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POINTWISE MEASURABILITY IN NONSTANDARD MODELS

DIETER LANDERS and LOTHAR ROGGE

Abstract.

In this paper we present a concept of pointwise measurability of functions. Formally, this concept is closely related to the concept of pointwise continuity of functions, and we hope that it will become similarly useful. The concept of pointwise measurability is introduced by nonstandard methods.

Let $\mathscr{P}(Z)$ be the power set of the set Z. A system $\mathscr{C} \subset \mathscr{P}(Z)$ is $\cup$-closed ($\cap$-closed) if $C, D \in \mathscr{C}$ implies $C \cup D \in \mathscr{C}$ ($C \cap D \in \mathscr{C}$). We denote by $\mathscr{T}(\mathscr{C})$ the smallest toplogy containing $\mathscr{C}$. If $\mathscr{C}_i \subset \mathscr{P}(Z_i)$, $i = 1,2$ we call a function $h: Z_1 \to Z_2$ $\mathscr{C}_1, \mathscr{C}_2$-measurable iff $h^{-1}(C_2) \in \mathscr{C}_1$ for all $C_2 \in \mathscr{C}_2$. Usually measurability is only defined for σ-algebras $\mathscr{C}_1$ and $\mathscr{C}_2$. To be more flexible we dropped this restriction. Thus, if $\mathscr{C}_1$ and $\mathscr{C}_2$ are topologies, $\mathscr{C}_1, \mathscr{C}_2$-measurability is the same as $\mathscr{C}_1, \mathscr{C}_2$-continuity.

In this paper we consider a superstructure containing two given sets X, Y and the set R of real numbers, and we work with a polysaturated nonstandard model for this superstructure. (We use the basic terminology and results of nonstandard analysis as given e.g. in [1], [2] and [6].)

DEFINITION. Let $\mathscr{A} \subset \mathscr{P}(X)$, $\mathscr{B} \subset \mathscr{P}(Y)$, and $g: {}^*X \to {}^*Y$ be an internal function. Put ${}^\sigma\mathscr{A} := \{{}^*A : A \in \mathscr{A}\}$ and ${}^\sigma\mathscr{B}$ analogously. The function g is called ${}^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in a point $x \in {}^*X$ iff for every $B \in \mathscr{B}$ with $g(x) \in {}^*B$ there exists $A \in \mathscr{A}$ with $x \in {}^*A$ such that $g({}^*A) \subset {}^*B$.

The following theorem shows that pointwise measurability for all $x \in {}^*X$ is the same as the usual measurability and is furthermore equivalent to continuity for appropriate topologies.

1. PROPOSITION. *Let $\mathscr{A} \subset \mathscr{P}(X)$ be $\cup,\cap$-closed with $X \in \mathscr{A}$ and $\mathscr{B} \subset \mathscr{P}(Y)$. For an internal function $g: {}^*X \to {}^*Y$ the following three conditions are equivalent:*

(i) *g is ${}^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable;*

(ii) *g is ${}^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in x for all $x \in {}^*X$;*

(iii) *g is $\mathscr{T}({}^\sigma\mathscr{A}), \mathscr{T}({}^\sigma\mathscr{B})$-continuous.*

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PROOF. (i) $\Rightarrow$ (iii) follows directly from the definition of the topologies $\mathscr{T}({}^{\sigma}\mathscr{A})$ and $\mathscr{T}({}^{\sigma}\mathscr{B})$.

(iii) $\Rightarrow$ (ii). Let $x \in {}^*X$ be fixed. Let $B \in \mathscr{B}$ with $g(x) \in {}^*B$. As g is $\mathscr{T}({}^{\sigma}\mathscr{A}), \mathscr{T}({}^{\sigma}\mathscr{B}))$-continuous in x, there exist $T \in \mathscr{T}({}^{\sigma}\mathscr{A})$ with $x \in T$ and $g(T) \subset {}^*B$. Since ${}^{\sigma}\mathscr{A}$ is $\cap$-closed, we obtain $T = \cup_{i \in I} {}^*A_i$ with suitable $A_i \in \mathscr{A}$. Hence there exists $A \in \mathscr{A}$ with $x \in {}^*A$ and $g({}^*A) \subset {}^*B$.

(ii) $\Rightarrow$ (i). Let $B \in \mathscr{B}$ be given. For each $x \in {}^*X$ with $g(x) \in {}^*B$ there exists $A_x \in \mathscr{A}$ with $x \in {}^*A_x$ and $g({}^*A_x) \subset {}^*B$. Hence $\cup_{g(x) \in {}^*B} {}^*A_x = g^{-1}({}^*B)$. As $g^{-1}({}^*B)$ is internal and the system $\{{}^*A_x : g(x) \in {}^*B\}$ contains at most as many sets as $\mathscr{A}$, we obtain by polysaturation that there exist finitely many A_{x_i}, $i = 1, \ldots, n$ with $\cup_{i=1}^{n} {}^*A_{x_i} = g^{-1}({}^*\mathrm{B})$. Hence $g^{-1}({}^*B) = {}^*A$ with $A = \cup_{i=1}^{n} A_{x_i}$. As $\mathscr{A}$ is $\cup$-closed, we have $A \in \mathscr{A}$.

2. COROLLARY. *Let $\mathscr{A} \subset \mathscr{P}(X)$ be $\cup,\cap$-closed with $X \in \mathscr{A}$ and $\mathscr{B} \subset \mathscr{P}(Y)$. For a function $f : X \to Y$ the following conditions are equivalent:*

(i) *f is $\mathscr{A},\mathscr{B}$-measurable;*

(ii) *${}^*f$ is ${}^{\sigma}\mathscr{A},{}^{\sigma}\mathscr{B}$-measurable in x for all $x \in {}^*X$.*

PROOF. Apply Proposition 1 to the internal function $g = {}^*f$ and use ${}^*f^{-1}({}^*B) = {}^*\mathrm{A}$ iff $f^{-1}(B) = A$.

Ross [5] showed that for an algebra $\mathscr{A}$ the following two conditions are equivalent:

(i) f is $\mathscr{A},\mathscr{B}$-measurable,

$(\mathrm{ii})_R$ ${}^*f(m_{\mathscr{A}}(x)) \subset m_{\mathscr{B}}({}^*f(x))$ for all $x \in {}^*X$,

where $m_{\mathscr{A}}(x) = \cap\{{}^*A : x \in {}^*A,\ A \in \mathscr{A}\}$ is the $\mathscr{A}$-monad of x, $m_{\mathscr{B}}(y)$ is defined analogously.

Using polysaturation it is easy to see that $(\mathrm{ii})_R$ is equivalent to condition (ii) of Corollary 2. Ross furthermore claimed ([5], II.1.9): If the equivalence of (i) and $(\mathrm{ii})_R$ holds for all f and all $\mathscr{B}$ then $\mathscr{A}$ is necessarily an algebra. This, however, contradicts the statement of Corollary 2, because the equivalence in Corollary 2 also holds for systems $\mathscr{A}$, which are not an algebra.

In our next Theorem we will weaken condtion (ii) of the preceding Corollary by assuming ${}^{\sigma}\mathscr{A},{}^{\sigma}\mathscr{B}$-measurability only for almost all points $x \in {}^*X$ where almost all refers to a suitable measure. To this aim we introduce the following notions. Let μ be a finite measure on a σ-algebra $\mathscr{A}$ on X. We denote by $\mathscr{A}_{\mu}$ the completion of $\mathscr{A}$ with respect to $\mu|\mathscr{A}$. It is well known that $\mathscr{A}_{\mu}$ is the σ-algebra of all $C \subset X$ such that $A \subset C \subset A \cup N$ for some $A, N \in \mathscr{A}$ with $\mu(N) = 0$.

If $\nu : {}^*\mathscr{A} \to {}^*[0, \infty)$ is a finite internal content, then $\nu_L : L(\nu) \to [0, \infty)$ denotes the Loeb-measure associated with $\nu|{}^*\mathscr{A}$; $L(\nu)$ is a σ-algebra containing ${}^*\mathscr{A}$, and $\nu_L|L(\nu)$ is a complete measure. $L(\nu)$ is called the system of ν_L-measurable sets. A set $C \subset X$ is *universally* Loeb-measurable iff $C \in L(\nu)$ for each finite internal content $\nu|{}^*\mathscr{A}$ (see e.g. [4], [3]).

3. THEOREM. *Let $\mathscr{A} \subset \mathscr{P}(X)$ be a σ-algebra, $\mathscr{B} \subset \mathscr{P}(Y)$ and $\mu|\mathscr{A}$ be a finite measure. For a fucntion $f: X \to Y$ the following conditions are equivalent:*

(i) *f is $\mathscr{A}_\mu, \mathscr{B}$-measurable;*

(ii) *$^*f$ is $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in x for $^*\mu_L$-a.a. $x \in {}^*X$.*

PROOF. (i) $\Rightarrow$ (ii): Since $f^{-1}(B) \in \mathscr{A}_\mu$ for $B \in \mathscr{B}$ there exist $A_B, N_B \in \mathscr{A}$ with $A_B \subset f^{-1}B \subset A_B \cup N_B$ and $\mu(N_B) = 0$. Put $N := \cup_{B \in \mathscr{B}} {}^*N_B$. Then $^*\mu_L(N) = 0$ according to Lema 5 below. We prove that $^*f$ is $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in x for all $x \notin N$. Let $x \notin N$. Take $B \in \mathscr{B}$ with $^*f(x) \in {}^*B$. Since $x \notin {}^*N_B$ and $^*f^{-1}({}^*B) \subset {}^*A_B \cup {}^*N_B$, we obtain $x \in {}^*A_B$. As futhermore $A_B \in \mathscr{A}$ and $^*f({}^*A_B) \subset {}^*B$, this implies that $^*f$ is $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in x.

(ii) $\Rightarrow$ (i): Let $B \in \mathscr{B}$. To show that $C := f^{-1}(B) \in \mathscr{A}_\mu$ observe that $^*C = \cup\{{}^*A : A \in \mathscr{A}, f(A) \subset B\} \cup \{x \in {}^*X : {}^*f(x) \in {}^*B$ and $A \in \mathscr{A}$, $x \in A \Rightarrow f(A) \not\subset B\}$.
The first of these sets is $^*\mu_L$-measurable by Lemma 5, and the second is $^*\mu_L$-measurable by (ii). Hence $^*C$ is $^*\mu_L$-measurable. Therefore for each fixed $n \in \mathbb{N}$ there exist D, $E \in {}^*\mathscr{A}$ with $D \subset {}^*C \subset E$ and $^*\mu(E - D) < \frac{1}{n}$. Using the transfer principle we obtain $C \in \mathscr{A}_\mu$.

Let $\mu|\mathscr{A}$ be a complete measure, i.e. $\mathscr{A} = \mathscr{A}_\mu$. Then Corollary 2 and Theorem 3 yield the following surprising fact: If $^*f$ is pointwise $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable $^*\mu_L$-a.e., then it is pointwise $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable everywhere. Theorem 3 shows in particular that the set of all $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurability points of $^*f$ is $^*\mu_L$-measurable for $\mathscr{A}_\mu, \mathscr{B}$-measurable functions f. It turns out, however (see Theorem 4), that we neither need any assumptions on f nor on the underlying Loeb-measure $^*\mu_L$; the set of measurability points is ν_L-measurable for each function f and each internal content ν.

4. THEOREM. *Let $\mathscr{A} \subset \mathscr{P}(X)$ be a σ-algebra and $\mathscr{B} \subset \mathscr{P}(Y)$ with $\bar{B} \in \mathscr{B}$ for all $B \in \mathscr{B}$. For each function $f: X \to Y$ the set of all $x \in {}^*X$, such that $^*f$ is $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in x, is universally Loeb-measurable.*

PROOF. Let F be the set of all points $x \in {}^*X$ such that $^*f$ is $^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in x. Then $F = \cap_{B \in \mathscr{B}} F_B$ with $F_B = \{x \in {}^*X : {}^*f(x) \in {}^*B \Rightarrow f(A) \subset B$ for some $A \in \mathscr{A}$ with $x \in {}^*A\}$. As $\bar{B} \in \mathscr{B}$ if $B \in \mathscr{B}$, we obtain that

$$F = \cap_{B \in \mathscr{B}} (F_B \cap F_{\bar{B}}). \tag{1}$$

By definition of F_B we have

$$F_B \cap F_{\bar{B}} = \cup\{{}^*A : A \in \mathscr{A}, f(A) \subset B\} \cup \cup\{{}^*A : A \in \mathscr{A}, f(A) \subset \bar{B}\}. \tag{2}$$

Now let $\nu|{}^*\mathscr{A}$ be a finite internal content. For each $C \subset Y$ choose $A_n \in \mathscr{A}$ with $f(A_n) \subset C$ and $\nu_L({}^*A_n) \uparrow \sup\{\nu_L({}^*A) : A \in \mathscr{A}, f(A) \subset C\}$; put $A_C = \cup_{n \in \mathbb{N}} A_n$. Then

$A_C \in \mathscr{A}$, $f(A_C) \subset C$ and

(3) $$\nu_L({}^*A - {}^*A_C) = 0 \text{ for each } A \in \mathscr{A} \text{ with } f(A) \subset C.$$

Put $\mathscr{S}_B := \{A \in \mathscr{A} : f(A) \subset B \text{ or } f(A) \subset \bar{B}\}$. By (1) and (2) we have

$$F - \cap_{B\in\mathscr{B}}({}^*A_B \cup {}^*A_{\bar{B}}) \subset \cup_{B\in\mathscr{B}} \cup_{A\in\mathscr{S}_B} {}^*(A - A_B \cup A_{\bar{B}}) =: N.$$

By (3) we obtain $\nu_L({}^*(A - A_B \cup A_{\bar{B}})) = 0$ for each $B \in \mathscr{B}$, $A \in \mathscr{S}_B$ and hence $\nu_L(N) = 0$ by Lemma 5. Put $D := \cap_{B\in\mathscr{B}}({}^*A_B \cup {}^*A_{\bar{B}})$. Then D is ν_L-measurable by Lemma 5. As $F - D \subset N$ and $\nu_L(N) = 0$, $F - D$ is ν-measurable, too. Hence $F = D \cup (F - D)$ is ν_L-measurable.

If $\mathscr{B}$ has not the property, that $\bar{B} \in \mathscr{B}$ for all $B \in \mathscr{B}$, then the assertion of the preceeding theorem need not be true any more. Let e.g. $X = Y = \mathsf{R}$, $\mathscr{A} = \{\emptyset, \mathsf{R}\}$ and $\mathscr{B} = \{B\}$ with $B \subset \mathsf{R}$, $B \notin \mathscr{A}$. If $f(x) = x$, $x \in \mathsf{R}$, then $\overline{{}^*B}$ is the set of all $x \in {}^*\mathsf{R}$ such that ${}^*f$ is ${}^\sigma\mathscr{A}, {}^\sigma\mathscr{B}$-measurable in x. However, $\overline{{}^*B} \notin L({}^*\mathscr{A}, {}^*\mu) = \{\emptyset, {}^*\mathsf{R}\}$ for each measure $\mu|\mathscr{A}$ with $0 < \mu(\mathsf{R}) < \infty$.

For the sake of completeness we cite a special case of Theorem 1 of [3].

5. Lemma. *Let $\mathscr{A}$ be an algebra on X and $\mathscr{S} \subset \mathscr{A}$ be a subsystem. Then the following assertions hold:*

(i) $\bigcup_{A\in\mathscr{S}} {}^*A$, $\bigcap_{A\in\mathscr{S}} {}^*A$ *are universally Loeb-measurable;*

(ii) $\nu_L({}^*A) = 0$ *for* $A \in \mathscr{S} \Rightarrow \nu_L\left(\bigcup_{A\in\mathscr{S}} {}^*A\right) = 0$.

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MATHEMATISCHES INSTITUT
DER UNIVERSITÄT ZU KÖLN
WEYERTAL 86-90
D-5000 KÖLN 41
GERMANY

FACHBEREICH MATHEMATIK
DER UNIVERSITÄT-GH-DUISBURG
LOTHARSTR. 65
D-4100 DUISBURG
GERMANY