K[[123], [124], [125], [135], [136], [236], [246], [256], [356], [456]] ⊕
134.267.589	[236] [134]	K[[123], [124], [134], [135], [136], [236], [246], [256], [356], [456]] ⊕
125.367.489	[235]	K[[123], [124], [125], [135], [235], [236], [246], [256], [356], [456]] ⊕
135.267.489	[235] [134]	K[[123], [124], [134], [135], [235], [236], [246], [256], [356], [456]] ⊕
145.267.389	[234]	K[[123], [124], [134], [234], [235], [236], [246], [256], [356], [456]] ⊕
126.347.589	[245]	K[[123], [124], [125], [135], [145], [245], [246], [256], [356], [456]] ⊕
136.247.589	[245] [134]	K[[123], [124], [134], [135], [145], [245], [246], [256], [356], [456]] ⊕
126.357.489	[245] [235]	K[[123], [124], [125], [135], [235], [245], [246], [256], [356], [456]] ⊕
136.257.489	[245] [235] [134]	K[[123], [124], [134], [135], [235], [245], [246], [256], [356], [456]] ⊕
146.257.389	[245] [234]	K[[123], [124], [134], [234], [235], [245], [246], [256], [356], [456]] ⊕
123.458.679	[346]	K[[123], [124], [125], [126], [136], [146], [246], [346], [356], [456]] ⊕
124.358.679	[346] [135]	K[[123], [124], [125], [135], [136], [146], [246], [346], [356], [456]] ⊕
134.258.679	[346] [134]	K[[123], [124], [134], [135], [136], [146], [246], [346], [356], [456]] ⊕
125.348.679	[346] [145]	K[[123], [124], [125], [135], [145], [146], [246], [346], [356], [456]] ⊕
135.248.679	[346] [145] [134]	K[[123], [124], [134], [135], [145], [146], [246], [346], [356], [456]] ⊕
123.468.579	[346] [236]	K[[123], [124], [125], [126], [136], [236], [246], [346], [356], [456]] ⊕
124.368.579	[346] [236] [135]	K[[123], [124], [125], [135], [136], [236], [246], [346], [356], [456]] ⊕
134.268.579	[346] [236] [134]	K[[123], [124], [134], [135], [136], [236], [246], [346], [356], [456]] ⊕
125.368.479	[346] [235]	K[[123], [124], [125], [135], [235], [236], [246], [346], [356], [456]] ⊕
135.268.479	[346] [235] [134]	K[[123], [124], [134], [135], [235], [236], [246], [346], [356], [456]] ⊕
145.268.379	[346] [234]	K[[123], [124], [134], [234], [235], [236], [246], [346], [356], [456]] ⊕
126.348.579	[346] [245]	K[[123], [124], [125], [135], [145], [245], [246], [346], [356], [456]] ⊕
136.248.579	[346] [245] [134]	K[[123], [124], [134], [135], [145], [245], [246], [346], [356], [456]] ⊕
126.358.479	[346] [245] [235]	K[[123], [124], [125], [135], [235], [245], [246], [346], [356], [456]] ⊕
136.258.479	[346] [245] [235] [134]	K[[123], [124], [134], [135], [235], [245], [246], [346], [356], [456]] ⊕
146.258.379	[346] [245] [234]	K[[123], [124], [134], [234], [235], [245], [246], [346], [356], [456]] ⊕
127.348.569	[345]	K[[123], [124], [125], [135], [145], [245], [345], [346], [356], [456]] ⊕
137.248.569	[345] [134]	K[[123], [124], [134], [135], [145], [245], [345], [346], [356], [456]] ⊕
127.358.469	[345] [235]	K[[123], [124], [125], [135], [235], [245], [345], [346], [356], [456]] ⊕
137.258.469	[345] [235] [134]	K[[123], [124], [134], [135], [235], [245], [345], [346], [356], [456]] ⊕
147.258.369	[345] [234]	K[[123], [124], [134], [234], [235], [245], [345], [346], [356], [456]] ⊕

lexicographic (with respect to the L-labeling) listing of chains σ_i of $\Lambda_{n,d}$ gives a shelling of $\Delta_{n,d}$. By an easy argument with a reversal of the order of the labels, reverse lexicographic listing works equally well. Furthermore, η_i in either case is given by the descent set of σ_i relative to the L-labeling. Now we can check that the listing of $\tau_1, \ldots, \tau_h$ given in the statement of the Theorem results in the desired reverse lexicographic listing of $\sigma_1, \ldots \sigma_h$, and that η_i gives T_i as described. The Theorem then follows from Lemma 1.

A similar theorem could be derived which constructs the lexicographic listing of the chains. We close by illustrating our construction for the bracket ring $\mathsf{B}_{6,3}$. This is the smallest case not covered by the results in [6]. The labeling of $C_3 \times C_3$ and the induced L-labeling of $\Lambda_{6,3}$ are given in Figure 1, with the labels replaced by $a, b, \ldots, i$. Using a LISP program, we computed a (reverse lexicographic) shelling of the lattice $\Lambda_{6,3}$. In Table 1, we list the standard 3 by 3 tableaux, and the resulting summands in the Stanley decomposition of the bracket ring $\mathsf{B}_{6,3}$. The tableaux are listed in the format 123.456.789 for the tableau with first row 1 2 3, etc.

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DEPARTMENT OF MATHEMATICS
CORNELL UNIVERSITY
ITHACA, NEW YORK 14853
U.S.A.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF FLORIDA
GAINESVILLE, FL32611
U.S.A.