MATH. SCAND. 67 (1990), 197–214.

CLASSIFICATION OF IRREDUCIBLE PROJECTIVE SURFACES OF SMOOTH SECTIONAL GENUS $\leqq 3$

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Let $S' \subset \mathbf{P}^N$ be an irreducible and reduced, possibly singular, complex projective surface. To classify such surfaces one of the possible way is to consider the smooth genus of the hyperplane section $H \in |O_{\mathbf{P}^N}(1)_{|S'}|$. This type of classification goes back to the work of Castelnuovo, Enriques and Scorza.

In recent time the second author has given a revised and more understandable version of their principal tool, the Adjunction Process (see [So1], [VdV] and [So2]). This had made possible the classification in the smooth case (see for instance [Li] and [Io]).

In this paper we approach the general case and we classify at least the normalizations of such surfaces whose hyperplane sections have desingularizations with genus less or equal then three (i.e. smooth genus $\leqq 3$).

We use the following notation. S' denotes an irreducible and reduced projective surface in $\mathbf{P}^N$. Let $\rho\colon S \to S'$ be the normalization, $\pi\colon \hat{S} \to S$ the minimal desingularization and let $L = \rho^* O_{\mathbf{P}^N}(1)|_S$, and $\hat{L} = \pi^* L$. Finally let $g(L) = 1/2\,[(K_S + L)\cdot L + 2]$. Note that a generic $C \in |L|$ is smooth.

We note that $(\hat{S}, \hat{L})$ is a pair consisting of a smooth surface and a nef and big line bundle such that there exist no smooth rational curves in $\hat{S}$, $E \subset \hat{S}$, such that $\hat{L}\cdot E = 0$ and $E \cdot E = -1$ (this is the minimality condition). In a previous paper, [$A + S1$], we called such pairs a-minimal and we worked out an "Adjunction Theory" for them. This results, which are recalled in 0.8, are the principal tools used in the paper.

The results we get are the following. (For genus 0 we have the theorem 0.6 proved in [Na] or [Fu]).

THEOREM A. *Let (S, L) be as above. Let $C \in |L|$ be smooth and suppose $g(L) = 1$. Then we have one of the following disjoint possibilities.*
a) *$(S, L) = (\hat{S}, \hat{L})$ is a geometrically ruled surface (scroll), $p\colon S = \mathbf{P}(\mathscr{E}) \to R_1$, over a curve R_1 of genus* 1. *($q(S) = q(\hat{S}) = 1$)*
b) *(S, L) is the cone over the polarized curve $(C, L_C = N_{C,S})$ and C is embedded in S as the infinite section ($q(S) = 0$, $q(\hat{S}) = 1$).*

Received June 23, 1988; in revised form January 8, 1990.

c) *S has only rational double points as singularities and* $-K_S \cong L$ (i.e. (S, L) *is a Gorenstein-del Pezzo surface*) $(q(S) = q(\hat{S}) = 0)$.

THEOREM B. *Let* (S, L) *as above. Let* $C \in |L|$ *be smooth and suppose* $g(L) = 2$. *Then we have one of the following possibilities.*
a) (S, L) *is a geometrically ruled surface (scroll),* $p: S = \mathbf{P}(\mathscr{E}) \to R_2$, *over a curve of genus* 2. $(q(S) = q(\hat{S}) = 2)$.
b) (S, L) *is the cone over a smooth polarized curve* $(C, N_{C,S}) = (C, L_C)$ *and* C *is embedded in* S *as the infinite section.* $(q(S) = 0$ *and* $q(\hat{S}) = 2)$.
c) $\pi: \hat{S} \to S$ *factors through* $\pi_1: \hat{S} \to \hat{S}_c$ *and* $\pi_2: \hat{S}_c \to S$, *where* $(\hat{S}_c, \pi_2{}^*L)$ *is a Gorenstein conic bundle over* $\mathbf{P}^1$; $\pi_2: \hat{S}_c \to S$ *precisely contracts less then or equal to two sections of* $\hat{S}_c \to \mathbf{P}^1$ $(\pi_1: \hat{S} \to \hat{S}_c$ *is the map* $\phi_c: \hat{S} \to \hat{S}_c$ *relative to* $(\hat{S}, \hat{L})$ *in* 0.8.1).

THEOREM C. *Let* (S, L) *as above. Let* $C \in |L|$ *be smooth and suppose* $g(L) = 3$. *Then we have one of the following possibilities (for definitions see* 0.5).
a) $(S, L) = (\hat{S}, \hat{L})$ *is a geometrically ruled surface (scroll),* $p: S = \mathbf{P}(\mathscr{E}) \to \mathscr{R}_3$, *over a curve of genus* 3 $(q(S) = q(\hat{S}) = 3)$.
b) (S, L) *is the cone over the polarized curve* $(C, N_{C,S}) = (C, L_C)$ *and* C *is embedded in* S *as the infinite section* $(q(S) = 0$ *and* $q(\hat{S}) = 3)$.
c) $\pi: \hat{S} \to S$ *factors through* $\pi_1: \hat{S} \to \hat{S}_c$ *and* $\pi_2: \hat{S}_c \to S$, *where* $(\hat{S}_c, \pi_2{}^*L)$ *is a Gorenstein conic bundle over* $\mathbf{P}^1$; $\pi_2: \hat{S}_c \to S$ *precisely contracts less then or equal to two sections of* $\hat{S}_c \to \mathbf{P}^1$ $(\pi_1: \hat{S} \to \hat{S}_c$ *is the map* $\phi_c: \hat{S} \to \hat{S}_c$ *relative to* $(\hat{S}, \hat{L})$ *in* 0.8.1).
d) $K_{\hat{S}} \otimes \hat{L}$ *is spanned by global sections,* $h^0(K_{\hat{S}} \otimes \hat{L}) = 3$, *and the associated map,* $\phi: \hat{S} \to \mathbf{P}^2$ *is a modification.*
e) $K_{\hat{S}} \otimes \hat{L}$ *is spanned by global sections,* $h^0(K_{\hat{S}} \otimes \hat{L}) = 3$, *and the associated map,* $\phi: \hat{S} \to \mathbf{P}^2$ *is a generically* 2 *to* 1 *map.* $\hat{S}_c$ *is a* 2 *sheeted cover of* $\mathbf{P}^2$ *branched along a quartic curve and* $\hat{L}_c$ *is the pullback of* $O_{\mathbf{P}^2}(2)$.
f) $(\hat{S}_c, \hat{L}_c)$ *is a conic bundle over a smooth elliptic curve.*
g) $S = S'$ *is a degree* 4 *(see* §2) *surface in* $\mathbf{P}^3$, *or* S' *is a degree* 5 *surface in* $\mathbf{P}^3$.

REMARK. We notice that the cases a) and b), i.e. the scroll and the cone over the curve of genus g, occur for all the genus, for $g \geqq 4$ as well. For every genus $\geqq 2$ we find also the Gorenstein conic bundles over $\mathbf{P}^1$ and possibly surfaces obtained contracting at most two of their sections.

In section 2 we classify also the singular surfaces of degree 4 (see the table A, see p. 208). One of our motivations for doing this classification comes from our work [A + S2] where we study when the *adjoint sheaf,* $K_S \otimes L$ is spanned by global sections. The degree four surfaces play a particular role in this study because the standard techniques to prove when a line bundle is spanned by global sections, e.g. Reider's theorem, apply only for degree $\geqq 5$. One consequence of the classification in this paper is in 1.13 that states when $K_S \otimes L$ is spanned at smooth points by global sections in the case $c_1(L)^2 = 4$.

The first author would like to thank the Consiglio Nazionale delle Ricerche and the second author would like to thank the National Science Foundation (DMS 8420 315) for their financial support. We would both like to thank the Max Planck Institut für Mathematik in Bonn, the University of Notre Dame and the University of Trento for their help in making this collaboration possible. We finally call attention to the work of Maeda on genus 3 ample divisor on smooth surfaces ([Ma]) and we would like also to thank prof. Fumio Sakai for helpful remarks.

§0. Notation and Background Material.

0.0. We work over the complex numbers. All spaces are complex analytic and all maps are holomorphic. By *surface* we mean an irreducible and reduced complex analytic space of dimension two. If S is a complex analytic space, we denote its holomorphic structure sheaf by O_S. We do not distinguish notationally between a locally free coherent analytic sheaf and its associated holomorphic vector bundle.

If S is a connected complex manifold then we have the dualizing sheaf $K_S = \Lambda^n T_S{}^*$ where $n = \dim S$ and $T_S{}^*$ is the cotangent bundle of S. If S is a normal variety then the dualizing sheaf $K_S = j_* K_{\mathrm{Reg}(S)}$ where j: $\mathrm{Reg}(S) \to S$ is the inclusion of the smooth points of S into S. In this case K_S is a reflexive rank one sheaf on S. When S is normal and projective and $\tau \in \mathsf{R}$ we use τK_S to denote the Weil divisor obtained by multiplying τ times the Weil divisor K_S. [Re] and $[A + K]$ are helpful references for these matters.

Given an effective normal Cartier divisor A on a normal Cohen-Macaulay variety S we have by ($[A + K]$, pg. 7): $K_S \otimes [A])_A \cong K_A$.

0.0.1. Let S be a variety and let π: $\hat{S} \to S$ be a resolution of singularities, i.e. $\hat{S}$ is a complex manifold and π is a surjective holomorphic map which gives a biholomorphism from $\hat{S} - \pi^{-1}(\mathrm{Sing}(S))$ to $S - \mathrm{Sing}(S)$. The Leray sheaves $R^i\pi_*(O_{\hat{S}})$ for $i \geqq 0$ are independent of the resolution. S is normal if and only if $\pi_*(O_{\hat{S}}) \cong O_S$.

We denote by $\boldsymbol{K}_S$ the Grauert-Riemenschneider canonical sheaf which is defined by $\boldsymbol{K}_S = \pi_* K_{\hat{S}}$ (The sheaf $\boldsymbol{K}_S$ does not depend on the desingularization π; see $[G + R]$).

Assume that S is normal. If $R^i\pi_*(O_{\hat{S}}) = 0$ for $i > 0$ then the singularities of S are said to be rational. It is a theorem of Kempf ([Ke], pg. 5O) that S has rational singularities if and only if S is Cohen-Macaulay and $\pi_*(K_{\hat{S}}) = \boldsymbol{K}_S \cong K_S$. We denote by Irr (S), the *irrational locus of* S, which is the union of the supports of the sheaves $R^i\pi_*(O_{\hat{S}})$ for $i > 0$.

If S is a surface, let π: $\hat{S} \to S$ be a minimal resolution of the singularities, minimal in the sense that the fibres of π contain no rational curves E satisfying

$E \cdot E = -1$. A fundamental fact is that

$$K_{\hat{S}} + \Delta = \pi^* K_S$$

where Δ is an effective divisor such that $\Delta_{red} = \pi^{-1}(\mathrm{Irr}(S))_{red}$ [cf. Sa].

Given a reflexive rank one sheaf F on S and an integer $t > 0$, we use F^t to denote $(F \otimes \ldots \otimes F)^{**}$ (t times) and F^{-t} to denote F^{*t}.

0.1. If F^t is invertible for some $t > 0$ and S is a normal projective variety then $c_1(F) \in H^2(S, \mathbf{Q})$ is well defined as $c_1(F^t)/t$. Such an F is said to be *numerically effective* (or *nef* for short) if $c_1(F)[C] \geqq 0$ for all effective curves C on S. If further $c_1(F)^{\dim S} > 0$ than F is said to be *big*. If there is a $t > 0$ such that F^t is invertible and spanned by global sections then F is said to be *semi-ample*. By going to a large enough positive multiple N of t we can then assume that there is a holomorphic map $\phi: S \to \mathbf{P}$ with connected fibres and normal image $Y = \phi(S)$ such that $F^N = \phi^* O_{\mathbf{P}}(1)$.

If S is Cohen-Macaulay and K_S in invertible, then S is said to be Gorenstein.

What follows is a list of results we will use in the paper, we will give references for proof and discussion of them; we need an easy consequence of the vanishing theorems of Kodaira, Ramanujam, Grauert-Riemenschneider and Kawamata-Vieweg.

0.2. VANISHING THEOREM. [So3] and [S-S]. *Let L be a nef and big line bundle on a normal projective variety S. Then*

a) $H^i(S, K_S \otimes L) = 0$ for $i > \max\{0, \dim \mathrm{Irr}(S)\}$,

b) $H^i(S, \boldsymbol{K}_S \otimes L) = 0$ for $i > 0$.

0.3. BERTINI'S THEOREM. (see [So3]. *Let S be a normal projective variety and let L be a line bundle on S spanned by a finite dimensional space V of global sections. Let $|V|$ denote the linear space of Cartier divisors associated to V. There is a Zariski open set $U \subset |U|$ of divisors D such that D is normal,* $\mathrm{Sing}(S) \supset \mathrm{Sing}(D)$ *and no irreducible components of* $\mathrm{Sing}(S)$ *belongs to D.*

0.4. CASTELNUOVO'S LEMMA. (see [G-H]. *If C is an irreducible curve embedded in $\mathbf{P}^{\ell-1}$ and C belongs to no linear hyperplane $\mathbf{P}^{\ell-2}$, then, with d the degree of C and g the genus of the desingularization of C:*

$$g \leqq [d - 2/(\ell - 2)] \cdot (d - \ell + 1 - [d - \ell/\ell - 2] \cdot (\ell - 2/2))$$

(where [ ] is the least integer function).

0.5. In this section we introduce some well known classes of surfaces which play an important role in the paper.

We denote $\mathbf{P}^n$ the n-dimensional space and by $O_{\mathbf{P}^n}(r)$ the rth-power of the hyperplane bundle.

A pair (S, L) consisting of an ample (nef and big) line bundle on a normal surface S is called a *(generically) polarized surface.*

A polarized surface (S, L) is called a *quadric if* S is biholomorphic to a possibly singular quadric $\mathbf{P}^3 \supset \mathbf{Q}$ and L is isomorphic to the restriction of the hyperplane bundle, $\mathcal{O}_{\mathbf{P}^3}(1)$, to Q.

S is called a *geometrically ruled surface* if S is a holomorphic $\mathbf{P}^1$-bundle, p: $S \to R$, over a non singular curve R. Equivalently, $\pi: S = \mathbf{P}(\mathscr{E}) \to R$ for some rank 2 locally free sheaf on R. We can chose $\mathscr{E}$ in such a way that there exists a section, E, of π such that: $\operatorname{Pic}(S) \approx \mathbf{Z}[E] \oplus \mathbf{Z}[f]$ where f is a fibre of π, and $f^2 = 0$, $f \cdot E = 1$ and $E^2 = -e = \deg \mathscr{E}$. ($\approx$ denotes numerical equivalence). For $r \geqq 0$ we denote as F_r the geometrically ruled surface $\mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(-r))$. F_r is the unique holomorphic $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ with a section E satisfing $E \cdot E = -r$. For $r \geqq 1$, E is unique and we denote the normal surface obtained from F_r by blowing down E, by F'_r. Note $F'_1 = \mathbf{P}^2$.

A polarized surface (S, L) is called again a *geometrically ruled surface* or a *scroll* if S is biholomorphic to a geometrically ruled surface, $\pi: S = \mathbf{P}(\mathscr{E}) \to R$, and $K_S \otimes L^2 \cong \pi^* \mathscr{M}$ for an ample line bundle $\mathscr{M}$ on R.

The usual definition of a scroll, (S, L), is that S is biholomorphic to a geometrically ruled surface and the restriction, L_f, of L to a fibre f of π is $\mathcal{O}_f(1)$. The only difference between these definitions is that our definition excludes the smooth quadric (S, L), where $S = \mathbf{P}^1 \times \mathbf{P}^1$ with L of degree 1 on the fibres of both projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. The chief virtue of our definition is that the projection of a scroll (S, L) is canonically associated to the scroll as the map associated to $\Gamma((K_S \otimes L)^N)$ for large N.

A polarized surface (S, L) is called a *Gorenstein-Del Pezzo surface* if $K_{S^{-1}} \cong L$ (see [Br] for a classification of these surfaces).

A *generically* polarized surface (S, L) is called a *generalized conic bundle* if there is a holomorphic surjection, $p: S \to R$, with connected fibres onto a smooth curve R with the property that, for some $k > 0$ and some very ample line bundle $\mathscr{L}$ on R, $(K_S \otimes L)^k \cong p^* \mathscr{L}$. A *conic bundle* is a generalized conic bundle such that L is ample relative to p. The reader should note that in either cases the general fibre f of the map p: $S \to R$ is a smooth rational curve with $L \cdot f = 2$.

Let L be a line bundle on a projective curve R which is ample and spanned by global sections. Assume that the map $f: R \to \mathbf{P}$ associated to a vector subspace $V \subset \Gamma(L)$ that spans L is generically one to one. Let $C = \mathbf{P}(\mathcal{O}_R \oplus L)$ and let ξ denote the tautological line bundle on C. Call $\phi: C \to \mathbf{P}$ the map associated to $\mathbf{C} \oplus V \subset \Gamma(\xi)$, where $\mathbf{C}$ corresponds to section $(\lambda, 0)$ of $\mathcal{O} \oplus L$ and V is as above. We denote by $C(R, L)$ the normalization of $\phi(C)$. Finally we denote by ξ_L the restriction of $\mathcal{O}_{\mathbf{P}}(1)$ to $C(R, L)$. We call the polarized surface $(C(R, L), \xi_L)$ the *cone over* (R, L). Note that ϕ is generically one to one. To see this note that ϕ restricted to any fibre of $C \to R$ is an embedding, and note that the restriction of ϕ of

C associated to the obvious quotient $O_R \oplus L \to L$ is the generically one to one map f.

The following is a very well known structure theorem (see [Na] or [Fu])

0,6. THEOREM. *Let L be an ample line bundle on a normal projective surface S spanned by global sections. Assume* $g(L) = 0$, *where* $2g(L) - 2 = (K_S + L) \cdot L$.

Then L is very ample and S has only rational singularities. Further either (S, L) *is* $(\mathbf{P}^2, O_{\mathbf{P}^2}(e))$, $e = 1$ *or* 2, *or a quadric, or a scroll over a smooth curve of genus* 0, *or a cone over* $(\mathbf{P}^1, O_{\mathbf{P}^1}(e))$, *with* $e \geqq 3$.

0.7. Let S be the normalization $\rho: S \to S'$, of an irreducible and reduced surface S' embedded in $\mathbf{P}^N$. Let L be the pullback of $L' = O_{\mathbf{P}^N}(1)$ to S. Let $g(L)$ be defined by $2g(L) - 2 = L \cdot L + K_S \cdot L$ and $g(L')$ be the arithmetic genus of a general $C' \in |L'|$. Note that if S' is a local complete intersection then $2g(L') - 2 = L' \cdot L' + K_{S'} \cdot L'$.

0.7.1 LEMMA. *If* $g(L) = g(L')$ *then* S' *has only isolated singularities. In particular, if* $g(L) = g(L')$ *and* S' *is a local complete intersection, then* ρ *is a biholomorphism.*

PROOF. A general $C' \in |L'|$ has a smooth $C \in |L|$ as inverse image $\rho^{-1}(C')$. Thus C is the normalization of C'. If $g(L) = g(L')$ then $\chi(O_{C'}) \cong \chi(O_C)$. From the Leray spectral sequence and the sequence $0 \to \rho_* O_{C'} \to O_C \to \mathcal{S} \to 0$, where $\mathcal{S}$ is a sky-scraper sheaf which is non trivial if C' is singular, we see that $\mathcal{S}$ is empty and C' is smooth. Since C' is a Cartier divisor, S' is smooth in a neighborhood of C'. Since C' is ample the singular set of S' must be finite. If S' is a local complete intersection and it is smooth except at a finite set then it is normal. ρ is a biholomorphism by Zariski's main theorem.

0.8. We recall now our main results in $[A + S\,1]$ regarding the structure theory of the pairs (S, L) consisting of a nef and big line bundle, L, on an irreducible normal Gorenstein surfaces. Let $\pi: \hat{S} \to S$ be a minimal resolution of the singularities and Δ be the effective divisor such that $K_{\hat{S}} + \Delta \doteq \pi^* K_S$ (see 0.0.1)

We say that the pair (S, L) is *a-minimal* if there are no smooth rational curves E on $\hat{S} - \Delta$ with $E \cdot E = -1$ and $\pi^* L \cdot E = 0$; that is the case if for instance L is ample. We say that (S, L) is *c-minimal* if it is a-minimal and if there are no smooth rational curves E on $\hat{S}$ with $\pi^* L \cdot E = 0$ and either a) $\Delta \cdot E = 1$ and $E \cdot E = -1$, or b) $\Delta \cdot E = 0$, $E \cdot E = -2$ and $\pi(E)$ is not a point.

We proved the following results, which motivate also the above definitions.

0.8.1. REDUCTION THEOREM. *Let* (S, L) *be a pair consisting of a nef and big line bundle, L, on a normal irreducible Gorenstein surface S. For* $i = a$ *or* c *there exists a nef and big line bundle,* L_i, *on a normal Gorenstein i-minimal surface* S_i, *and a bimeromorphic holomorphic map* $\phi_i: S \to S_i$ *with* $L \cong \phi_i^* L_i$ *such that the positive dimensional fibres of* ϕ_i *consist of rational curves.*

Further

a) *if* $h^0(K_S^n \otimes L^n) \neq 0$ *for some* $n > 0$ *then* $K_{S_a} \otimes L_a$ *is nef and* (S_i, L_i) *is uniquely determined by* (S, L),

b) *if* $h^0(K_S^n \otimes L^n) = 0$ *for all* $n > 0$, *then* (S_a, L_a) *is either* $(\mathrm{P}^2, O_{\mathrm{P}^2}(e)) = 1$ *or* 2, *or a quadric in* P^3, π: $\hat{\mathrm{Q}} \to \mathrm{Q}$, *with* $L_a = \pi^* O_{\mathrm{P}^2}(1)_{|\mathrm{Q}}$, *or a scroll.*

We would emphasize that the theorem says in particular the following

0.8.2. THEOREM. *Let* (S, L) *be a generically polarized Gorenstein surface and suppose it is a-minimal. The following are equivalent:*

a) (S, L) *is neither a scroll nor a quadric nor the minimal resolution of a quadric, nor equal to* $(\mathrm{P}^2, O_{\mathrm{P}^2}(e))$ *for* $e = 1$ *or* 2,

b) $K_S \otimes L$ *is numerically effective,*

c) $h^0((K_S \otimes L)^N) \neq$ *for some* $N > 0$.

0.8.3. MAIN THEOREM. *Let* (S, L) *be an a-minimal generically polarized Gorenstein surface. Assume that* $h^0(K_{S^n} \otimes L^n) \neq 0$ *for some* $n > 0$. *Then there is an* $N > 0$ *such that* $(K_S \otimes L)^N$ *is spanned by global sections and such that the map* ϕ: $S \to \mathrm{P}$ *associated to* $\Gamma((K_S \otimes L))^N$ *has connected fibres and a normal image. The map* ϕ: $S \to S' = \phi(S)$ *can be factorized* $\phi = \phi' \circ \phi_c$ *where* ϕ_C: $S \to S_C$ *is the map in the reduction theorem above,* (S_c, L_c) *is c-minimal and* ϕ': $S_c \to S'$ *is the map associated to* $\Gamma((K_{Sc} \otimes L_c)^N)$. *One of the following holds.*

i) *If* $\dim \phi(S) = 0$ *then* (S, L) *is a Gorenstein-Del Pezzo surface.*

ii) *If* $\dim \phi(S) = 1$ *then* (S, L) *is a generalized conic bundle and* ϕ: $S \to \mathrm{P}$ *is the projection in the definition of conic bundle. Further* (S_c, L_c) *is a conic bundle.*

iii) *If* $\dim \phi(S) = 2$, *then the image* $S' = \phi(S)$ *is a normal Gorenstein surface.* $L_{S - \varphi^{-1}(\mathrm{F})}$ *extends to a nef and big line bundle* L' *on* S' *such that* $\phi^*(K_{S'} \otimes L') \cong (K_S \otimes L)$ *and* $K_{S'} \otimes L'$ *is ample on* S'.

0.8.4. THEOREM. *Let* (S, L) *be a generically polarized Gorenstein surface. Assume that L is spanned by global sections and the genus,* $g(L)$ *of a smooth* $C \in |L|$ *equals* $h^{1,0}(S)$. *Then* $h^0((K_S \otimes L)^N) = 0$ *for all N positive integer. In particular the associated a-minimal pair of* (S, L) *is classified in theorem* (0.8.2a).

We need also the following easy inverse result

0.8.5. PROPOSITION. *In the hypothetis of* 0.8.4, *if* $h^0((K_S \otimes L)) = 0$ *then* $g(L) = h^{1,0}(S)$.

PROOF. Let $C \in |L|$ a general section. By the residue sequence $0 \to K_S \to K_S \otimes L \to K_C \to 0$, and the vanishing $h^1(K_S \otimes L) = 0$, if $h^0(K_S \otimes L) = 0$, then $g(L) = h^0(K_C) = h^1(K_S) = h^{1,0}(S)$.

From 0.8.2 and 0.8.3. it clearly results the following

0.9.1. PROPOSITION. *Let* $(\hat{S}, \hat{L})$ *be an a-minimal generically polarized Goren-*

stein surface. If $g(\hat{L}) = 1$ then either

a) *$(\hat{S}, \hat{L})$ is geometrically ruled over an elliptic curve, or*

b) *$K_{\hat{S}} = \hat{L}^{-1}$ and $\hat{S}$ is the minimal desingularization, $\pi: \hat{S} \to S$ of a normal Gorenstein surface with rational singularities such that $\hat{L} = \pi^* L$ for an ample line bundle L on S with $K_S = L^{-1}$* (i.e. *S is a Gorenstein-Del Pezzo surface*).

0.9.2. LEMMA. *Let $\hat{S}$, $\hat{L}$, S, L be as in* 0.9.1b) *and assume that $\hat{L} \cdot \hat{L} = 4$ and that $\hat{L}$ is spanned by global sections. Let $\rho: S \to \mathbf{P}$ be the map associated to $\Gamma(L)$. Then $h^0(L) = 5$, ρ is an embedding with image the intersection of two quadric hypersurfaces.*

PROOF. Clearly a smooth $C \in |L|$ is an elliptic curve and $h^1(O_S) = 0$ by 0.2, since L is ample and $L - K_S = 2L$ is ample. Using these facts and considering $0 \to O_S \to L \to L_C \to 0$ we see that $h^0(L) = 1 + h^0(L_C) = 1 + 4 = 5$.

Note $\deg(\rho) \cdot \deg \rho(S) = L \cdot L = 4$, where $\deg(\rho)$ is the generic fibre degree of ρ. Since $\deg \rho(S) \geqq 5 - 2$ we conclude that $\deg(\rho) = 1$.

Consider the sequence $0 \to O_{\mathbf{P}^4}(1) \otimes \mathscr{I}_{\rho(S)} \to O_{\mathbf{P}^4}(1) \to L \to 0$.
Clearly $\Gamma(O_{\mathbf{P}^4}(1)) \cong \Gamma(L)$. Consider $0 \to O_{\mathbf{P}^4}(2) \otimes \mathscr{I}_{\rho(S)} \to O_{\mathbf{P}^4}(2) \to 2L \to 0$.
Since $2L - K_S = 3L$ we know by 0.2 that $h^0(2L) = \chi(2L)$. By Riemann-Roch $\chi(2L) = 13$. Since $\Gamma(O_{\mathbf{P}^4}(2)) = 15$ we conclude that $\dim \Gamma\ (O_{\mathbf{P}^4}(2) \otimes \mathscr{I}_{\rho(S)}) \geqq 2$. Thus there are two linearly independent quadrics Q_1 and Q_2 with $Q_1 \supset \rho(S)$, $Q_2 \supset \rho(S)$. Both are irreducible since $\rho(S)$ is not contained in any linear $\mathbf{P}^3$. $Q_1 \cap Q_2$ is codimension 2 and $\supset \rho(S)$. Since both $Q_1 \cap Q_2$ and $\rho(S)$ are of degree 4 we conclude that $\rho(S) = Q_1 \cap Q_2$.

Finally note that $\rho(S)$ is normal. To see this, note that a smooth $C \in |L|$ has genus 1. Also, a curve section of $\rho(S)$ has arithmetic genus $\{(2 + 2 - 4) \cdot 2 \cdot 2 + 2\}/2 = 1$. Since these numbers are equal we are done by 0.7.1.

0.10. *Iterating the Adjunction Process.* Typically we have a nef and big line bundle $\hat{L}$ on an a-minimal smooth surface $\hat{S}$, such that $\mathscr{L} = K_{\hat{S}} \otimes \hat{L}$ is also nef and big. If we apply the results in 0.8 we obtain in this case the following

0.10.1. LEMMA. *If $K_{\hat{S}} \otimes \mathscr{L}$ is not* nef, e.g. $(K_{\hat{S}} + \mathscr{L})^2 < 0$, *then either $(\hat{S}, \mathscr{L})$ is one of the pairs in* 0.8.2.a), *or there is a smooth rational curve E on $\hat{S}$ with self intersection -1 such that $\mathscr{L} \cdot E = 0$. In the latter case let $\pi: \hat{S} \to S^\#$ be the contraction of E. There is a big nef and big line bundle $L^\#$ on $S^\#$ such that $\hat{L} = \pi^* L^\# - E$ and $\hat{L} \cdot E = 1$, $\mathscr{L} = \pi^* \mathscr{L}^\#$ where $\mathscr{L}^\# = K_{S^\#} \otimes L^\#$.*

In particular after a sequence of such contractions we obtain a pair (Z, M), where M is nef and big on the smooth surface Z and either $2K_Z \otimes M$ is nef, or $(Z, K_Z \otimes M)$ is one of the pairs in 0.8.2.a). (Note that $(Z, K_Z \otimes M)$ is the a-minimal model for $(\hat{S}, K_{\hat{S}} \otimes \hat{L})$).

An important consequence of the above is the following.

0.10.2. LEMMA. *Let $\hat{L}$, $\hat{S}$, $\mathscr{L}$, M, Z be as above. Assume $\hat{S}$ is birationally ruled. If*

$\hat{q} = h^1(O_{\hat{S}}) \geqq 1$ *and* $(K_{\hat{S}} + \mathscr{L})^2 < K_{\hat{S}}{}^2 - (8 - 8\hat{q})$ *then* Z *is geometrically ruled and* $M_{|f} = O_{|f}(3)$.

The key point in proving this lemma is to note that $1 + (K_{\hat{S}} + \mathscr{L})^2 = (K_{S^*} + \mathscr{L}^*)^2$.

§1. Surfaces of Degree 4.

1.0. In this section we give a classification of the irreducible surfaces of degree 4 in P sufficient for the needs of this paper and which is used in [A + S 2], to study when the adjunction bundle is spanned by global sections. The table A in the introduction summarizes this classification. (See also the related articles [Um], [Ur] and [Sa])

S' will always denote an irreducible surface in P of degree 4 and $O_{S'}(1)$ will denote the restriction of $O_{\mathrm{P}}(1)$ to S'. S will denote the normalization $\rho: S \to S'$ of S' and $L = \rho^* O_{S'}(1)$. $\hat{S}$ will denote the minimal desingularization of S, $\pi: \hat{S} \to S$ and $\hat{L} = \pi^* L$. We let $\delta = (K_{\hat{S}} + \hat{L})^2$, $g = g(L) = g(\hat{L})$, $q = h^1(O_S)$ (respectively $\hat{q} = h^1(O_{\hat{S}})$) and $p_g = h^2(O_S)$ (respectively $p_{\hat{g}} = h^2(O_{\hat{S}})$). Note that

1.1. $g \leqq 3$ and $\hat{q} \leqq g$.

The first inequality follows immediately from Castelnuovo's inequality 0.4. and the second from the vanishing theorem 0.2.

Using the proposition 0.8.2 we have

1.2. Either $(\hat{S}, \hat{L})$ belongs to the list in 0.8.2.a), and in particular (S, L) is in the class consisting of cones, scrolls, quadrics and $(\mathrm{P}^2, O(1))$, $(\mathrm{P}^2, O(2))$, or $K_{\hat{S}} + \hat{L}$ is nef.

Therefore from here on we *assume that* $K_{\hat{S}} + \hat{L}$ *is nef.*

One consequence of this, by 0.8.4, is

1.3. $\hat{q} < g$, and in particular $g > 0$.

If $g = 1$ then by 0.9.1 and 0.9.2, $K_{\hat{S}} = \hat{L}^{-1}$, $K_S = L^{-1}$, S is Gorenstein-Del Pezzo with rational singularities, and ρ embeds S as the intersection of 2 quadrics in P^4.

We are therefore reduced to the cases of $g = 2$ and $g = 3$. We have, by the Index theorem, since $(\hat{L})^2 > 0$, that $(K_{\hat{S}} + \hat{L})^2 (\hat{L})^2 \leqq ((K_{\hat{S}} + \hat{L}) \cdot \hat{L})^2$, that is

1.4. $\delta \leqq (g - 1)^2$

Assume that $g = 2$. By 1.4, $\delta \leqq 1$ and therfore $\delta = 0$ or $\delta = 1$.

If $\delta = 0$, then by 0.8.3 $(\hat{S}, \hat{L})$ is a generalized conic bundle, $f: \hat{S} \to C$, over a smooth curve of genus $\hat{q}$ and $K_{\hat{S}} + \hat{L} = f^* M$, where degree $M = g - 1$. Note that, by 1.3, $\hat{q} = 0$ or 1. If $\hat{q} = 0$ then M, and hence $K_{\hat{S}} + \hat{L}$ is spanned by global sections. Now assume $\hat{q} = 1$. Since $\delta = 0$ implies $K_{\hat{S}}{}^2 = 0$, we conclude that $\hat{S}$ is geometrically ruled over an elliptic curve by f.

Claim. Such a geometrically ruled $\hat{S}$ does not exist.

Proof of the claim. First note $\hat{S} = \mathrm{P}(\mathscr{E})$, with

a) $\mathscr{E} = O_C \oplus M$ degree $M \geqq 0$, or

b) $\mathscr{E}$ is the unique non split extension: $0 \to O_{\hat{S}} \to \mathscr{E} \to O_{\hat{S}} \to 0$, or

c) $\mathscr{E}$ is the unique non split extension: $0 \to O_{\hat{S}} \to \mathscr{E} \to M \to 0$ with degree $M = 1$.

Let E be the section of $\mathrm{P}(\mathscr{E})$ corresponding to the subbundle M of $\mathscr{E}$ in case a). Then $E^2 = -r = -\deg M \leqq 0$. $\hat{L} \approx 2E + bf$ and $4 = \hat{L}^2 = 4b - 4r$. Since $b - 2r = \hat{L} \cdot E \geqq 0$, we conclude $4 = 4b - 4r \geqq 4r$ or $r = 0$ or 1.

Note that $r \neq 1$ since, if $r = 1$, then $b = 2$, and $\hat{L} \cdot (E + 2f) = 2$. But $h^0[E + 2f] = h^0(A \oplus B) = 3$, where A is a degree 1 line bundle and B is a degree 2 line bundle. Since B is spanned by global sections and A has one section there are infinitely smooth elliptic curves $D \in |(E + 2f)|$. Since α is generically $1 - 1$ it is impossible that $\hat{L} \cdot (E + 2f) = 2$.

If $r = 0$ then $\hat{L} \approx 2E + f$ and $\hat{L} \cdot E = 1$, which contradicts the fact that $\hat{L}$ is spanned by global sections.

In case b), $\hat{L} \approx 2\xi + bf$ where ξ is the tautological line bundle satisfying $\xi^2 = 0$. Thus, since $\hat{L}^2 = 4$, $b = 1$. Again letting C be the section of $\mathscr{E} \to O_{\hat{S}} \to 0$, we have $\hat{L} \cdot C = 1$ on nothing $[C] = \xi$. This also contradicts that $\hat{L}$is spanned by global sections.

In case c), let ξ denote the tautological line bundle satisfying $\xi^2 = 1$. $\hat{L}^2 = 4$ and $\hat{L} \approx 2\xi + bf$ imply $\hat{L} = 2\xi + \tau$ where τ is a Chern class zero line bundle. Note $2\xi + \tau - K_{\mathrm{P}(\mathscr{E})} \approx 4\xi - f$ is ample by proposition 2.2.1 chap. V [Ha]. Thus by Kodaira's vanishing theorem $h^i(\hat{L}) = h^i(K_{\mathrm{P}(\mathscr{E})} + (\hat{L} - K_{\mathrm{P}(\mathscr{E})})) = 0$ for $i > 0$ and $h^0(\hat{L}) = \chi(\hat{L})$. By Riemann-Roch $\chi(\hat{L}) = 4/2 - (-4 + 2)/2 + 0 = 3$. This contradicts $h^0(\hat{L}) \geqq 4$ and finishes the proof of the claim.

Now assume $\delta = 1$. Then $K_{\hat{S}}{}^2 = 1$ and $\hat{L} \cdot (2K_{\hat{S}} + \hat{L}) = 0$.

By the Hodge Index theorem it follows from $(2K_{\hat{S}} + \hat{L})^2 = 4 - 8 + 4 = 0$ that $2K_{\hat{S}} \approx \hat{L}^{-1}$. From this we conclude that $\hat{S}$ is rational and $2K_{\hat{S}} = \hat{L}^{-1}$. Note the curves D such that $0 = \hat{L} \cdot D$ are -2 rational curves. Thus S is Gorenstein with rational double points. Then $2K_S = L^{-1}$. A simple computation shows $h^0(K_{S^{-1}}) = h^0(K_S \otimes K_S{}^{-2}) = h^0(K_S \otimes L) = \chi(K_S \otimes L) = \chi(K_{S^{-1}}) = 2$. Thus $|K_{S^{-1}}|$ is a pencil of curves of arithmetic genus 1 which are irreducible since $K_{S^{-1}} \cdot K_{S^{-1}} = 1$. Thus $K_{S^{-1}}$ is spanned by global sections except at the point where all these curves meet. Since these curves are Cartier, both S and the curves are smooth at this point. By Bertini's theorem an infinite number of these curves are smooth elliptic curves. Since $L \cdot D = 2$ for $D \in |K_{S^{-1}}|$ and since ρ is generically one to one we get a contradiction.

Now assume $g = 3$. The first thing to note is that $h^0(\hat{L}) = 4$, since $h^0(\hat{L}) \geqq 4$ and $h^0(\hat{L}) \leqq 1 + h^0(\hat{L}_C) \leqq 4$ for a smooth $C \in |L|$. Next note that, by 0.7, ρ is an embedding of S into P^3.

1.5. LEMMA. *If* $g = 3$ *and* $\hat{q} > 0$, *then* $p_g = 0$.

PROOF. Consider

1.5.1. $$0 \to K_{\hat{S}} \to K_{\hat{S}} \otimes \hat{L} \to K_C \to 0$$

where C is a smooth and general element of $|\hat{L}|$. Note that since $h^0(\hat{L}) \to h^0(\hat{L}_C)$ has a 3-dimensional image for general $C \in |\hat{L}|$, $\Gamma(K_{\hat{S}} \otimes \hat{L}) \to \Gamma((K_{\hat{S}} \otimes \hat{L})_C = K_C$ has at least a 3-dimensional image if $\Gamma(K_{\hat{S}}) \neq 0$. Since $\hat{q} \neq 0$, $h^1(K_{\hat{S}} \otimes \hat{L}) = 0$ by 0.2, and $h^0(K_C) = g = 3$, we conclude that dim image $(\Gamma(K_{\hat{S}} \otimes \hat{L}) \to \Gamma(K_C)) = 3 - \hat{q} \leqq 2$. This proves the lemma.

1.6. LEMMA. *Either* $K_{\hat{S}} = O_{\hat{S}}$ *or* $K_{\hat{S}^2} < 0$.

PROOF. $\hat{L} \cdot \hat{L} = 4$ and $g = 3$ imply $\hat{L} \cdot K_{\hat{S}} = 0$. By the Hodge index theorem either $K_{\hat{S}} \approx 0$ or $K_{\hat{S}}{}^2 < 0$. If $K_{\hat{S}} \approx 0$ then $K_{\hat{S}}{}^t \cong O_{\hat{S}}$ for some $t > 0$. If $K_{\hat{S}} \neq O_{\hat{S}}$ then, by 1.5.1, $h^0(K_{\hat{S}} \otimes \hat{L}) = 3 - \hat{q} \leqq 3$. By 0.2, $\chi(K_{\hat{S}} \otimes \hat{L}) = h^0(K_{\hat{S}} \otimes \hat{L}) \leqq 3$. Since $K_{\hat{S}} \approx 0, \chi(\hat{L}) = \chi(K_{\hat{S}} \otimes \hat{L}) \leqq 3$. But $K_{\hat{S}}{}^{t-1} \otimes \hat{L} \approx \hat{L}$ is nef and big and thus $h^0(\hat{L}) = \chi(\hat{L}) \leqq 3$, contradicting $h^0(\hat{L}) \geqq 4$.

1.7. LEMMA. *If* $K_{\hat{S}} = O_{\hat{S}}$ *then* $\hat{S}$ *is a* $K - 3$ *surface. Further* $K_S = O_S$, $q = 0$ *and* S *is a Gorenstein* $K - 3$ *surface with rational singularities.*

PROOF. Consider the sequence $0 \to O_S \to \hat{L} \to \hat{L}_C \to 0$ for a smooth $C \in |\hat{L}|$. Since $\hat{L} = K_{\hat{S}} \otimes \hat{L}$, $h^1(\hat{L}) = 0$. Thus $h^0(\hat{L}) = 1 + h^0(\hat{L}_C) - \hat{q}$. Since $h^0(\hat{L}_C) = h^0((K_{\hat{S}} \otimes \hat{L})_C) = h^0(K_C) = 3$ and $h^0(\hat{L}) \geqq 4$ we conclude that $\hat{q} = 0$ and thus $\hat{S}$ is a $K - 3$ surface. If $\hat{L} \cdot D = 0$ for some irreducible curve D then $D^2 < 0$ and $K_{\hat{S}} \cdot D = O_{\hat{S}} \cdot D = 0$ imply D is a smooth rational curve with $D^2 = -2$. From this the rest of the lemma follows immediately.

1.8. LEMMA. If $K_{\hat{S}^2} < 0$ *then* $\hat{S}$ *is birationally ruled or rational.*

PROOF. Assume that $mK_{\hat{S}}$ is effective for some $m > 0$. Then $mK_{\hat{S}} \cdot (K_{\hat{S}} + \hat{L})$ is nef. But $mK_{\hat{S}} \cdot (K_{\hat{S}} + \hat{L}) = mK_{\hat{S}} \cdot K_{\hat{S}} < 0$.

1.9. LEMMA. $K_{\hat{S}} \cdot K_{\hat{S}} + 4 = \delta \leqq 4$, *with equality only if* $K_{\hat{S}} = O_{\hat{S}}$.

PROOF. Use 1.4 to conclude $\delta \leqq 4$. By the Hodge index theorem, exactly as in the case of $g = 2$ and $\delta = 1$, we conclude $K_{\hat{S}} \approx 0$. Use 1.6.

1.10. LEMMA. $\hat{q} \leqq 1$.

PROOF. $\hat{q} \leqq g - 1 = 2$. If $\hat{q} = 2$ then $\hat{p}_g = 0$ by lemma 1.5. Then $\hat{S}$ is ruled over a genus 2 curve by 1.6 and 1.8. Thus $K_{\hat{S}} \cdot K_{\hat{S}} \leqq 8(1 - \hat{q}) = -8$. But, by 1.9, $\delta \geqq 0$ implies $K_{\hat{S}} \cdot K_{\hat{S}} \geqq -4$. This contradiction establishes the lemma.

1.11. LEMMA. *If* $\delta = 0$ *then* $(\hat{S}, \hat{L})$ *is a conic bundle and* $K_{\hat{S}} \otimes \hat{L}$ *is spanned by global sections.*

TABLE A

$K_{\hat{S}} + \hat{L}$ not nef				(S, L) and $(\hat{S}, \hat{L})$ are in the class consisting of cones, scrolls and $(\mathbf{P}^2, \mathcal{O}(2))$	
$K_{\hat{S}} + \hat{L}$ nef					
g	$(K_{\hat{S}} + \hat{L})^2$	$h^0(\hat{L}) = h^0(L)$	$\hat{q}$		
1	0	5	0	$K_{\hat{S}} = \hat{L}^{-1}$, $K_S = L^{-1}$, (S, L) is Gorenstein-Del Pezzo, with rational singularities, ρ embeds S as the intersection of 2 quadrics.	
2	0	4	0	$(\hat{S}, \hat{L})$ is a conic bundle over $\mathbf{P}^1$.	
3	0	4	0	$(\hat{S}, \hat{L})$ is a conic bundle over $\mathbf{P}^1$.	ρ gives an embedding of S into $\mathbf{P}^3$.
3	0	4	1	$(\hat{S}, L)$ is a conic bundle over an elliptic curve	
3	4	4	0	$\hat{S}$ is a K-3 surface, S is a Gorenstein K-3 surface with only rational sigularities.	
3	1, 2, 3	4	0	$\hat{S}$ is rational.	
3	2	4	1	This is the special example in described in [Um].	

PROOF. If $\delta = 0$ then, since $g = 3$, it follows from the fact that $K_{\hat{S}} + \hat{L}$ is nef and 0.8.3, that $(\hat{S}, \hat{L})$ is a conic bundle, $f\colon \hat{S} \to D$, and $K_{\hat{S}} \otimes \hat{L} = f^*M$, where M is a line bundle of degree 2 on the smooth curve D. Since $\hat{q} = \text{genus}(D) \leqq 1$, by 1.10 we conclude that M, and hence $K_{\hat{S}} \otimes \hat{L}$, is spanned by global sections.

Putting together the above we have the classification summarized in the table A.

We notice that for $g = 3$ and $\hat{q} = 1$ we have that $\hat{S}$ is birationally equivalent to an elliptic ruled surfaces, and that S is a quartic surface in P^3 embedded by L. A description of (S, L) in this case is given in [Um].

The following theorem follows from the classification we have just obtained.

1.13. THEOREM. *Let S be the normalization of an irreducible degree four surface in* P^N. *Let L be the pullback of* $O_{\mathrm{P}^n(1)}$. *If* (S, L) *is neither* $(\mathrm{P}^2, O(2))$, *nor a scroll nor a cone, then* $K_S \otimes L$ *is spanned by global sections outside the singular points.*

PROOF. By the classification either $K_S \otimes L$ is spanned by global sections or $(\hat{S}, \hat{L})$ is a conic bundle over P^1. By 1.11, $K_{\hat{S}} \otimes \hat{L}$ is spanned by global sections in this case. Since the direct image of $K_{\hat{S}} \otimes \hat{L}$ is a subsheaf of $K_S \otimes L$with cokernel supported on the singular points, we are done.

§2. Surfaces with low sectional genus.

2.0. In this chapter we give the proofs of the theorems A, B and C stated in the introduction.

We use the same notation as before: $S' \subset \mathrm{P}^n$ is a surface in P^n and $L' = O(1)_{|S'}$, $\rho\colon S \to S'$ its normalization and $L = \rho^* L'$.

$\hat{S}$ is the minimal desingularization of S, $\pi\colon \hat{S} \to S$ and $\hat{L} = \pi^* L$.

Let $C \in |L|$ a section. The adjunction formula gives $g(C) = 1/2[(K_S + L)\cdot L + 2] = g(L)$.

We use also the notation $q(S) = h^1(S, K_S)$ and $p_g(S) = h^0(S, K_S)$.

2.0.1. LEMMA. *Let* (S, L) *(resp.* $(\hat{S}, \hat{L})$*) as above. Then* $h^0(K_S \otimes L) = g(L) - q(S) + p_g(S)$ *(resp* $h^0(K_{\hat{S}} \otimes \hat{L}) = g(\hat{L}) - q(\hat{S}) + p_g(\hat{S})$*).*

PROOF. The equality follows easily from the long exact sequence associated to

$$0 \to K_S \to K_S \otimes L \to K_C \to 0$$
$$(\text{resp. } 0 \to K_{\hat{S}} \to K_{\hat{S}} \otimes \hat{L} \to K_{\hat{C}} \to 0,\ \hat{C} = \pi^{-1}(C))$$

and the vanishing theorem 0.2.

PROOF OF THEOREM A. By the adjunction formula and the ampleness of L, we have that $K_S \cdot L < 0$ and therefore $h^0(K_S^N) = 0$ for all $N > 0$; in particular $p_g(S) = 0$ (the same for $p_g(\hat{S}) = 0$).

If $h^0(K_{\hat{S}} \otimes \hat{L}) = 0$, then we apply proposition 0.8.5. Therefore we are either in case a) or in b) depending if S is or not smooth, or, differently, if $q(S) = 1$ or $q(S) = 0$.

If $h^0(K_{\hat{S}} \otimes \hat{L}) \geqq 1$, then $h^0(K_S \otimes L) \geqq 1$.

Choose a smooth generic $C \in |L|$. Note $(K_S \otimes L)_C \cong K_C \cong O_C$. Thus a general $s \in \Gamma(K_S \otimes L)$ is zero only on a finite set in $S \backslash C$. Thus $(K_S \otimes L)_U \cong O_U$ for $U = S - F$ where F is finite. Therefore if i: $U \to S$ is the inclusion $K_S \cong i * K_U \cong i * L_U^{-1} \cong L^{-1}$. A similar argument shows $K_{\hat{S}} \cong \hat{L}^{-1}$. Thus by the vanishing theorem $q(S) = h^1(K_S) = h^1(L^{-1}) = 0$.

Tensoring the exact sequence $0 \to \boldsymbol{K}_S \to K_S \to \mathscr{S} \to 0$ with L using the fact that $h^0(\boldsymbol{K}_S \otimes L) = h^0(K_{\hat{S}} \otimes \hat{L}) = 1$ and the vanishing theorem 0.2, we can conclude that $\mathscr{S} = 0$; as noticed in 0.0.1., this implies that S has only rational (Gorenstein) singularities.

Proof of the theorem B. Using the Castelnuovo inequality 0.3 we can suppose $\hat{L} \cdot \hat{L} = L \cdot L \geqq 4$. By §2 we can assume that $\hat{L} \cdot \hat{L} \geqq 5$. With this, by the adjunction formula and the fact that L and $\hat{L}$ are nef, we have $K_{\hat{S}} \cdot \hat{L} < 0$ and $K_S \cdot L < 0$; this implies $h^0(K_{S^N}) = h^0(K_{\hat{S}^N}) = 0$ for all $N > 0$. In particular $p_g(S) = p_g(\hat{S}) = 0$.

By means of lemma 2.0.1. and what above we have $h^0(K_{\hat{S}} \otimes \hat{L}) = 2 - q(\hat{S})$.

If $q(\hat{S}) = 2$, and thus $h^0(K_{\hat{S}} \otimes \hat{L}) = 0$, proposition 0.8.5 applies and we conclude directly that (S, L) is either in b) or in a) depending S is or is not singular.

If $q(\hat{S}) = 1$ then there is an Albanese map from $\hat{S}$ to a curve R of genus 1. Since $h^0(K_{\hat{S}}^N) = 0$ for all $N > 0$, $\hat{S}$ is ruled and therefore the general fibre of the Albanese map is P^1, i.e. $\hat{S}$ is a ruled surface over a genus 1. This implies $K_{\hat{S}} \cdot K_{\hat{S}} \leqq 0$. By our assumption, $q(\hat{S}) = 1$, we have also $h^0(K_{\hat{S}} + \hat{L}) = 1$; applying theorem 0.8.2, we have $0 \leqq (K_{\hat{S}} + \hat{L})^2 = K_{\hat{S}} \cdot K_{\hat{S}} + 2(K_{\hat{S}} + \hat{L}) \cdot \hat{L} - \hat{L} \cdot \hat{L} \leqq 4 - \hat{L} \cdot \hat{L}$. This is impossible since $\hat{L} \cdot \hat{L} \geqq 5$.

We finally assume $q(\hat{S}) = 0$. Thus $h^0(K_{\hat{S}} \otimes \hat{L}) = 2$ and, by the theorem 0.8.2 $(K_{\hat{S}} + \hat{L})$ is nef. By theorem 0.8.3 we have that $n(K_{\hat{S}} + \hat{L})$ is spanned by global sections for $n \gg 0$. Moreover we have $(K_{\hat{S}} + \hat{L}) \cdot (K_{\hat{S}} + \hat{L}) = 0$, otherwise the Hodge Index theorem will imply $(\hat{L} \cdot \hat{L})((K_{\hat{S}} + \hat{L}) \cdot (K_{\hat{S}} + \hat{L})) \leqq ((K_{\hat{S}} + \hat{L}) \cdot \hat{L})^2 = 4$, and thus $\hat{L} \cdot \hat{L} \leqq 4$. Therefore $n(K_{\hat{S}} + \hat{L})$ gives a map, $\hat{p}$: $\hat{S} \to R$, from $\hat{S}$ into a smooth curve R of genus 0, i.e. into P^1. It is immediate to see that the generic fibre, f, is a smooth rational P^1 such that $\hat{L} \cdot f = 2$, i.e. by 0.8.3 $(\hat{S}_c, \hat{L}_c)$ is a conic bundle over P^1, p: $\hat{S}_c \to \mathsf{P}^1$.

Let C be an irreducible curve contracted by π_2. Since a general fibre, f, of the conic bundle structure, p: $\hat{S}_c \to \mathsf{P}^1$, deforms we conclude that $C \cdot f > 0$. We want to show that C is a section, e.g. that $C \cdot f = 1$. Assume $C \cdot f > 1$. Since $\pi_2(C)$ is a point and $C \cdot f > 1$, it follows that $\pi_2(f)$ cannot be smooth for a general fibre f of p. But for a general fibre f of p, $\pi_2(f)$ is an irreducible and reduced conic; in particular $\pi_2(f)$ must be smooth. This contradiction shows that $C \cdot f = 1$.

Next we claim that there cannot be 3 sections C_1, C_2, C_3 collapsed by π_2. Let $C_{\hat{1}}, C_{\hat{2}}, C_{\hat{3}}$ be the proper transforms of C_1, C_2, C_3 in $\hat{S}$. Let $C = \pi(f)$ for a generic fibre of $\hat{p}$. Since C is a conic, $h^0(L_{|C}) \leqq 3$, and moreover $h^0(L) \geqq 4$, since otherwise $S = \mathbf{P}^2$. Therefore there exists a $D \in |L|$ such that $C \subset D$. This implies $C_{\hat{1}} \cup C_{\hat{2}} \cup C_{\hat{3}} \cup f \subset \pi^{-1}(D) \in |\hat{L}|$. thus $\hat{L} \cdot f = \pi^{-1}(D) \cdot f \geqq 3$, which contradicts $\hat{L} \cdot f = 2$.

Proof of the theorem C. Using the Castelnuovo inequality 0.3 we can suppose $\hat{L} \cdot \hat{L} = L \cdot L \geqq 6$ except if the general section $C \in |L|$ is a plane curve of degree 4 or 5. This will imply S' to be a hypersurface of degree 4 or 5 in $\mathbf{P}^3$ and, for degree 4, $S' = S$ (see §2).

As in the previous proofs, since L and $\hat{L}$ are nef, by the adjunction formula we have $K_{\hat{S}} \cdot \hat{L} < 0$ and $K_S \cdot L < 0$ and therefore $h^0(K_S^N) = h^0(K_{\hat{S}}^N) = 0$ for all $N > 0$. In particular $p_g(S) = p_g(\hat{S}) = 0$ and $\hat{S}$ is ruled, $p: \hat{S} \to R$.

By means of lemma 2.0.1. we have $h^0(K_{\hat{S}} \otimes \hat{L}) = 3 - q(\hat{S})$.

If $q(\hat{S}) = 3$ then proposition 0.8.5. will apply and we conclude that (S, L) (resp. (S', L')) is either in a) or in b) depending if S is or is not singular.

Suppose now $q(\hat{S}) < 3$, i.e. $h^0(K_{\hat{S}} \otimes \hat{L}) \neq 0$. By theorem 0.8.2 $(K_{\hat{S}} \otimes \hat{L})$ is nef; thus two cases are possible:

$$1)\ (K_{\hat{S}} + \hat{L})^2 = 0 \qquad 2)\ (K_{\hat{S}} + \hat{L})^2 > 0.$$

Suppose first $(K_{\hat{S}} + \hat{L})^2 = 0$. Then as in the proof of theorem B, since $(K_{\hat{S}} \otimes \hat{L})$ is nef, by the theorem 0.8.3 we can show that $\hat{S}$ is a conic bundle over a curve R of genus $q(\hat{S})$.

We have the following inequalities if $q(\hat{S}) \geqq 1$:

$$-2 \leqq \hat{L} \cdot \hat{L} - 8 = K_{\hat{S}} \cdot K_{\hat{S}} \leqq 8(1 - q(\hat{S}))$$

(the last works for ruled surfaces). Therefore $q(\hat{S}) = 0$ or 1.

If $q(\hat{S}) = 0$ we are in the case c (see the analogous proof for theorem B). If $q(\hat{S}) = 1$ we are in the case f).

Assume now $(K_{\hat{S}} + \hat{L})^2 > 0$. By the Hodge Index theorem, $(K_{\hat{S}} + \hat{L})^2 (\hat{L})^2 \leqq ((K_{\hat{S}} + \hat{L}) \cdot \hat{L})^2 = 16$. Since we have $6 \leqq \hat{L} \cdot \hat{L}$, the following two cases are possible:

$$\alpha)\ (K_{\hat{S}} + \hat{L})^2 = 1 \text{ and } 6 \leqq \hat{L} \cdot \hat{L} \leqq 16$$
$$\beta)\ (K_{\hat{S}} + \hat{L})^2 = 2 \text{ and } 6 \leqq \hat{L} \cdot \hat{L} \leqq 8.$$

Note $q(\hat{S}) = 0$ or 1. If $q(\hat{S}) = 2$ then by Hurwitz the ruling $p_C: C \to R$ would give a two to one map for a general $C \in |\hat{L}|$. Thus $(K_{\hat{S}} + \hat{L})^2$ would be $= 0$, contradicting our hypothesis.

Therefore we have $h^0(K_{\hat{S}} + \hat{L}) = g(\hat{L}) - q(\hat{S}) \geqq 2$.

By the definition of arithmetical genus we have easily $g(K_{\hat{S}} + \hat{L}) + g(\hat{L}) = (K_{\hat{S}} + \hat{L})^2 + 2$, i.e. $g(K_{\hat{S}} + \hat{L}) = (K_{\hat{S}} + \hat{L})^2 - 1$.

Suppose we are in the case α) and thus that $(K_{\hat{S}} + \hat{L})^2 = 1$ and $g(K_{\hat{S}} + \hat{L}) = 0$. The general divisor $D \in |(K_{\hat{S}} + \hat{L})|$ can be written as $D = \mathscr{X} + \mathscr{I}$ with $\mathscr{X}$ the moving part and $\mathscr{I}$ the fixed part. $\mathscr{X} \cdot \mathscr{X} \geqq 0$ and therefore $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{X} > 0$, by the Hodge Index theorem, and $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{I} \geqq 0$, since $(K_{\hat{S}} + \hat{L})$ is nef. Moreover

$$(K_{\hat{S}} + \hat{L}) \cdot \mathscr{I} = (K_{\hat{S}} + \hat{L})^2 - (K_{\hat{S}} + \hat{L}) \cdot \mathscr{X} = 1 - (K_{\hat{S}} + \hat{L}) \cdot \mathscr{X}.$$

This implies

*) $$(K_{\hat{S}} + \hat{L}) \cdot \mathscr{I} = 0 \text{ and } (K_{\hat{S}} + \hat{L}) \cdot \mathscr{X} = 1.$$

In particular, since $\mathscr{X}$ is the moving part, it is irreducible.

Using the exact sequence

$$0 \to (-K_{\hat{S}} - \hat{L}) \to O_{\hat{S}} \to O_D \to 0,$$

the fact that $h^1(-K_{\hat{S}} - \hat{L}) = 0$ (theorem 0.2.), and $h^1(O_D) = g(K_{\hat{S}} + \hat{L}) = 0$, we obtain that $h^1(O_{\hat{S}}) = q(\hat{S}) = 0$ and therefore $h^0(K_{\hat{S}} + \hat{L}) = h^0(\mathscr{X}) = 3$.

This implies, using the exact sequence

$$0 \to O_{\hat{S}} \to \mathscr{X} \to \mathscr{X}_{|\mathscr{X}} \to 0$$

that $\mathscr{X}$ is spanned by global sections ($g(\mathscr{X}) = 0$ and $\mathscr{X} \cdot \mathscr{X} \geqq 0$). Since $\mathscr{X}$ is irreducible $h^0(\mathscr{X}_{|\mathscr{X}}) = 2$, and thus $\mathscr{X} \cdot \mathscr{X} = 1$. By *), $\mathscr{X} \cdot \mathscr{I} = \mathscr{I} \cdot \mathscr{I} = 0$. The Index theorem forces $\mathscr{I}$ to be 0.

So far we have proved that $(K_{\hat{S}} + \hat{L}) = [\mathscr{X}]$ is spanned by global sections. $h^0(K_{\hat{S}} + \hat{L}) = 3$ and $(K_{\hat{S}} + \hat{L})^2 = 1$. Therefore we have a map associated to $|(K_{\hat{S}} + \hat{L})|$ that is a birational morphism, $\phi: \hat{S} \to \mathbf{P}^2$, from $\hat{S}$ to $\mathbf{P}^2$.

We now consider the second case, β), in which we assume $(K_{\hat{S}} + \hat{L})^2 = 2, 6 \leqq \hat{L} \cdot \hat{L} \leqq 8$ and $g(K_{\hat{S}} + \hat{L}) = 1$. We have

$$8(1 - q(\hat{S})) \geqq K_{\hat{S}} \cdot K_{\hat{S}} = (K_{\hat{S}} + \hat{L})^2 - 2(K_{\hat{S}} + \hat{L}) \cdot \hat{L} + \hat{L} \cdot \hat{L} = -6 + \hat{L} \cdot \hat{L} \geqq 0.$$

Therefore either $q(\hat{S}) = 1$ and $\hat{L} \cdot \hat{L} = 6$ or $q(\hat{S}) = 0$.

In the first case, since $q(\hat{S}) = 1$ and $K_{\hat{S}} \cdot K_{\hat{S}} = 0$, we have that $\hat{S}$ is a geometrically ruled surface over a curve, R_1, of genus 1, $p: \hat{S} = \mathbf{P}(\mathscr{E}) \to R_1$. It is straightforward to show that $\hat{L} \approx 3\sigma + f$, where f is a fibre of p, and σ is a section of p with $\sigma^2 = 0$, and such that any other section has self intersection at least 0. Noting that $\hat{L} \cdot \sigma = 1$ we see $\hat{L}$ is not spanned by global sections.

Thus we can suppose $q(\hat{S}) = 0, 8 \geqq \hat{L} \cdot \hat{L} \geqq 6$ and $(K_{\hat{S}} + \hat{L})^2 = 2$.

Let $D = \mathscr{X} + \mathscr{I}$ be a general divisor in $|(K_{\hat{S}} + \hat{L})|$ decomposed in its moving part, $\mathscr{X}$, plus its fixed part $\mathscr{I}$.

We have $3 = h^0(K_{\hat{S}} + \hat{L}) = h^0(\mathscr{X})$. Two different cases are possible, since $(K_{\hat{S}} + \hat{L})^2 = 2$:

a) $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{X} = 1$ and $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{I} = 1$; here by the index theorem $\mathscr{X} \cdot \mathscr{X} = 0$,

b) $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{X} = 2$ and $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{I} = 0$; here by the index theorem $\mathscr{X} \cdot \mathscr{X} \leqq 2$.

In the first case a) we first notice that $\mathscr{X}$ is irreducible and, using the exact sequence $0 \to O_{\hat{S}} \to \mathscr{X} \to \mathscr{X}_{|\mathscr{X}} \to 0$ and $h^0(\mathscr{X}_{|\mathscr{X}}) = 2$. But $\mathscr{X} \cdot \mathscr{X} = 0$ implies the contradiction $h^0(\mathscr{X}_{|\mathscr{X}}) \leqq 1$.

Suppose then the equalities in b) are true. If $\mathscr{X} \cdot \mathscr{X} = 2$, then by the Hodge index theorem $K_{\hat{S}} + \hat{L} \approx \mathscr{X}$ and $\mathscr{I}$ is trivial. Here $g(\mathscr{X}) = g(K_{\hat{S}} + \hat{L}) = 1$. From this, the exact sequence $0 \to O_{\hat{S}} \to \mathscr{X} \to \mathscr{X}_{|\mathscr{X}} \to 0$ and $h^0(|\mathscr{X}|) = h^0(K_{\hat{S}} + \hat{L})$, we see that $|\mathscr{X}|$ is spanned by global sections, giving a map $\phi: \hat{S} \to \mathbf{P}^2$, which is 2 to 1.

Now assume $\mathscr{X} \cdot \mathscr{X} \leqq 1$. In this case $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{X} = 2$ tells us that $\mathscr{X}$ has at most two components, $\mathscr{X} = \mathscr{X}_1 + \mathscr{X}_2$. If this is so, then since neither stays fixed $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{X}_i = 1$.

If $\mathscr{X}_1 \cdot \mathscr{X}_2 \neq 0$ then, by Bertini's theorem, they meet in base points, therefore $(\mathscr{X}_1 + \mathscr{X}_2)^2 \geqq 2$, which is absurd.

If $\mathscr{X}_1 \cdot \mathscr{X}_2 = 0$ then $\mathscr{X} \cdot \mathscr{X} = \mathscr{X}_1 \cdot \mathscr{X}_1 + \mathscr{X}_2 \cdot \mathscr{X}_2$. Since $\mathscr{X} \cdot \mathscr{X} \leqq 1$ and $\mathscr{X}_i \cdot \mathscr{X}_i \geqq 0$, we conclude either $\mathscr{X}_i \cdot \mathscr{X}_i = 0$ for $i = 1, 2$ or, after renaming, $\mathscr{X}_1 \cdot \mathscr{X}_1 = 1$, $\mathscr{X}_2 \cdot \mathscr{X}_2 = 0$. The latter is absurd because by the Hodge index theorem this would imply $\mathscr{X}_2 \approx 0$ which is absurd since $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{X}_i = 1$. In the former case $(\mathscr{X}_1 - \mathscr{X}_2) \cdot (K_{\hat{S}} + \hat{L}) = 0$ and we conclude that $\mathscr{X}_1 \approx \mathscr{X}_2$, $\mathscr{X} \approx 2\mathscr{X}_1$. Thus $\mathscr{X}_i \cdot \mathscr{I} = 1$ for $i = 1, 2$ and $\mathscr{I}^2 = -2$. Since $4 = (K_{\hat{S}} + \hat{L}) \cdot \hat{L} = (2\mathscr{X}_1 + \mathscr{I}) \cdot \hat{L}$we conclude $\hat{L} \cdot \mathscr{X}_i = 1$ and $\hat{L} \cdot \mathscr{I} = 2$ or $\hat{L} \cdot \mathscr{X}_i = 2$ and $\hat{L} \cdot \mathscr{I} = 0$. Since $|\mathscr{X}|$ has finite base locus, L is spanned by global sections, ρ is generically one to one, we conclude that the $\mathscr{X}_i$ are smooth rational curves and thus $K_{\hat{S}} \cdot \mathscr{X}_i = -2$. This gives the contradiction $1 = (K_{\hat{S}} + \hat{L}) \cdot \mathscr{X}_1 = -2 + \hat{L} \cdot \mathscr{X}_1 = -1$ or 0.

Therefore we can suppose that a general $\mathscr{X}$ has one component. Here $\mathscr{X} \cdot \mathscr{X} = 0$ or $\mathscr{X} \cdot \mathscr{X} = 1$. Let first $\mathscr{X} \cdot \mathscr{X} = 1$; if $\hat{L} \cdot \mathscr{X} = 1$ then $\mathscr{X}$ is smooth and has genus zero. Thus $K_{\hat{S}} \cdot \mathscr{X} = -3$ and we get the contradiction $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{X} = -2$. If $\hat{L} \cdot \mathscr{X} \geqq 2$, then $K_{\hat{S}} \cdot \mathscr{X} \leqq 0$. Since $K_{\hat{S}} \cdot \mathscr{X} + \mathscr{X} \cdot \mathscr{X}$ is even and $\mathscr{X} \cdot \mathscr{X} = 1$ we conclude $K_{\hat{S}} \cdot \mathscr{X} = -3$ and $g(\mathscr{X}) = 0$ or $K_{\hat{S}} \cdot \mathscr{X} = -1$ and $g(\mathscr{X}) = 1$. In the first case, since $(K_{\hat{S}} + \hat{L}) \cdot K_{\hat{S}} = -2$, we have $K_{\hat{S}} \cdot \mathscr{I} = 1$. From $(K_{\hat{S}} + \hat{L}) \cdot \mathscr{I} = 0$ we have $\hat{L} \cdot \mathscr{I} = -1 < 0$, which is absurd. In the second case, since $\mathscr{X}$ is irreducible, $h^0(\mathscr{X}_{|\mathscr{X}}) = 1$; we have the absurd using the exact sequence $0 \to O_{\hat{S}} \to \mathscr{X} \to \mathscr{X}_{|\mathscr{X}} \to 0$ and the fact that $h^0(\mathscr{X}) = 3$.

Thus we come down to the final case $\mathscr{X} \cdot \mathscr{X} = 0$ and $\mathscr{X}$ irreducible. In this case, by $0 \to O_{\hat{S}} \to [\mathscr{X}] \to [\mathscr{X}]_{|\mathscr{X}} \to 0$, we see $h^0([\mathscr{X}]) \leqq 1 + h^0([\mathscr{X}]_{\mathscr{X}}) \leqq 2$ since $[\mathscr{X}]_{\mathscr{X}}$ is a degree 0 line bundle on an irreducible curve. This contradicts $3 = h^0([\mathscr{X}]) = h^0(K_{\hat{S}} + \hat{L})$.

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