MATH. SCAND. 67 (1990), 273–289.

THE TRANSMISSION PROPERTY

GERD GRUBB and LARS HÖRMANDER

Introduction.

A pseudo-differential operator P in a C^∞ manifold X is said to have *the transmission property* with respect to an open subset Y with C^∞ boundary, if

$$P_Y u = r_Y P e_Y u$$

has a C^∞ extension to $\bar{Y}$ whenever $u \in r_Y C_0^\infty(X)$. (Here r_Y denotes restriction to Y; and e_Y denotes extension by 0 on $X \setminus Y$, i.e., $e_Y u = u$ on Y and $e_Y u = 0$ on $X \setminus Y$ when u is a function on Y.)

For the case where P has a polyhomogeneous (classical) symbol, Boutet de Monvel [1] has given a necessary and sufficient condition on the symbol of P for the transmission property to hold. An extended version is proved in Hörmander [7, Th. 18.2.18]. Related conditions are studied in works of Višik and Èskin, cf. e.g. Èskin [4]. The purpose of this note is to study the transmission property for more general classes of pseudo-differential operators, with symbols in $S^m_{\rho,\delta}$ (see [7, p. 94] for a definition).

For symbols in $S^m_{1,0}$ a condition was given in Boutet de Monvel [2, (2.2)] which assures that P has the transmission property with respect to both Y and $X \setminus \bar{Y}$ (or, equivalently: P as well as its adjoint P^* have the transmission property with respect to Y; cf. Corollary 1.8 below). We shall introduce a weaker condition below in Section 1, that we show is necessary and sufficient for the transmission property with respect to Y alone, for general symbols in $S^m_{\rho,\delta}(\mathsf{R}^n \times \mathsf{R}^n)$ with $0 \leqq \delta < \varrho \leqq 1$ when Y is a halfspace in R^n. A characterization of operators with the two-sided transmission property is obtained as a corollary.

In Section 2, we determine the mapping properties of P_Y in Sobolev spaces $\bar{H}_{(s)}(Y)$, when P has the transmission property. To get global estimates over $Y = \mathsf{R}^n_+$ we assume a certain uniformity in the condition for the transmission property (this covers local estimates also). It is found that the customary continuity of P from $H^{\mathrm{comp}}_{(s)}(X)$ to $H^{\mathrm{loc}}_{(s-m)}(X)$ only generalizes to P_Y with a loss of $(1-\varrho)(s-\frac{1}{2})$ derivatives for $s > \frac{1}{2}$ (see Theorem 2.5). This stems from a related

Received December 8, 1989

property (Theorem 2.4) of the associated Poisson operator $K_p^+ : v \mapsto r_Y P(v(x') \otimes \delta(x_n))$, that we study in detail.

1. C^∞ mapping properties.

We shall denote $r_Y C_0^\infty(X)$ by $C_{(0)}^\infty(\bar{Y})$ and $r_Y C^\infty(X)$ by $C^\infty(\bar{Y})$. Since the question of whether $P_Y u$ has a C^∞ extension to $\bar{Y}$, whenever $u \in C_{(0)}^\infty(\bar{Y})$, is local and invariant under diffeomorphisms, it is clear that results for general manifolds will follow from results for the case

$$X = \mathbb{R}^n,\ Y = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}.$$

(When $X = \mathbb{R}^n$, $Y = \mathbb{R}^n_\pm$, $\mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x_n < 0\}$, we usually write $r^\pm$ and $e^\pm$ instead of $r_{\mathbb{R}^n_\pm}$ and $e_{\mathbb{R}^n_\pm}$, respectively; and we write P_+ for $P_{\mathbb{R}^n_+}$.) We can then consider an arbitrary $P = p(x, D) = \operatorname{Op}(p(x, \xi))$ with $p(x, \xi) \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for some $0 \leqq \delta < \varrho \leqq 1$ (although this class is not invariant under a change of variables unless in addition $\delta \geqq 1 - \varrho$). Recall that the condition $p(x, \xi) \in S^m_{\rho,\delta}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ means that with $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$

$$|p^{(\alpha)}_{(\beta)}(x, \xi)| \leqq C_{\alpha,\beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|} \text{ for all } \alpha \in \mathbb{N}^{n_2}, \beta \in \mathbb{N}^{n_1},$$

where $p^{(\alpha)}_{(\beta)}(x, \xi) = \partial^\alpha_\xi \partial^\beta_x p(x, \xi)$, and recall that for $p \in S^m_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ and $u \in \mathscr{S}(\mathbb{R}^n)$,

$$p(x, D)u = (2\pi)^{-n} \iint e^{i(x-y)\cdot\xi} p(x, \xi) u(y)\, dy d\xi \tag{1.1}$$

(further details are given in [7, Section 18.1]). $S^{-\infty}_{\rho,\delta} = \cap_m S^m_{\rho,\delta} = S^{-\infty}_{1,0}$ is also called $S^{-\infty}$. It is well known that for each (x, ξ'), where $\xi' = (\xi_1, \ldots, \xi_{n-1})$, the distribution in z_n

$$\tilde{p}(x, z_n, \xi') = \frac{1}{2\pi} \int e^{iz_n\xi_n} p(x, \xi)\, d\xi_n = \mathscr{F}^{-1}_{\xi_n \to z_n} p(x, \xi) \tag{1.2}$$

is C^∞ for $z_n \in \mathbb{R} \setminus \{0\}$ and rapidly decreasing for $z_n \to \pm\infty$; for

$$z_n^k D^j_{z_n} D^\beta_x D^{\alpha'}_{\xi'} \tilde{p} = \mathscr{F}^{-1}_{\xi_n \to z_n} (-D_{\xi_n})^k \xi_n^j D^\beta_x D^{\alpha'}_{\xi'} p$$

is a bounded continuous function when $\varrho k \geqq j + m + \delta|\beta| - \varrho|\alpha'| + 2$, since $\partial^k_{\xi_n} \xi_n^j \partial^\beta_x \partial^{\alpha'}_{\xi'} p$ has an integrable majorant then. The sufficient condition of [2] for the transmission property in case $\varrho = 1$, $\delta = 0$ amounts primarily to requiring that $\tilde{p}(x', 0, z_n, \xi')$ is C^∞ for $z_n \to +0$ and for $z_n \to -0$. (The formulation used in [2] and [5] is more complicated but was reduced to this in Grubb [6].) However, as observed already in [1, Remarque (2.3.3)], [2, p. 23] and [4], the transmission property is only related to smoothness for $z_n \to +0$; the condition when $z_n \to -0$ was added in [2, 5, 6] to obtain a calculus closed under adjoints. We shall here make this precise for general $S^m_{\rho,\delta}$ symbols. To do so we introduce the following spaces of symbols on $\mathbb{R}^{n-1} \times \mathbb{R}$:

DEFINITION 1.1. For $m \in \mathsf{R}$, $\varrho, \delta \in [0,1]$, $S^m_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$ denotes the space of symbols $a(x', \xi_n) \in S^m_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R})$ such that $r^+\tilde{a}(x', x_n) = r^+\mathscr{F}^{-1}_{\xi_n \to x_n} a(x', \xi_n)$ extends to a C^∞ function of $(x', x_n) \in \bar{\mathsf{R}}^n_+$.

When $a(x', \xi_n) \in S^m_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$, then the expressions $x_n^N D_x^\beta r^+ \tilde{a}(x', x_n)$ will generally only be bounded locally in x'. This is of no importance in the discussion of the transmission property, which is local, but for the study of global Sobolev estimates in Section 2 it will be convenient to have a uniform version of the estimates, and we shall then use the following symbol space:

DEFINITION 1.2. For $m \in \mathsf{R}$, $\varrho, \delta, \in [0,1]$, $S^m_{\rho,\delta,\mathrm{utr}}(\mathsf{R}^{n-1} \times \mathsf{R})$ denotes the space of symbols $a(x', \xi_n) \in S^m_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R})$ such that $x_n^N D_x^\beta r^+ \tilde{a}(x', x_n)$ is bounded on R^n_+ for all $N \in \mathsf{N}$, $\beta \in \mathsf{N}^n$, hence $a(x', \xi_n) \in S^m_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$.

An example of a symbol in $S^1_{1,0,\mathrm{tr}} \backslash S^1_{1,0,\mathrm{utr}}$ is $a(x', \xi_n) = \langle \xi_n \rangle \chi(\xi_n / \langle x' \rangle)$ where $\chi \in C_0^\infty(\mathsf{R})$ and $\chi(0) \neq 0$.

The definitions are elucidated by the following:

LEMMA 1.3. *$a \in S^m_{\rho,\delta,\mathrm{utr}}$ if and only if a has a decomposition $a = a_0 + a_1$ where $a_0 \in S^{-\infty}(\mathsf{R}^{n-1} \times \mathsf{R})$ and $a_1 \in S^m_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R})$ with* supp $\tilde{a}_1 \subset \bar{\mathsf{R}}^n_-$. *The map $a \mapsto (i\xi_n - 1)a$ is an isomorphism $S^m_{\rho,\delta,\mathrm{utr}}(\mathsf{R}^{n-1} \times \mathsf{R}) \to S^{m+1}_{\rho,\delta,\mathrm{utr}}(\mathsf{R}^{n-1} \times \mathsf{R})$ for every m, and this is also true if* utr *is replaced by* tr.

PROOF. If we define $\tilde{a}_0$ as a Seeley extension (see [8]) of $r^+\tilde{a}$ to R^n, the first statement is obvious and the decomposition is in fact provided by linear operators. It is clear that if $b(x', \xi_n) = (i\xi_n - 1)a(x', \xi_n)$, then $\tilde{b}(x) = (\partial_n - 1)\tilde{a}(x)$, so $a \in S^m_{\rho,\delta,\mathrm{utr}}(\mathsf{R}^{n-1} \times \mathsf{R})$ implies $b \in S^{m+1}_{\rho,\delta,\mathrm{utr}}(\mathsf{R}^{n-1} \times \mathsf{R})$. On the other hand, if $b \in S^{m+1}_{\rho,\delta,\mathrm{utr}}(\mathsf{R}^{n-1} \times \mathsf{R})$, $\mathrm{supp}\tilde{b} \subset \bar{\mathsf{R}}^n_-$, and a is defined by $a(x', \xi_n) = (i\xi_n - 1)^{-1} b(x', \xi_n)$, then $\mathrm{supp}\tilde{a} \subset \bar{\mathsf{R}}^n_-$ since $\tilde{a}$ is small when $x_n \to \infty$ and $(\partial_n - 1)\tilde{a} = 0$ in R^n_+. This proves the second statement, and the variant for the non-uniform spaces is an immediate consequence.

We shall also need that the symbol spaces $S^m_{\rho,\delta,\mathrm{tr}}$ as well as their uniform versions admit asymptotic sums as usual:

LEMMA 1.4. *Let $0 < \varrho \leqq 1, 0 \leqq \delta \leqq 1$, and let $b_j \in S^{\mu_j}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$, $\mu_j \to -\infty$ as $j \to \infty$. If $b \sim \sum b_j$ in $S^m_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R})$ for some m, it follows that $b \in S^{\max \mu_j}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$.*

PROOF. It is no restriction to assume that μ_j is decreasing, and that $\mu_j < -2 - j$. This implies that each distribution $\tilde{b}_j = \mathscr{F}^{-1}_{\xi_n \to x_n} b_j$ belongs to C^j; cutting b_j off sufficiently far away as in the proof of [7, Prop. 18.1.3] we can make sure that

$$|D^\alpha \tilde{b}_j| \leqq 2^{-j}, \quad |\alpha| \leqq j,$$

and that $b = \sum b_j$ exists and is an asymptotic sum in the sense defined there. Since $r^+\tilde{b}_j$ extends to a C^∞ function on $\bar{\mathsf{R}}^n_+$ for every j, it follows that $\tilde{b} = \sum \tilde{b}_j = \mathscr{F}^{-1}_{\xi_n \to x_n} b$ also has this property.

If $u \in \mathscr{S}(\mathsf{R}^n)$ and $u_+ = r^+u$, then

$$(1.3)\quad \begin{aligned} a(x', \xi_n) = \mathscr{F}_{x_n \to \xi_n} e^+ u_+(x', x_n) \sim \sum_{j \in \mathsf{N}} a_j(x')(i\xi_n - 1)^{-j-1} \text{ in } S^{-1}_{1,0}(\mathsf{R}^{n-1} \times \mathsf{R}), \\ \text{with } a_j(x') = \partial_n^j(u(x)e^{-x_n})|_{x_n=0}; \end{aligned}$$

as is seen by integration by parts in $a(x', \xi_n) = \int_0^\infty e^{-ix_n\xi_n + x_n}(e^{-x_n}u(x', x_n))dx_n$, using that $e^{-ix_n\xi_n + x_n} = (1 - i\xi_n)^{-1}\partial_{x_n} e^{-ix_n\xi_n + x_n}$. (We could of course take an expansion in powers of $\lambda\xi_n - 1$ for any $\lambda \in \mathsf{C}\setminus\mathsf{R}$, but $\lambda = i$ will allow us to use Lemma 1.3 later on.) It is shown in [7, (18.2.16)] that the partial Fourier transform $b(x', \xi_n) = \mathscr{F}_{x_n \to \xi_n} Pe^+u_+$ of Pe^+u_+ is a symbol in $S^{m-1}_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R})$ satisfying

$$(1.4)\quad b(x', \xi_n) \sim \sum_{j \in \mathsf{N}} (\langle iD_{y'}, D_{\xi'}\rangle - iD_{x_n}D_{\xi_n})^j p(x, \xi)a(y', \xi_n)/j!|_{y'=x', x_n=0, \xi'=0}.$$

The proof given in detail for $\varrho = 1, \delta = 0$, extends without difficulty to general ϱ, δ with $0 \leqq \delta < \varrho \leqq 1$. Note that the application of $\langle D_{y'}, D_{\xi'}\rangle$ to $p(x, \xi)a(y', \xi_n)$ in (1.4) lowers the degree by at least ϱ, while application of $D_{x_n}D_{\xi_n}$ lowers it by at least $\varrho - \delta$, so (1.4) defines a symbol in $S^{m-1}_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R})$ modulo $S^{-\infty}(\mathsf{R}^{n-1} \times \mathsf{R})$. By separating the terms in (1.4) we shall now prove a necessary condition for the transmission property:

THEOREM 1.5. *Let $0 \leqq \delta < \varrho \leqq 1$ and let $m \in \mathsf{R}$. If $p \in S^m_{\rho,\delta}(\mathsf{R}^n \times \mathsf{R}^n)$ and the asymptotic sum (1.4) is in $S^{m-1}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$ for every a given by (1.3) with $u \in C_0^\infty(\mathsf{R}^n)$, it follows that for all α and $\beta \in \mathsf{N}^n$,*

$$(1.5)\quad p^{(\alpha)}_{(\beta)}(x', 0, 0, \xi_n) \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R}).$$

PROOF. Throughout the proof, $S^m_{\rho,\delta,\mathrm{tr}}$ stands for $S^m_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$ and $S^m_{\rho,\delta}$ stands for $S^m_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R})$. If $v \in C_0^\infty(\mathsf{R}^{n-1})$, k is a positive integer, and we choose $u \in C_0^\infty(\mathsf{R}^n)$ such that $u(x) = v(x')x_n^{k-1}e^{x_n}/(k-1)!$ when $|x_n| < 1$, then the function a in (1.3) is $\sim v(x')(i\xi_n - 1)^{-k}$. The symbol (1.4) is in $S^{m-k}_{\rho,\delta}$ by the calculus, and it is in $S^{m-1}_{\rho,\delta,\mathrm{tr}}$ by hypothesis, hence it is in $S^{m-k}_{\rho,\delta,\mathrm{tr}}$. Since we can take $v = 1$ on any compact set, and Definition 1.1 is a local condition in the x' variables, we may take $v = 1$. The product by $(i\xi_n - 1)^k$ is in $S^m_{\rho,\delta,\mathrm{tr}}$, that is,

$$(i\xi_n - 1)^k \sum_{j \in \mathsf{N}} (-iD_{x_n}D_{\xi_n})^j(p(x, \xi)(i\xi_n - 1)^{-k})/j!|_{x_n=0, \xi'=0} \in S^m_{\rho,\delta,\mathrm{tr}}.$$

(The sum, as all the following ones, is of course an asymptotic one.) Since iD_{x_n}

only acts on p, we can write this condition in the form

$$\sum_j \sum_{0 \leqq \nu \leqq j} (iD_{x_n})^j(-D_{\xi_n})^{j-\nu} p(x', 0, 0, \xi_n) k \ldots \tag{1.6}$$
$$(k + \nu - 1)(i\xi_n - 1)^{-\nu}/\nu!(j - \nu)! \in S^m_{\rho,\delta,\mathrm{tr}}$$

This holds for any positive integer k. Note that each term in (1.6) is a polynomial in k of degree $\leqq \nu \leqq j$ and lies in $S^{m-j(\rho-\delta)}_{\rho,\delta}$.

We claim that (1.6) is true for any value of k if it is true for infinitely many. To prove this, we fix a large integer N, and choose N distinct values $k_1, \ldots, k_N$ of k for which (1.6) is known to hold. By the Lagrange interpolation formula one has for all polynomials q of degree $< N$ and all $\kappa \in \mathbf{C}$:

$$q(\kappa) = \sum_{i=1}^{N} q(k_i) L_i(\kappa), \quad L_i(\kappa) = \prod_{1 \leqq j \leqq N, j \neq i} \frac{\kappa - k_j}{k_i - k_j}.$$

For fixed κ we consider the linear combination in $S^m_{\rho,\delta,\mathrm{tr}}$ obtained by taking $k = k_i$ in (1.6), multiplying by $L_i(\kappa)$ and adding for $i = 1, \ldots, N$. The terms with $j < N$ give the corresponding expressions with k replaced by κ, and the terms with $j \geqq N$ are in $S^{m-(\rho-\delta)N}_{\rho,\delta}$, so (1.6) is valid, modulo $S^{m-(\rho-\delta)N}_{\rho,\delta}$, with k replaced by κ. Letting $N \to \infty$, we conclude from Lemma 1.4 that (1.6) is valid with k replaced by any κ. In particular, we have for $\kappa = 0$:

$$\sum_{j=0}^{\infty} (-iD_{x_n} D_{\xi_n})^j p(x', 0, 0, \xi_n)/j! \in S^m_{\rho,\delta,\mathrm{tr}}. \tag{1.7}$$

If a pseudo-differential operator P in a C^∞ manifold X has the transmission property with respect to an open subset Y with C^∞ boundary, then any product $Q_1 P Q_2$ with differential operators Q_1 and Q_2 has the weaker transmission property that $r_Y Q_1 P Q_2 e_Y u$ has a C^∞ extension to $\bar{Y}$ whenever $u \in C^\infty_{(0)}(\bar{Y})$ and u vanishes of order m_2 on ∂Y, where m_2 is the order of Q_2; for $Q_2 e_Y u = e_Y Q_2 u$ then. Since

$$p_{(j)}(x, D) = [iD_j, p(x, D)] \quad \text{and} \quad p^{(j)}(x, D) = [p(x, D), ix_j], \tag{1.8}$$

if follows that $p^{(\alpha)}_{(\beta)}(x, D)$ for arbitrary α and β has this weakened transmission property, for functions vanishing of order β_n when $x_n = 0$. The functions u used in the first part of the proof vanish of order $k - 1$ at $x_n = 0$. Hence (1.6) is valid with p replaced by $p^{(\alpha)}_{(\beta)}$ and m replaced by $m - \varrho|\alpha| + \delta|\beta|$ if $k \geq \beta_n + 1$, so (1.7) is also valid with these substitutions for p and m. If we apply (1.7) to $(-iD_{x_n} D_{\xi_n})^k p^{(\alpha)}_{(\beta)}(x, \xi)$ it follows that

$$\sum_{j=0}^{\infty} (-iD_{x_n} D_{\xi_n})^{j+k} p^{(\alpha)}_{(\beta)}(x', 0, 0, \xi_n)/j! \in S^{m+k(\delta-\rho)-\rho|\alpha|+\delta|\beta|}_{\rho,\delta,\mathrm{tr}}.$$

We multiply by $(-1)^k/k!$ and sum for $k = 0, 1, \ldots$, noting that

$$\sum_{k+j=l} (-1)^k/k!j! = (1-1)^l/l! = 0 \quad \text{if} \quad l > 0.$$

In view of Lemma 1.4 again it follows that $p^{(\alpha)}_{(\beta)}(x', 0, 0, \xi_n) \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta,\mathrm{tr}}$.

THEOREM 1.6. *Let* $0 \leqq \delta \leqq 1$ *and let* $m \in \mathsf{R}$. *The following conditions on* $p \in S^m_{\rho,\delta}(\mathsf{R}^n \times \mathsf{R}^n)$ *are equivalent:*

(i) $p(x, D)$ *has the transmission property with respect to* R^n_+.

(ii) (1.5) *is valid for all* $\alpha, \beta \in \mathsf{N}^n$.

(iii) (1.5) *is valid when* $\beta' = 0$, $\alpha_n = 0$.

(iv) *For all* $\alpha, \beta \in \mathsf{N}^n$,

$$r^+ \mathscr{F}^{-1}_{\xi_n \to z_n} p^{(\alpha)}_{(\beta)}(x', 0, 0, \xi_n) \in C^\infty(\bar{\mathsf{R}}^n_+). \tag{1.9}$$

(v) (1.9) *is valid when* $\beta' = 0$, $\alpha_n = 0$.

(vi) *For every* $j \in \mathsf{N}$, $\alpha' \in \mathsf{N}^{n-1}$, *and every fixed* $\xi' \in \mathsf{R}^{n-1}$,

$$(x', \xi_n) \mapsto \partial^j_{x_n} \partial^{\alpha'}_{\xi'} p(x', 0, \xi', \xi_n) \in S^{m+\delta j-\rho|\alpha'|}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R}). \tag{1.10}$$

PROOF. We have proved in Theorem 1.5 that (i) implies (ii). (ii) and (iii) are equivalent since $\partial^{\beta'}_{x'} \partial^j_{\xi_n} S^\mu_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R}) \subset S^{\mu-\rho j+\delta|\beta'|}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$ in view of Definition 1.1, and (iv) and (v) are just reformulations of (ii) and (iii). To see that (ii) implies (i), let $u_+ \in C^\infty_{(0)}(\bar{\mathsf{R}}^n_+)$ and insert a from (1.3) into (1.4). This gives

$$\mathscr{F}_{x_n \to \xi_n} P e^+ u_+ \sim \sum_{j,k \in \mathsf{N}} (\langle iD_{y'}, D_{\xi'} \rangle - iD_{x_n} D_{\xi_n})^j p(x', 0, 0, \xi_n) a_k(y')(i\xi_n - 1)^{-k-1}/j!|_{y'=x'},$$

where Lemma 1.3 and (iii) show that each term is in $S^{m-1}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$. The order goes to $-\infty$ so that by Lemma 1.4 the asymptotic series defines a symbol is $S^{m-1}_{\rho,\delta,\mathrm{tr}}(\mathsf{R}^{n-1} \times \mathsf{R})$. Then $P_+ u_+ = r^+ P e^+ u_+$ is in $C^\infty(\bar{\mathsf{R}}^n_+)$ in view of Definition 1.1.

We shall finally show that ξ' can be taken $\neq 0$ in (1.10). By Taylor's formula, the function in (1.10) is for fixed ξ' an asymptotic sum,

$$\partial^j_{x_n} \partial^{\alpha'}_{\xi'} p(x', 0, \xi', \xi_n) \sim \sum_{\beta' \in \mathsf{N}^{n-1}} \partial^j_{x_n} \partial^{\alpha'}_{\xi'} \partial^{\beta'}_{\xi'} p(x', 0, 0, \xi_n) \xi'^{\beta'}/\beta'!, \tag{1.11}$$

where $\partial^j_{x_n} \partial^{\alpha'}_{\xi'} \partial^{\beta'}_{\xi'} p(x', 0, 0, \xi_n) \in S^{m-\rho|\alpha'+\beta'|+\delta j}(\mathsf{R}^{n-1} \times \mathsf{R})$. Then it follows from Lemma 1.4 that (ii) implies (vi). When $\xi' = 0$ the condition (1.10) is the condition (iii), which completes the proof.

We shall say that $p(x, \xi)$ has the transmission property with respect to R^n_+ when the conditions in Theorem 1.6 are satisfied. In preparation for Section 2 we also introduce (cf. Definition 1.2):

DEFINITION 1.7. $p(x,D)$ (as well as $p(x,\xi)$) is said to have the uniform transmission property with respect to R^n_+, if the conditions in Theorem 1.6 are satisfied with $S_{\rho,\delta,\mathrm{tr}}$ replaced by $S_{\rho,\delta,\mathrm{utr}}$.

An example of a symbol with the transmission property which lacks the uniform transmission property is $a(x,\xi) = \langle\xi\rangle\chi(\langle\xi\rangle/\langle x\rangle)$ where $\chi\in C_0^\infty(\mathrm{R})$ and $\chi(0)\neq 0$.

If $p(x,\xi)$ vanishes for x' outside a compact subset of R^{n-1}, the uniform transmission property is of course equivalent to the transmission property, so estimates over compact sets for operators having the transmission property follow from estimates for operators with the uniform transmission property.

Theorem 1.6 has an easy consequence concerning two-sided transmission:

COROLLARY 1.8. *The following conditions on $p\in S^m_{\rho,\delta}(\mathrm{R}^n\times\mathrm{R}^n)$ are equivalent:*
(i) *$p(x,D)$ has the transmission property with respect to R^n_+ and R^n_-.*
(ii) *$p(x,\xi)$ and $p(x,-\xi)$ satisfy the conditions in Theorem 1.6.*
(iii) *$p(x,\xi)$ and $\bar{p}(x,\xi)$ satisfy the conditions in Theorem 1.6.*
(iv) *$p(x,D)$ and its adjoint have the transmission property with respect to R^n_+.*
(v) *For every $j\in\mathrm{N}$ and $\alpha'\in\mathrm{N}^{n-1}$,*

$$r^\pm\mathscr{F}^{-1}_{\xi_n\to z_n}\partial^j_{x_n}\partial^{\alpha'}_{\xi'}p(x',0,0,\xi_n)\in C^\infty(\bar{\mathrm{R}}^n_\pm). \tag{1.12}$$

PROOF. Clearly, $\mathrm{Op}(p(x,\xi))$ has the transmission property with respect to R^n_- if and only if $U\,\mathrm{Op}(p(x,\xi))U$ has the transmission property with respect to R^n_+, where U is the mapping

$$U: u(x)\mapsto u(-x);$$

and here $U\,\mathrm{Op}(p(x,\xi))U = \mathrm{Op}(p(-x,-\xi))$, cf. (1.1). Note also that if $\mathscr{F}^{-1}_{\xi_n\to x_n}f(x',\xi_n) = g(x',x_n)$, then $\mathscr{F}^{-1}_{\xi_n\to x_n}f(x',-\xi_n) = g(x',-x_n)$. Then the equivalence of (i), (ii) and (v) follows from the equivalence of (i), (iii) and (v) in Theorem 1.6, in view of the fact that (1.5) takes place at $x_n = 0$. Now (iii) is included in the equivalences by the observation that (cf. (1.1))

$$\overline{\mathrm{Op}(p(x,-\xi))u} = \mathrm{Op}(\bar{p}(x,\xi))\bar{u};$$

and (iv) is included in view of the usual formula for the symbol of the adjoint of a pseudodifferential operator (cf. [7, (18.1.11)]), together with Lemma 1.4.

Also in this corollary, the conditions on the set of $\partial^j_{x_n}\partial^{\alpha'}_{\xi'}$-derivatives can be replaced by conditions on the full set of $\partial^\beta_x\partial^\alpha_\xi$-derivatives. In particular, the conditions (i)–(v) are equivalent to the $\mathscr{H}$-condition in [6], stating that $p^{(\alpha)}_{(\beta)}(x',0,\xi',\xi_n)$ is in $\mathscr{H}$ as a function of ξ_n, locally uniformly with respect to x' for all α, β and ξ'. Here $\mathscr{H}$ is the space of symbols $a(\xi_n)$ having an asymptotic expansion $a(\xi_n)\sim\sum_{j\in\mathrm{Z},j\leq d}a_j\xi_n^j$ as $|\xi_n|\to\infty$, for some $d\in\mathrm{Z}$; this is equivalent to $r^\pm\tilde{a}\in r^\pm\mathscr{S}(\mathrm{R})$.

2. Sobolev space estimates and Poisson operators.

For any $p \in S^m_{\rho,\delta}(\mathsf{R}^n \times \mathsf{R}^n)$ the operator

$$u \mapsto r^+ p(x,D)e^+ u = p(x,D)_{\mathsf{R}^n_+} u = p(x,D)_+ u \tag{2.1}$$

is continuous from $\bar{H}_{(s)}(\mathsf{R}^n_+)$ to $\bar{H}_{(s-m)}(\mathsf{R}^n_+)$ if $-\frac{1}{2} < s < \frac{1}{2}$. This follows by combination of the well known facts that $u \mapsto e^+ u$ is continuous from $\bar{H}_{(s)}(\mathsf{R}^n_+)$ to $H_{(s)}(\mathsf{R}^n)$ when $-\frac{1}{2} < s < \frac{1}{2}$ and that $p(x,D)$ is continuous from $H_{(s)}(\mathsf{R}^n)$ to $H_{(s-m)}(\mathsf{R}^n)$ for all $s \in \mathsf{R}$. For symbols in $S^m_{1,0}$ satisfying the transmission condition of [2], the conclusion has been extended to arbitrary $s \geqq \frac{1}{2}$. We shall show here that the result extends to $S^m_{\rho,\delta}$ symbols having the uniform transmission property, with a certain loss of differentiability depending on ϱ. Let us begin with some preliminary remarks.

The first step is to lift the information from the case $-\frac{1}{2} < s < \frac{1}{2}$ to slightly larger values of s, by estimating $D_j p(x,D)_+ u$ for $j = 1, \ldots, n$. Observing that

$$D_n e^+ u = e^+ D_n u - i\gamma_0 u(x') \otimes \delta(x_n), \quad \text{where } \gamma_0 u = u|_{x_n=0}, \tag{2.2}$$

and defining

$$K_p^+ v = r^+ K_p v, \quad K_p v = p(x,D)(v(x') \otimes \delta(x_n)), \tag{2.3}$$

we have (cf. (1.8))

$$\begin{aligned} D_j r^+ p(x,D)e^+ u &= r^+([D_j, p(x,D)]e^+ u + p(x,D)e^+ D_j u - i\delta_{jn} p(x,D)(\gamma_0 u \otimes \delta)) \\ &= -ip_{(j)}(x,D)_+ u + p(x,D)_+ D_j u - i\delta_{jn} K_p^+ \gamma_0 u; \end{aligned} \tag{2.4}$$

and we shall discuss the continuity of the terms separately. Since $p_{(j)}$ is of order $m + \delta$ and p is of order m, the contributions from the first two terms are estimated by

$$\begin{aligned} \|p_{(j)}(x,D)_+ u\|_{(\frac{1}{2}-\varepsilon-\delta-m)} &\leqq C\|u\|_{(s)}, \quad \text{for } s \geqq \tfrac{1}{2}, \text{ all } \varepsilon > 0, \\ \|p(x,D)_+ D_j u\|_{(s-1-m)} &\leqq C\|u\|_{(s)}, \quad \text{for } \tfrac{1}{2} < s < \tfrac{3}{2}. \end{aligned} \tag{2.5}$$

The first estimate is even better than the second if $\frac{1}{2} < s < \frac{3}{2} - \delta$. The third term (for $j = n$) contains the trace operator $\gamma_0\colon \bar{H}_{(s)}(\mathsf{R}^n_+) \to H_{(s-\frac{1}{2})}(\mathsf{R}^{n-1})$ and the Poisson operator $-iK_p^+$, and it turns out that the continuity properties of the latter are decisive. They will now be studied.

We can write $p(x,D)$ in the form

$$p(x,D)u = (2\pi)^{-n} \int_{\mathsf{R}^{2n}} e^{i(x-y)\cdot\xi} q(x', y_n, \xi) u(y)\, dy d\xi,$$

where $q \in S^m_{\rho,\delta}$ is related to p by

$$q(x,\xi) \sim \sum_{j=0}^{\infty} (-iD_{x_n} D_{\xi_n})^j p(x,\xi)/j!,$$

(cf. [7, Theorem 18.5.10]); clearly q also has the uniform transmission property. Then, writing $\mathscr{F}^{-1}_{\xi_n \to x_n} q(x',0,\xi) = \tilde{q}(x',0,x_n,\xi')$ as usual (cf. (1.2)), we have for $v \in \mathscr{S}(\mathrm{R}^{n-1})$,

$$K_p^+ v(x',x_n) = (2\pi)^{-n} r^+ \int_{\mathrm{R}^{2n}} e^{i(x-y)\cdot\xi} q(x',y_n,\xi) v(y')\delta(y_n)\, dy d\xi$$

$$= (2\pi)^{-n} r^+ \int_{\mathrm{R}^n} e^{ix\cdot\xi} q(x',0,\xi)\hat{v}(\xi')\, d\xi \tag{2.6}$$

$$= (2\pi)^{1-n} \int_{\mathrm{R}^{n-1}} e^{ix'\cdot\xi'} r^+ \tilde{q}(x',0,x_n,\xi')\hat{v}(\xi')\, d\xi';$$

as in the case of $S^m_{1,0}$ symbols studied in [2, 5, 6]. We may therefore in the following restrict the attention to symbols $p(x',\xi)$ independent of x_n.

Recall that the uniform transmission property then means that $\partial^{\alpha'}_{\xi'} p(x',0,\xi_n)$ is the sum of a symbol is $S^{-\infty}(\mathrm{R}^{n-1} \times \mathrm{R})$ and one with inverse Fourier transform vanishing in R^n_+ (Lemma 1.3). To use this in the study of K_p^+, one considers Taylor expansions for large N,

$$p(x',\xi',\xi_n) = p_N(x',\xi',\xi_n) + r_N(x',\xi',\xi_n), \quad \text{with} \tag{2.7}$$

$$p_N(x',\xi',\xi_n) = \sum_{|\gamma'|<N} \partial^{\gamma'}_{\xi'} p(x',0,\xi_n)\xi'^{\gamma'}/\gamma'!, \tag{2.8}$$

$$r_N(x',\xi',\xi_n) = \sum_{|\gamma'|=N} \frac{N}{\gamma'!}\xi'^{\gamma'} \int_0^1 (1-h)^{N-1}\, \partial^{\gamma'}_{\xi'} p(x',h\xi',\xi_n) dh. \tag{2.9}$$

The terms in p_N are $O(\langle\xi_n\rangle^{m-\rho|\gamma'|}|\xi'|^{|\gamma'|})$ but in general no better, so the Taylor expansion is only useful where $|\xi'|$ is $O(\langle\xi_n\rangle^\rho)$. Using a cutoff function we shall essentially only have to consider the Taylor expansion in this set.

We begin by studying K_p^+ in the special case where

$$\partial^{\alpha'}_{\xi'} p(x',0,\xi_n) \in S^{-\infty}(\mathrm{R}^{n-1} \times \mathrm{R}), \quad \text{for all } \alpha' \in \mathrm{N}^{n-1}. \tag{2.10}$$

(2.10) is much stronger than (1.9) and we shall therefore be able to estimate the operator K_p without the restriction r^+. In the study of the continuity of K_p we can consider the operator $\mathscr{F}_{x_n \to \xi_n} K_p$ as a vector valued pseudo-differential operator on R^{n-1}, going from $C_0^\infty(\mathsf{R}^{n-1})$ to $C^\infty(\mathsf{R}^{n-1}; X)$ where $X = L^2(\mathsf{R})$, with symbol $k_p(x', \xi') = p(x', \xi', \xi_n) \in L^2_{\xi_n}(\mathsf{R})$. Sobolev space continuity properties of $S^m_{\rho,\delta}$ ps.d.o.s extend to the Hilbert space valued case (cf. [7, p. 169]): If $k \in C^\infty(\mathsf{R}^{n-1} \times \mathsf{R}^{n-1}; X)$, and X is a Hilbert space, then the estimates

$$\|k^{(\alpha')}_{(\beta')}(x', \xi')\|_X \leqq C_{\alpha',\beta'} \langle \xi' \rangle^{m-\rho|\alpha'|+\delta|\beta'|} \quad \text{for } \alpha', \beta' \in \mathsf{N}^{n-1}, \tag{2.11}$$

imply continuity of $k(x', D')$ from $H_{(m)}(\mathsf{R}^{n-1})$ to $H_{(0)}(\mathsf{R}^{n-1}; X)$. We recall that it suffices to have the estimates for $|\alpha'|$ and $|\beta'|$ in a finite set. For a recent account of precise results see Coifman and Meyer [3, p. 30], where it is shown that for $0 \leqq \delta \leqq \varrho \leqq 1$, $\delta < 1$, it suffices to have the estimates for $|\alpha'|, |\beta'| \leqq [\frac{n}{2}] + 1$.

PROPOSITION 2.1. *Let $m \in \mathsf{R}$ and $0 \leqq \delta < \varrho \leqq 1$. Let $p(x', \xi) \in S^m_{\rho,\delta}(\mathsf{R}^n \times \mathsf{R}^n)$ be independent of x_n, and assume that* (2.10) *holds. Then K_p defined by* (2.3) *is continuous:*

$$K_p : H_{(s)}(\mathsf{R}^{n-1}) \to H_{(t)}(\mathsf{R}^n), \tag{2.12}$$

where $t = \varrho s - m - \frac{1}{2}$ if $s > 0$, $t = s - m - \frac{1}{2}$ if $s < 0$. For $s = 0$, the continuity holds with $t = -m - \frac{1}{2}$ if $\varrho = 1$ (and with $t = -m - \frac{1}{2} - \varepsilon$, any $\varepsilon > 0$, if $\varrho < 1$). Hence $K_p^+ = r^+ K_p$ is continuous

$$K_p^+ : H_{(s)}(\mathsf{R}^{n-1}) \to \bar{H}_{(t)}(\mathsf{R}^n_+), \tag{2.13}$$

for the same values of s and t.

For each s, there exist M_1 and $M_2 \in \mathsf{N}$ such that the continuity statements above remain valid if instead of (2.10) *it is only assumed that $p^{(\alpha')}_{(\beta')}(x', 0, \xi_n)$ is $O(\langle \xi_n \rangle^{-M_1})$ for $|\alpha'|, |\beta'| \leqq M_2$.*

PROOF. We shall first consider the case $t = 0$, and use afterwards that $\langle D \rangle^t = \mathrm{Op}(\langle \xi \rangle^t)$ is a homeomorphism of $H_{(t)}$ onto L^2 for all $t \in \mathsf{R}$, and that $\langle D \rangle^t p(x', D)$ satisfies the hypotheses with m replaced by $m + t$.

If $\chi \in C_0^\infty(\mathsf{R})$, $\chi = 1$ in a neighborhood of 0, then for $\kappa \geqq 1$,

$$\chi(\xi_n / \langle \xi' \rangle^\kappa) \in S^0_{1/\kappa, 0}. \tag{2.14}$$

In fact, in the support of a derivative of order $\neq 0$ we have $|\xi_n| \sim \langle \xi' \rangle^\kappa$, so $\langle \xi \rangle \sim \langle \xi' \rangle^\kappa + \langle \xi' \rangle \sim \langle \xi' \rangle^\kappa \sim |\xi_n|$ there; and differentiation brings out a derivative of a factor $\xi_n / \langle \xi' \rangle^\kappa$ which gains at least a factor $\langle \xi' \rangle^{-1} \sim \langle \xi \rangle^{-1/\kappa}$.

We split p into a sum of three terms, $p = p_1 + p_2 + p_3$ where

$$\begin{aligned} &p_1 = \chi(\xi_n / \langle \xi' \rangle) p, \ p_2 = (1 - \chi(\xi_n / \langle \xi' \rangle^\kappa)) p, \\ &p_3 = (\chi(\xi_n / \langle \xi' \rangle^\kappa) - \chi(\xi_n / \langle \xi' \rangle)) p, \end{aligned} \tag{2.15}$$

taking $\kappa = 1/\varrho$; note that

$$\begin{gathered}|\xi_n| \leqq C_1\langle\xi'\rangle \text{ in supp } p_1, \quad |\xi_n| \geqq C_2\langle\xi'\rangle^\kappa \text{ in supp } p_2, \\ C_1'\langle\xi'\rangle \leqq |\xi_n| \leqq C_2'\langle\xi'\rangle^\kappa \text{ in supp } p_3\end{gathered} \tag{2.16}$$

For p_1 we have immediately

$$\left(\int |p_{1(\beta')}^{(\alpha')}(x',\xi)|^2\,d\xi_n\right)^{\frac12} \leqq \left(\int_{|\xi_n|\leqq C_1\langle\xi'\rangle} \langle\xi\rangle^{2(m-\rho|\alpha'|+\delta|\beta'|)}\,d\xi_n\right)^{\frac12}$$
$$\leqq C_{\alpha',\beta'}\langle\xi'\rangle^{m+\frac12-\rho|\alpha'|+\delta|\beta'|},$$

which implies that K_{p_1} is continuous from $H_{(m+\frac12)}(\mathsf{R}^{n-1})$ to $L^2(\mathsf{R}^n)$.

For p_2 we have by Taylor expansion up to order N, cf. (2.7),

$$|p_2(x',\xi',\xi_n)| = |p_{2,N}(x',\xi',\xi_n) + r_{2,N}(x',\xi',\xi_n)| \leqq C_N'\langle\xi'\rangle^N\langle\xi_n\rangle^{m-\rho N};$$

where we used that $\partial_{\xi'}^{\gamma'} p_2(x',0,\xi_n)$ is $O(\langle\xi_n\rangle^{m-\rho N})$ for any N, and that $\langle\xi\rangle \sim \langle\xi_n\rangle + \langle\xi'\rangle \sim \langle\xi_n\rangle + |\xi_n|^\rho \sim \langle\xi_n\rangle$ in supp p_2. Similarly, we have

$$|p_{2(\beta')}^{(\alpha')}(x',\xi)| \leqq C_{\alpha',\beta',N}\langle\xi'\rangle^N\langle\xi_n\rangle^{m-\rho|\alpha'|+\delta|\beta'|-\rho N}.$$

Taking N so large that the exponent of $\langle\xi_n\rangle$ is $< -\frac12$, i.e.,

$$N > (m+\tfrac12 - \varrho|\alpha'| + \delta|\beta'|)/\varrho, \tag{2.17}$$

we obtain in view of (2.16),

$$\left(\int |p_{2(\beta')}^{(\alpha')}(x',\xi)|^2 d\xi_n\right)^{\frac12} \leqq C_{\alpha',\beta'}'\langle\xi'\rangle^{\kappa(m+\frac12-\rho|\alpha'|+\delta|\beta'|)}$$
$$= C_{\alpha',\beta'}'\langle\xi'\rangle^{\kappa(m+\frac12)-|\alpha'|+\kappa\delta|\beta'|}.$$

Note that $0 \leqq \kappa\delta = \delta/\varrho < 1$. By the continuity properties of operators of type $1, \delta/\varrho$, we can conclude that K_{p_2} is continuous from $H_{(\kappa(m+\frac12))}(\mathsf{R}^{n-1})$ to $L^2(\mathsf{R}^n)$.

Consider finally p_3. For each fixed ξ_n, we may regard $v \mapsto p_3(x',D',\xi_n)v$ as a pseudodifferential operator in R^{n-1}. Since $|\xi_n| \sim \langle\xi\rangle$ in supp p_3 in view of (2.16), the symbol has the bounds

$$|p_{3(\beta')}^{(\alpha')}(x',\xi',\xi_n)| \leqq C_{\alpha',\beta'}''|\xi_n|^{m-\rho|\alpha'|+\delta|\beta'|}.$$

If U is the unitary dilation operator $(Uf)(x') = \lambda^{(n-1)/2}f(\lambda x')$, for some $\lambda > 0$, then

$$U^{-1}a(x',D')U = a(\lambda^{-1}x',\lambda D').$$

Thus $p_3(x',D',\xi_n)$ is unitarily equivalent to $p_4(x',D',\xi_n)$ defined by

$$p_4(x',\xi',\xi_n) = p_3(x'\langle\xi_n\rangle^{-\rho}, \xi'\langle\xi_n\rangle^{\rho}, \xi_n).$$

Since

$$|p^{(\alpha')}_{4(\beta')}(x',\xi',\xi_n)| \leqq C''_{\alpha',\beta'}|\xi_n|^{m+(\delta-\rho)|\beta'|} \leqq C''_{\alpha',\beta'}|\xi_n|^m, \tag{2.18}$$

the Calderón-Vaillancourt theorem on continuity of operators of type 0, 0 proves that the norm of $p_4(y', D', \xi_n)$ as an operator in $L^2(\mathsf{R}^{n-1})$ is $\leqq C|\xi_n|^m$; the unitarily equivalent operator $p_3(x', D', \xi_n)$ has the same norm. Thus

$$\int |p_3(x', D', \xi_n)v(x')|^2\, dx' \leqq C|\xi_n|^{2m} \int_{C_1'\langle\xi'\rangle \leqq |\xi_n| \leqq C_2'\langle\xi'\rangle^\kappa} |\hat v(\xi')|^2\, d\xi',$$

for we can replace $\hat v(\xi')$ by 0 in the set where the symbol vanishes. Hence

$$\begin{aligned}\|K_{p_3}v\|^2_{L^2(\mathsf{R}^n)} &= (2\pi)^{-1}\iint |p_3(x', D', \xi_n)v(x')|^2\, dx'\, d\xi_n \\ &\leqq C \iint_{C_1'\langle\xi'\rangle \leqq |\xi_n| \leqq C_2'\langle\xi'\rangle^\kappa} |\xi_n|^{2m}|\hat v(\xi')|^2\, d\xi'\, d\xi_n.\end{aligned} \tag{2.19}$$

If $m + \frac{1}{2} > 0$, the integral with respect to ξ_n can be estimated by $\langle\xi'\rangle^{\kappa(2m+1)}$, so the right-hand side is bounded by $C'\|v\|^2_{(\kappa(m+\frac{1}{2}))}$. If $m + \frac{1}{2} < 0$ we get instead a bound by $C'\|v\|^2_{(m+\frac{1}{2})}$.

Altogether, this shows that K_p is continuous from $H_{(s)}(\mathsf{R}^{n-1})$ to $L^2(\mathsf{R}^n)$ for $s = \kappa(m + \frac{1}{2}) > 0$ resp. for $s = m + \frac{1}{2} < 0$, proving the main statement of the proposition for $t = 0$. Applying this to $\langle D\rangle^t K_p$, we find that (2.12) holds when $s = \kappa(m + t + \frac{1}{2}) > 0$, hence $t = \varrho s - m - \frac{1}{2}$, resp. when $s = m + t + \frac{1}{2} < 0$ and hence $t = s - m - \frac{1}{2}$. This completes the proof of (2.12) for $s \neq 0$. The statement for $s = 0$ follows by interpolation if $\varrho = 1$ and is obvious otherwise. (2.13) follows immediately by restriction.

The last observation is seen by inspection of the proof (cf. (2.17)), using the remarks before the proposition.

It will be shown below in Example 2.6 that these estimates are generally best possible, at least when $s \neq 0$.

When studying general symbols $p(x', \xi) \in S^m_{\rho,\delta}$ satisfying the uniform transmission condition, we must cut off the terms in the Taylor expansion (2.8) without destroying the vital information on them given by the uniform transmission condition. To do so we shall construct a cutoff function with suitable properties in the next lemma.

LEMMA 2.2. 1° *The analytic function*

$$f(z) = \exp(-(1-z)^{-\frac{1}{4}}\exp(-(1+z)^{-\frac{1}{4}})), \quad |z| < 1, \tag{2.20}$$

where the square roots are taken in the right half plane, extends to a C^∞ *function in the closed unit disc such that* f *and* $1 - f$ *vanish of infinite order at* 1 *and at* -1 *respectively.*

2° *There is an analytic function* $\chi(t)$, $\operatorname{Im} t > 0$, *with a* C^∞ *extension to* $\{t \in \mathbf{C} \mid \operatorname{Im} t \geqq 0\}$, *such that* $\chi(t)$ *vanishes of infinite order when* $t \to 0$, *and* $\chi(t) = 1 + O(|t|^{-N})$, $\chi^{(j)}(t) = O(|t|^{-N})$, *for* $j > 0$ *and every* N *as* $|t| \to \infty$ *in* $\{t \in \mathbf{C} \mid \operatorname{Im} t \geqq 0\}$. χ *is the Fourier-Laplace transform of* $\delta + \psi$, *where* $\psi \in \mathscr{S}(\mathbf{R})$ *and* $\operatorname{supp} \psi \subset \bar{\mathbf{R}}_-$.

PROOF. We just have to examine f at ± 1. The function $g(w) = \exp(-1/\sqrt{w})$, $\operatorname{Re} w > 0$, extends to a C^∞ function in the right half plane vanishing of infinite order at 0, for $|\exp(-1/\sqrt{w})| \leqq \exp(-1/\sqrt{2|w|})$, and differentiation of g produces at most negative powers of w. This shows the statement at -1, since $(1-z)^{\frac{1}{2}}$ is close to $\sqrt{2}$ when z is close to -1. When z is close to 1 then $\exp(-(1+z)^{-\frac{1}{2}})$ is close to $\exp(-1/\sqrt{2}) > 0$ so this factor does not affect the infinite order vanishing at 1. This proves 1°.

To prove 2°, we can choose $\chi(t) = f((1+it)/(1-it))$, where f is given by (2.20). The last statement follows from the Paley-Wiener theorem since χ is analytic when $\operatorname{Im} t > 0$.

The cutoff function is used as follows:

LEMMA 2.3. *Let* $q(x', \xi_n) \in S^{m-\varrho|\gamma'|}_{\rho,\delta}(\mathbf{R}^{n-1} \times \mathbf{R})$, *where* $\gamma' \in \mathbf{N}^{n-1}$, *and let* χ *be the function given by Lemma 2.2. Let* $\kappa = 1/\varrho$. *Then*

$$q_1(x', \xi) = \chi(\xi_n/\langle \xi' \rangle^\kappa) q(x', \xi_n) \xi'^{\gamma'} \in S^m_{\rho,\delta}(\mathbf{R}^{n-1} \times \mathbf{R}^n), \tag{2.21}$$

and $q_1(x', \xi) - q(x', \xi_n)\xi'^{\gamma'}$ *and all its derivatives with respect to* ξ' *are of order* $-\infty$ *when* $\xi' = 0$. *If* $\mathscr{F}^{-1}_{\xi_n \to x_n} q$ *vanishes for* $x_n > 0$, *then this is also true for* $\mathscr{F}^{-1}_{\xi_n \to x_n} q_1$.

PROOF. We have proved (2.21) before when $\chi = 1$ and $\chi = 0$ in a neighborhood of ∞ and of 0 respectively (see (2.14) ff.) — which we cannot require now since χ must be analytic when $\operatorname{Im} \xi_n > 0$. Since $|\chi(t)| \leqq C_N |t|^N$ for every $N \geqq 0$, we have

$$|q_1(x', \xi)| \leqq C_N \langle \xi_n \rangle^{m-\rho|\gamma'|} |\xi'|^{|\gamma'|} |\xi_n/\langle \xi' \rangle^\kappa|^N.$$

With $N = \varrho|\gamma'|$ we get an estimate by $C\langle \xi_n \rangle^m \leqq C'\langle \xi \rangle^m$ if either $m \geqq 0$ or $\langle \xi_n \rangle \geqq \langle \xi' \rangle$. Assume now that $m < 0$ and that $\langle \xi_n \rangle < \langle \xi' \rangle$. Then $|\xi_n|/\langle \xi' \rangle^\kappa \leqq \langle \xi' \rangle^{1-\kappa}$. If $\varrho < 1$ then $1 - \kappa < 0$, and taking N large we get an estimate by any negative power of $\langle \xi' \rangle$. If $\varrho = 1$ we just take $N = |\gamma'| - m$ (recall that $m < 0$ now) and obtain the bound $C\langle \xi' \rangle^m \leqq C'\langle \xi \rangle^m$.

Differentiations of q_1 with respect to x' act directly on q, so it is sufficient to estimate $q_1^{(\alpha)}$, $|\alpha| \neq 0$. The terms where χ is not differentiated are $O(\langle \xi \rangle^{m-\rho|\alpha|})$ by our estimates of q_1 above. Let us now estimate a term of the form

$$\partial_\xi^{\alpha^{(1)}} \chi(\xi_n/\langle\xi'\rangle^\kappa)\, \partial_\xi^{\alpha^{(2)}} q(x', \xi_n)\, \partial_\xi^{\alpha^{(3)}} \xi'^{\gamma'}, \quad \alpha^{(1)} \neq 0, \alpha^{(1)} + \alpha^{(2)} + \alpha^{(3)} = \alpha.$$

Here χ is differentiated, and a derivative of $\chi(t)$ can be estimated by $|t|$ raised to any positive or negative power. Differentiation of $\xi_n/\langle\xi'\rangle^\kappa$ will remove a factor ξ_n, which is equivalent to a factor $\langle\xi'\rangle^{-\kappa}$ in the estimates because of the presence of derivatives of χ, or else we gain a factor $1/\langle\xi'\rangle$. Altogether we can estimate the term by

$$C_N|\xi_n/\langle\xi'\rangle^\kappa|^N \langle\xi'\rangle^{|\gamma'| - |\alpha^{(1)}| - |\alpha^{(3)}|} \langle\xi_n\rangle^{m - \rho|\gamma'| - \rho|\alpha^{(2)}|}$$

for any $N \in \mathsf{R}$. When $N = \varrho(|\gamma'| - |\alpha^{(1)}| - |\alpha^{(3)}|)$ this is bounded by

$$C|\xi_n|^N \langle\xi_n\rangle^{m - \rho|\gamma'| - \rho|\alpha^{(2)}|} \leqq C'\langle\xi\rangle^{m - \rho|\alpha|},$$

if $|\xi_n| > \langle\xi'\rangle$. If $|\xi_n| \leqq \langle\xi'\rangle$ we can estimate by any power of $1/\langle\xi'\rangle$ if $\varrho < 1$. If $\varrho = 1$ this is still true if $|\xi_n| < 1$, so we may replace $|\xi_n|$ by $\langle\xi_n\rangle$ in the estimates and take $N = -m + |\gamma'| + |\alpha^{(2)}|$ to complete the proof that $|q_1^{(\alpha)}(x', \xi)| \leqq C_\alpha \langle\xi\rangle^{m - \rho|\alpha|}$. The remaining statements are obvious consequences of the properties of χ.

We can then finally estimate the Poisson operator K_p^+ in general:

THEOREM 2.4. *Let $m \in \mathsf{R}$ and $0 \leqq \delta < \varrho \leqq 1$. Let $p(x, \xi) \in S^m_{\rho,\delta}(\mathsf{R}^n \times \mathsf{R}^n)$ have the uniform transmission property with respect to R^n_+. Then K_p^+ defined by* (2.3) *is continuous:*

$$K_p^+ : H_{(s)}(\mathsf{R}^{n-1}) \to \bar{H}_{(t)}(\mathsf{R}^n_+),$$

where $t = \varrho s - m - \frac{1}{2}$ if $s > 0$, $t = s - m - \frac{1}{2}$ if $s < 0$. For $s = 0$, the continuity holds with $t = -m - \frac{1}{2}$ if $\varrho = 1$ (and with $t = -m - \frac{1}{2} - \varepsilon$, any $\varepsilon > 0$, if $\varrho < 1$).

PROOF. In view of (2.6), we may assume that $p(x, \xi) = p(x', \xi)$ is independent of x_n. The Taylor expansion (2.7)–(2.9) will be used for large N in the following way: Let $p = p_{1,N} + p_{2,N}$, where

$$p_{1,N} = p - \chi p_N, \quad p_{2,N} = \chi p_N,$$

with $\chi = \chi(\xi_n/\langle\xi'\rangle^\kappa)$, $\chi(t)$ as in Lemma 2.2. Then $p_{1,N}$ and $p_{2,N}$ are in $S^m_{\rho,\delta}(\mathsf{R}^{n-1} \times \mathsf{R}^n)$ by Lemma 2.3. Moreover, since $p = p_N + r_N$,

$$p_{1,N} = (1 - \chi)p_N + r_N,$$

where $(1 - \chi)p_N$ and all its derivatives are of order $-\infty$ at $\xi' = 0$, by Lemma 2.3, while $\partial_{\xi'}^{\alpha'} \partial_{x'}^{\beta'} r_N = 0$ when $\xi' = 0$ if $|\alpha'| < N$. Then we can apply Proposition 2.1 to $p_{1,N}$ and find that for each s, N can be taken so large that $K_{p_{1,N}}$ and $K^+_{p_{1,N}}$ have the continuity property described there.

By Lemma 1.3 we can write $\partial_{\xi'}^{\gamma'} p(x', 0, \xi_n) = q_{\gamma'}(x', \xi_n) + r_{\gamma'}(x', \xi_n)$ where $q_{\gamma'} \in S^{-\infty}$ and $\tilde{r}_{\gamma'}(x', x_n)$ vanishes in R^n_+; and by Lemma 2.3, these properties are preserved under multiplication by $\xi'^{\gamma'} \chi$. Then if we set

$$q = \chi \sum_{|\gamma'| < N} q_{\gamma'}(x', \xi_n) \xi'^{\gamma'} / \gamma'!$$

it follows (cf. (2.6) and (2.8)) that $K^+_{p_2, N} = K^+_q$ and that $q \in S^{-\infty}$. Here K_q is continuous from $H_{(s)}(\mathsf{R}^{n-1})$ to $H_{(t)}(\mathsf{R}^n)$ for any t if $s > -\frac{1}{2}$.

Altogether we get the statement in the theorem by taking N large enough adapted to each s.

We now have the ingredients to treat (2.1).

THEOREM 2.5. *Let $0 \leqq \delta < \varrho \leqq 1$ and $m \in \mathsf{R}$, and let $p(x, \xi) \in S^m_{\varrho,\delta}(\mathsf{R}^n \times \mathsf{R}^n)$ have the uniform transmission property with respect to R^n_+. Then the operator $p(x, D)_+ = p(x, D)_{\mathsf{R}^n_+}$ is continuous:*

$$p(x, D)_+ : \bar{H}_{(s)}(\mathsf{R}^n_+) \to \bar{H}_{(t)}(\mathsf{R}^n_+) \tag{2.22}$$

where $t = s - m - (1 - \varrho)(s - \frac{1}{2}) = \frac{1}{2} - m + \varrho(s - \frac{1}{2})$ if $s > \frac{1}{2}$, and $t = s - m$ if $-\frac{1}{2} < s < \frac{1}{2}$. For $s = \frac{1}{2}$, (2.22) holds with $t = s - m$ when $\varrho = 1$ and $t < s - m$ when $\varrho < 1$.

PROOF. The statement is true for $-\frac{1}{2} < s < \frac{1}{2}$, even without the transmission condition, as was observed at the beginning of the section. If $s > \frac{1}{2}$ then in view of (2.4),

$$\begin{aligned} \|p(x, D)_+ u\|_{(t+1)} \leqq C \sum_{1 \leqq j \leqq n} (\|p_{(j)}(x, D)_+ u\|_{(t)} + \|p(x, D)_+ D_j u\|_{(t)}) \\ + C\|p(x, D)_+ u\|_{(t)} + C\|K^+_p \gamma_0 u\|_{(t)}. \end{aligned} \tag{2.23}$$

By Theorem 2.4,

$$\|K^+_p \gamma_0 u\|_{(\varrho(s - \frac{1}{2}) - m - \frac{1}{2})} \leqq C\|\gamma_0 u\|_{(s - \frac{1}{2})} \leqq C'\|u\|_{(s)}, \quad \text{for } s > \tfrac{1}{2}. \tag{2.24}$$

The estimate in the second line of (2.5) is at least equally strong since $\varrho \leqq 1$; and the same holds for the estimate in the first line of (2.5) when

$$\tfrac{1}{2} - \delta - m > \varrho(s - \tfrac{1}{2}) - m - \tfrac{1}{2},$$

i.e., when $s < \frac{1}{2} + (1 - \delta)/\varrho$ (where $(1 - \delta)/\varrho > 0$); recall that (2.5) also requires $s < \frac{1}{2} + 1$. Inserting $t = \varrho(s - \frac{1}{2}) - m - \frac{1}{2}$ in (2.23), we find that the desired estimate holds for $\frac{1}{2} < s < \frac{1}{2} + \tau$, where

$$\tau = \min\{(1 - \delta)/\varrho, 1\}. \tag{2.25}$$

We shall show that the estimate extends to all $s > \frac{1}{2}$ by an inductive argument that increases s by τ in each step. Assume that the desired estimate has been shown for all ps.d.o.s of type ϱ, δ with the uniform transmission property when $s \leqq s_0$, for some $s_0 > \frac{1}{2}$. Again we use (2.23), where Theorem 2.4 and the inductive hypothesis imply:

$$\|K_p\gamma_0 u\|_{(\rho(s-\frac{1}{2})-m-\frac{1}{2})} \leqq C\|u\|_{(s)}, \quad \text{for } s > \tfrac{1}{2},$$

$$\|p_{(j)}(x, D)_+ u\|_{(\rho(s-\frac{1}{2})-m-\delta+\frac{1}{2})} \leqq C\|u\|_{(s)}, \quad \text{for } \tfrac{1}{2} < s \leqq s_0,$$

$$\|p(x, D)_+ D_j u\|_{(\rho(s-\frac{3}{2})-m+\frac{1}{2})} \leqq C\|u\|_{(s)}, \quad \text{for } \tfrac{1}{2} < s \leqq s_0 + 1, s \neq \tfrac{3}{2}.$$

The third line gives an estimate which is at least as strong as the first when $s \leqq s_0 + 1$, and this is also true for the second line when

$$\varrho(s - \tfrac{1}{2}) - m - \tfrac{1}{2} \leqq \varrho(s_0 - \tfrac{1}{2}) - m - \delta + \tfrac{1}{2}, \text{ i.e., when}$$

$$s \leqq s_0 + (1 - \delta)/\varrho.$$

Inserting $t = \varrho(s - \frac{1}{2}) - m - \frac{1}{2}$ in (2.23), we find the desired estimate for $s \leqq s_0 + \tau$, cf. (2.25). This shows the induction step, so we can conclude that the asserted estimate holds for all $s > \frac{1}{2}$. The last statement in the theorem follows by interpolation.

The following example shows that the result is the best possible in general.

Example 2.6. Take p equal to $\chi(\xi_n/\langle\xi'\rangle^\kappa)$ as in (2.14), with $\chi \in C_0^\infty$, $\chi = 1$ in a neighborhood of 0 and $\kappa = 1/\varrho > 1$. Then $p \in S^0_{\rho,0}$ even satisfies (2.10). We shall show that if $p(D)_+$ is continuous from $\bar{H}_{(s)}(\mathbf{R}^n_+)$ to $\bar{H}_{(t)}(\mathbf{R}^n_+)$ for some $s > \frac{1}{2}$, then $t \leqq s - (1 - \varrho)(s - \frac{1}{2}) = \frac{1}{2} + \varrho(s - \frac{1}{2})$. It is sufficient to do so when $\frac{1}{2} < s < \frac{3}{2}$, for if a better estimate is valid for some $s > \frac{1}{2}$ it follows by interpolation that this is true for every $s > \frac{1}{2}$. In the proof we may assume that $t \leqq s$.

Since $p(D)$ is of order 0, $p(D)_+$ is continuous from $\bar{H}_{(s-1)}(\mathbf{R}^n_+)$ to $\bar{H}_{(s-1)}(\mathbf{R}^n_+) \subset \bar{H}_{(t-1)}(\mathbf{R}^n_+)$. If $p(D)_+$ is continuous from $\bar{H}_{(s)}(\mathbf{R}^n_+)$ to $\bar{H}_{(t)}(\mathbf{R}^n_+)$, then $D_n p(D)_+$ is continuous from $\bar{H}_{(s)}(\mathbf{R}^n_+)$ to $\bar{H}_{(t-1)}(\mathbf{R}^n_+)$, so it follows from (2.3) with $j = n$ that $K_p^+\gamma_0$ is continuous from $\bar{H}_{(s)}(\mathbf{R}^n_+)$ to $\bar{H}_{(t-1)}(\mathbf{R}^n_+)$. Since γ_0 is surjective from $\bar{H}_{(s)}(\mathbf{R}^n_+)$ to $H_{(s-\frac{1}{2})}(\mathbf{R}^{n-1})$, this implies that K_p^+ is continuous from $H_{(s-\frac{1}{2})}(\mathbf{R}^{n-1})$ to $\bar{H}_{(t-1)}(\mathbf{R}^n_+)$. Recall that $K_p^+ v = r^+ p(D)(v \otimes \delta(x_n))$ when $v \in \mathscr{S}(\mathbf{R}^{n-1})$. Set $\tau = t - 1$. Then

$$\|K_p^+ v\|_{(\tau)} = \sup |\langle K_p^+ v, \bar{w}\rangle|/\|w\|_{(-\tau)},$$

where $w \in \mathscr{S}$ and $\operatorname{supp} w \subset \bar{\mathbf{R}}^n_+$. We take a function $\varphi \in C_0^\infty(\mathbf{R}_+)$ and test with w defined by $\hat{w}(\xi) = \hat{\varphi}(\xi_n/\langle\xi'\rangle^\kappa)\psi(\xi')$, $\psi \in \mathscr{S}$. Since $2\tau < 1$ and $\operatorname{supp} w \subset \bar{\mathbf{R}}^n_+$ we have:

$$(2\pi)^n\|w\|^2_{(-\tau)} = \int |\hat{\varphi}(\xi_n/\langle\xi'\rangle^\kappa)\psi(\xi')|^2\langle\xi\rangle^{-2\tau}\,d\xi'\,d\xi_n \leqq C\int |\psi(\xi')|^2\langle\xi'\rangle^{(1-2\tau)\kappa}\,d\xi',$$

$$\begin{aligned}\langle K_p^+ v, \bar{w}\rangle &= \langle p(D)(v\otimes \delta), \bar{w}\rangle \\ &= (2\pi)^{-n}\int \chi(\xi_n/\langle\xi'\rangle^\kappa)\hat{v}(\xi')\bar{\hat{\varphi}}(\xi_n/\langle\xi'\rangle^\kappa)\bar{\hat{\psi}}(\xi')\,d\xi'\,d\xi_n \\ &= C'\int \hat{v}(\xi')\bar{\hat{\psi}}(\xi')\langle\xi'\rangle^\kappa\,d\xi', \quad C' = (2\pi)^{-n}\int \chi(\xi_n)\bar{\hat{\varphi}}(\xi_n)\,d\xi_n.\end{aligned}$$

We have $C' \neq 0$ for a suitable choice of φ. It follows that

$$\left|\int \hat{v}(\xi')\bar{\hat{\psi}}(\xi')\langle\xi'\rangle^\kappa\,d\xi'\right| \leqq C\|K_p^+ v\|_{(t-1)}\left(\int |\hat{\psi}(\xi')|^2\langle\xi'\rangle^{(1-2\tau)\kappa}\,d\xi'\right)^{\frac{1}{2}},$$

hence

$$\int |\hat{v}(\xi')|^2\langle\xi'\rangle^{\kappa(1+2\tau)}\,d\xi' \leqq C^2\|K_p^+ v\|^2_{(t-1)} \leqq C'\|v\|^2_{(s-\frac{1}{2})},$$

which proves the claim that $\kappa(t-\frac{1}{2}) \leqq s-\frac{1}{2}$. Note that the example shows also that the auxiliary results Proposition 2.1 and Theorem 2.4 are precise.

The results on $p(x, D)_+$ and K_p^+ may be useful in a study of boundary problems for hypoelliptic operators.

REFERENCES

1. L. Boutet de Monvel, *Comportement d'un opérateur pseudo-différentiel sur une variété à bord*, I–II, J. Analyse Math. 17 (1966), 241–304.
2. L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math. 126 (1971), 11–51.
3. R. R. Coifman and Y. Meyer, *Au delà des opérateurs pseudo-différentiels*, Astérisque no. 57, Soc. Math. France, Paris, 1978.
4. G. Èskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations*, Amer. Math. Soc., Providence, R. I., 1981.
5. G. Grubb, *Functional Calculus of Pseudo-differential Boundary Problems*, Progress in Math. Vol. 65, Birkhäuser, Boston, 1986.
6. G. Grubb, *Parabolic pseudo-differential boundary problems and applications*, to appear in the proceedings of the CIME meeting *Microlocal Analysis* 1989 (Springer Lecture Note).
7. L. Hörmander, *The Analysis of Linear Partial Differential Operators*, Vol. III, Springer Verlag, Berlin, New York, 1985.
8. R. T. Seeley, *Extensions of C^∞ functions defined in a half space*, Proc. Amer. Math. Soc. 15 (1964), 625–626.

KØBENHAVNS UNIVERSITETS MATEMATISKE INSTITUT,
UNIVERSITETSPARKEN 5
DK-2100 KØBENHAVN Ø
DANMARK

MATEMATISKA INSTITUTIONEN
LUNDS UNIVERSITET, BOX 118
S-221 00 LUND
SVERIGE