MATH. SCAND. 67 (1990), 193–196.

BETTI NUMBERS OF BUCHSBAUM COMPLEXES

ANDERS BJÖRNER and TAKAYUKI HIBI

Abstract.

Given non-negative integers $\beta_0, \beta_1, \ldots, \beta_d$ we construct a d-dimensional Buchsbaum complex Δ over $\mathbf{Z}$ such that $\tilde{H}_i(\Delta; \mathbf{Z}) \cong \mathbf{Z}^{\beta_i}$ for all $0 \leqq i \leqq d$. This demonstrates (via [6]) the existence of Stanley-Reisner rings with arbitrarily prescribed Betti numbers for local cohomology.

Let Δ be a finite simplicial complex of dimension d, $\tilde{H}_i(\Delta; F)$ the ith reduced simplicial homology group of Δ over a field F, and β_i the dimension of $\tilde{H}_i(\Delta; F)$ as a vector space over F. The (reduced) *Betti number sequence* of Δ over F is $\beta(\Delta; F) := (\beta_0, \beta_1, \ldots, \beta_d)$.

It is not difficult to prove that, for every sequence $\beta_0, \beta_1, \ldots, \beta_d$ of non-negative integers, there exists a d-dimensional simplicial complex Δ such that $\beta(\Delta; F) = (\beta_0, \beta_1, \ldots, \beta_d)$ for all fields F. Here we should remark that if, moreover, Δ is Buchsbaum, then $\tilde{H}_{i-1}(\Delta; F)$ is isomorphic to the i-th local cohomology module of the Stanley-Reisner ring $F[\Delta]$ of Δ over F ($1 \leqq i \leqq d$), see Schenzel [6, p. 134].

On the other hand, given integers $d \geqq 0$ and $\beta_0, \beta_1, \ldots, \beta_d \geqq 0$, there exists a Buchsbaum local ring A of dimension $d + 1$ with maximal ideal m such that the dimension of the ith local cohomology module $H^i_m(A)$ as a vector space over A/m is equal to β_i for every $0 \leqq i \leqq d$ (cf. Goto [3]). Thus it seems to be reasonable to expect that, for every sequence $\beta_0, \beta_1, \ldots, \beta_d$ of non-negative integers, there exists a d-dimensional Buchsbaum complex Δ such that $\beta(\Delta; F) := (\beta_0, \beta_1, \ldots, \beta_d)$ for all fields F. It is the purpose of this paper to prove that this is indeed the case.

THEOREM. *Given non-negative integers $\beta_0, \beta_1, \ldots, \beta_d$ there exists a d-dimensional Buchsbaum complex Δ over $\mathbf{Z}$ such that $\tilde{H}_i(\Delta; \mathbf{Z}) \cong \mathbf{Z}^{\beta_i}$ for all $0 \leqq i \leqq d$. (It follows that Δ is Buchsbaum over F and has Betti number sequence $\beta(\Delta; F) = (\beta_0, \beta_1, \ldots, \beta_d)$ for every field F.)*

Now, let us recall some fundamental definitions on simplicial complexes. All simplicial complexes Δ considered in this paper are finite. Each element σ of Δ is

Received December 7, 1989

called a *face*. The *dimension* of Δ, denoted by $\dim \Delta$, is $\max\{\#(\sigma); \sigma \in \Delta\} - 1$, where $\#(\sigma)$ is the cardinality of σ as a set. We say that Δ is *pure* if every facet (maximal face) of Δ has the same cardinality. The *link* of a face $\sigma \in \Delta$ is the subcomplex $\mathrm{link}_\Delta(\sigma) := \{\tau \in \Delta; \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}$.

Let F be a field. A simplicial complex Δ is called *Buchsbaum* over F if
(i) Δ is pure (cf. Remark 1 below), and
(ii) $\tilde{H}_i(\mathrm{link}_\Delta(\sigma); F) = 0$ for every non-empty face σ of Δ and for every $i \neq \dim(\mathrm{link}_\Delta(\sigma))$.

Moreover, we can define Buchsbaum complex over Z (the set of integers) in the obvious way. Note that Δ is Buchsbaum over Z if and only if Δ is Buchsbaum over every field F. For example, if the geometric realization of Δ is a manifold (with or without boundary), then Δ is Buchsbaum over Z. For the ring-theoretic background to the Buchsbaum property, see [5] or [6].

In what follows the Buchsbaum property and all homology groups will be considered over Z.

LEMMA 1. *Suppose that Δ_1, Δ_2 and $\Delta_1 \cap \Delta_2$ are Buchsbaum complexes of dimension d. Then $\Delta_1 \cup \Delta_2$ is also a Buchsbaum complex of dimension d. Furthermore, if $\Delta_1 \cap \Delta_2$ is acyclic then $\tilde{H}_i(\Delta_1 \cup \Delta_2) \cong \tilde{H}_i(\Delta_1) \oplus \tilde{H}_i(\Delta_2)$ for all $0 \leqq i \leqq d$.*

PROOF. The simple argument is based on the (reduced) Mayer-Vietoris exact sequence:

$$(*) \quad \ldots \to \tilde{H}_i(\Delta_1 \cap \Delta_2) \to \tilde{H}_i(\Delta_1) \oplus \tilde{H}_i(\Delta_2) \to \tilde{H}_i(\Delta_1 \cup \Delta_2) \to \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2) \to \ldots$$

Consider $\mathrm{link}_{\Delta_1}(\sigma)$ and $\mathrm{link}_{\Delta_2}(\sigma)$ instead of Δ_1 and Δ_2 in (*), and the vanishing of lower-dimensional homology on $\mathrm{link}_{\Delta_1 \cup \Delta_2}(\sigma)$ follows immediately, since

$$\mathrm{link}_{\Delta_1 \cup \Delta_2}(\sigma) = \mathrm{link}_{\Delta_i}(\sigma), \text{ if } \sigma \in \Delta_i - \Delta_{3-i}\ (i = 1, 2),$$

$$\mathrm{link}_{\Delta_1 \cup \Delta_2}(\sigma) = \mathrm{link}_{\Delta_1}(\sigma) \cup \mathrm{link}_{\Delta_2}(\sigma), \text{ if } \sigma \in \Delta_1 \cap \Delta_2,$$

$$\mathrm{link}_{\Delta_1 \cap \Delta_2}(\sigma) = \mathrm{link}_{\Delta_1}(\sigma) \cap \mathrm{link}_{\Delta_2}(\sigma), \text{ if } \sigma \in \Delta_1 \cap \Delta_2.$$

If $\tilde{H}_i(\Delta_1 \cap \Delta_2) = 0$ for every i, then (*) gives $\tilde{H}_i(\Delta_1 \cup \Delta_2) \cong \tilde{H}_i(\Delta_1) \oplus \tilde{H}_i(\Delta_2)$ for all $0 \leqq i \leqq d$ as required.

LEMMA 2. *For every $0 \leqq k \leqq d$, there exists a d-dimensional Buchsbaum complex Γ_k such that*

$$\tilde{H}_i(\Gamma_k) \cong \begin{cases} \mathbf{Z} & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

PROOF. Let Γ_k be any triangulation of the space $S^k \times I^{d-k}$, the product of the k-sphere with $d - k$ copies of the unit interval. Since $S^k \times I^{d-k}$ is a manifold (with boundary if $k \neq d$), the simplicial complex Γ_k is a Buchsbaum complex. Furthermore, Γ_k has the same homology as S^k. (In fact, S^k is a strong deformation retract of Γ_k.)

We are now in the position to prove the theorem. First note that a d-simplex Δ is Buchsbaum over Z with $\tilde{H}_i(\Delta; \mathbf{Z}) = 0$ for all $0 \leqq i \leqq d$. Thus, it suffices to show that if there exists a d-dimensional Buchsbaum complex Δ with $\tilde{H}_i(\Delta; \mathbf{Z}) \cong \mathbf{Z}^{\beta_i}$ $(0 \leqq i \leqq d)$, and if $0 \leqq k \leqq d$, then there exists a d-dimensional Buchsbaum complex Δ' with $\tilde{H}_i(\Delta'; \mathbf{Z}) \cong \mathbf{Z}^{\beta_i}$ $(0 \leqq i \leqq d, i \neq k)$ and $\tilde{H}_k(\Delta'; \mathbf{Z}) \cong \mathbf{Z}^{\beta_k + 1}$. For this, choose any facet σ of Δ and any facet τ of the simplicial complex Γ_k of Lemma 2. Identifying σ with τ yields a new complex Δ' which is Buchsbaum over Z with $\tilde{H}_i(\Delta'; \mathbf{Z}) \cong \mathbf{Z}^{\beta_i}$ $(0 \leqq i \leqq d, i \neq k)$ and $\tilde{H}_k(\Delta'; \mathbf{Z}) \cong \mathbf{Z}^{\beta_k + 1}$ by Lemma 1, as desired.

REMARK 1. (On the definition of Buchsbaum complexes). Let us call a finite simplicial complex Δ *locally Cohen-Macaulay* if $\mathrm{link}_\Delta(v)$ is Cohen-Macaulay for every vertex (0-dimensional face) v of Δ, cf. [1]. The definition given before Lemma 1 can then be restated as follows: Δ is Buchsbaum (over F) if and only if Δ is pure and locally Cohen-Macaulay (over F). The purity condition has been overlooked by several authors on Buchsbaum complexes, and its necessity was pointed out by Miyazaki [5, Remark, p. 251]. We want to remark that for connected complexes the purity condition is not needed.

LEMMA 3. *A simplicial complex Δ is Buchsbaum if and only if Δ is locally Cohen-Macaulay and all connected components have the same dimension.*

PROOF. Assume that Δ is locally Cohen-Macaulay and that all connected components of Δ are d-dimensional. We must show that they are pure. This comes down to showing that a connected locally Cohen-Macaulay complex is pure, which can be done with the standard argument for showing that a Cohen-Macaulay complex is pure: Let σ, τ be facets of Δ, and pick $x \in \sigma, y \in \tau$. Since Δ is connected there exists an edge path $x = x_0 - x_1 - x_2 - \ldots - x_{n-1} - x_n = y$. Let $\sigma_0 = \sigma$, $\sigma_{n+1} = \tau$, and for $1 \leqq i \leqq n$ let σ_i be a facet containing $\{x_{i-1}, x_i\}$. Since $\mathrm{link}_\Delta(x_i)$ is pure, $\dim(\sigma_i) = \dim(\sigma_{i+1})$ for every $0 \leqq i \leqq n$.

The argument just given can also be adapted to prove that every pair of facets in a connected Buchsbaum complex can be connected by a sequence of facets with successive intersections of codimension one (a "dual path").

REMARK 2. (On f-vectors of Buchsbaum complexes). The sequences (3, 1) and (4, 4, 1) are f-vectors of unique simplicial complexes. The first of these complexes is locally Cohen-Macaulay but not pure, the second is not even locally Cohen-Macaulay. This shows that not all possible f-vectors and (f, β)-pairs [2] can be realised by Buchsbaum (or locally Cohen-Macaulay) complexes. It would, of course, be of great interest to find a characterization of f-vectors and (f, β)-pairs for Buchsbaum complexes (and for the more general classes of pure comlexes (cf.

[4]) and of locally Cohen-Macaulay complexes). Stanley [7] has characterized the f-vectors (and hence also the (f, β)-pairs) for Cohen-Macaulay complexes.

REMARK 3. (A strengthening of the main result). For good choices of the triangulations Γ_k, the complex Δ constructed to prove the Theorem can be made "homotopy Buchsbaum" in the following sense: $\text{link}_\Delta(\sigma)$ is topologically $(d - 1 - \sharp(\sigma))$-connected for every non-empty face $\sigma \in \Delta$, and Δ is homotopy equivalent to a $(\beta_0, \beta_1, \ldots, \beta_d)$-wedge of spheres.

REMARK 4. It seems difficult to strengthen the main result in other directions than as in Remark 3. For instance, it cannot for any $d \geqq 1$ be asserted that Δ is a manifold. One obvious constraint for this is Poincaré duality, which for a connected manifold with $\beta_d = 1$ forces a symmetry condition $\beta_i = \beta_{d-i}$, $0 < i < d$. If d is odd then Poincaré symmetry is both necessary and sufficient for nonnegative integers $\beta_0, \beta_1, \ldots, \beta_d$ to be the Betti number sequence of some connected closed orientable manifold. However, if d is even special constraints on the middle Betti number $\beta_{d/2}$ may exist, depending on d.

REFERENCES

1. K. Baclawski, *Cohen-Macaulay ordered sets*, J. Algebra 63 (1980), 226–258.
2. A. Björner and G. Kalai, *An extended Euler-Poincaré theorem*, Acta Math. 161 (1988), 279–303.
3. S. Goto, *On Buchsbaum rings*, J. Algebra 67 (1980), 272–279.
4. T. Hibi, *What can be said about pure 0-sequences?*, J. Combin. Theory Ser. A 50 (1989), 319–322.
5. M. Miyazaki, *Characterizations of Buchsbaum complexes*, Manuscripta Math. 63 (1989), 245–254.
6. P. Schenzel, *On the number of faces of simplicial complexes and the purity of Frobenius*, Math. Z. 178 (1981), 125–142.
7. R. Stanley, *Cohen-Macaulay complexes*, in *Higher Combinatorics* (M. Aigner, ed.), Reidel, Boston, 1977, pp. 51–62.

DEPARTMENT OF MATHEMATICS
ROYAL INSTITUTE OF TECHNOLOGY
S-100 44 STOCKHOLM
SWEDEN

DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCE
NAGOYA UNIVERSITY
CHIKUSA-KU, NAGOYA 464-01
JAPAN