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THE LEVI PROBLEM IN NON-LOCALLY CONVEX SEPARABLE TOPOLOGICAL VECTOR SPACES*

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1. Introduction.

In the last 15 years some efforts have been devoted towards developing the field of complex Analysis in non-locally convex topological vector spaces. Although the results obtained are very few compared with those obtained for the locally convex spaces, we may say that some progress has been achieved (see Lelong [19, 20], Bochnak & Siciak [6, 7] and the author [1, 2, 3, 4, 5]).

In this paper we consider one of the interesting problems in complex Analysis, that is, the Levi problem. We shall prove that, *every pseudoconvex domain is a domain of holomorphy* provided that our spaces have the bounded approximation property. It has been assumed that the Levi problem is as useful to the field as the Hahn-Banach theorem to Functional Analysis. In fact there is some analogy in the sense that the Levi problem tells us about the richness of the space $H(E)$ of holomorphic functions on a given topological space E, while the Hahn-Banach theorem tells us about the richness of the dual space of continuous linear functionals E'.

A topological vector space E will simply be called a *Levi space* if the Levi problem has a solution for E. Several mathematicians have solved the Levi problem for domain spread over finite and infinite dimensional locally convex spaces E (see for example Gruman [14], Gruman & Kiselman [15], Dineen, Noverraz & Schottenloher [13], Colombeaux & Mujica [9], Dineen [11] and Schottenloher [26]. As for a counterexample, see Josefson [17]). We point out that the solution given by Gruman & Kiselman [15] represents one of the fundamental results in this field.

The author [1, 2] has solved the Levi problem for different classes of non-locally convex spaces. In [2] the radius of convergence technique was used to solve the radius of convergence problem and to obtain as well the Levi

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problem solution for certain classes of metrizable topological vector spaces. In [1] the solution of the Levi problem was extended to cover the following classes of spaces: (i) all metrizable spaces with finite-dimensional Schauder decomposition, (f.d. decomposition), (ii) all locally pseudoconvex Fréchet spaces with the bounded approximation property (b.a.p.).

In this paper we propose and prove theorems concerning the following:

(1) The Levi problem for domains spead over locally pseudoconvex topological vector spaces (Lps) (not necessarily metrizable or locally convex) with the b.a.p.

(2) The Levi problem in some separable p-Banach spaces $(0 < p < 1)$ in particular, some separable Banach spaces (when $p = 1$).

(3) The well-known non-locally convex spaces:

l^{Φ}, $l^{+}_{p_0}$ $(0 < p_0 \leqq 1)$, $l_{(p_n)}$ $(0 < p_n \leqq 1)$, $\bigcup_{1>p>0} l_p$, $s(E)$ and some complemented subspaces of H^p $(0 < p < 1)$ are examples of Levi spaces.

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2. Surjective Limits In Locally Pseudoconvex Spaces.

The concept of surjective limit in a locally pseudoconvex space Lps can be formulated conveniently, as for locally convex spaces, as follows: a Lps E is a *surjective limit* of the Lps $(E_i)_{i\in A}$, and we write $E = \varprojlim_{i\in A} E_i$, if there exists a continuus linear surjection $\pi_i: E \to E_i$, for each $i \in A$, and the inverse images of the neigthbourhood of 0 in E_i, as i ranges over A, form a subbase for the neighbourhoods system at 0 in E. This means that for each neighbourhood W of 0 in E, there exists an $i \in A$ and a neighbourhood V of 0 in E_i such that $\pi_i^{-1}(V) \subset W$. Now $E = \varprojlim_{i\in A} E_i$ is called an *open surjective limit* if π_i is an open mapping for each $i \in A$.

EXAMPLE 2.1. Let $\mathrm{CP}(E)$ denote the collection of all continuous pseudonorms (i.e. p_α-seminorm for some $0 < p_\alpha \leqq 1$) on a Lps E. For each $q \in \mathrm{CP}(E)$ let E_q denote the vector space E equipped with the topology generated by q, and also let π_q denote the canonical map from E onto $E_q/q^{-1}(0)$. $(E_q/q^{-1}(0), \pi_q)$ is called *the canonical pseudo-normed surjective representation* of E.

In what follows we show that a Lps E with the approximation property (a.p.) is closed under the operation of surjective limit.

LEMMA 2.1. *The collection of locally pseudoconvex spaces with the approximation property is closed under the operation of surjective limit.*

PROOF. The a.p. of a topological vector space E does not depend on whether E is locally convex or not. It does depend on the existence of a family $(\pi_i)_{i\in A}$ of finite-rank continuous linear functionals π_i: $E \to E$ which can be used to approximate the identity mapping I. In connection with this, we may follow here Dineen's method [11, Example 2.3] of a locally convex case.

The following proposition allows to reduce our study of the Levi problem in Lps to spaces which have continuous p_α-norms ($0 < p_\alpha \leqq 1$).

PROPOSITION 2.2. *Let E be a Hausdorff locally pseudoconvex space with an equicontinuous f.d. decomposition (π_i). Then E is a surjective limit of spaces E_i, $i \in A$, where E_i has a continuous p_i-norm and an equicontinuous f.d. decomposition.*

This can be established in a manner similar to that of Dineen [11, Example 2.4] noticing Example 2.1.

The next result is equivalent to show that: if π: $E \to E/q^{-1}(0)$ is the quotient mapping on a Lps E where q is a continuous p-seminorm, then $U = \pi^{-1}(\pi(U))$ for every pseudoconvex domain U containing a closed ball $B_q(0, \delta)$, $\delta > 0$.

LEMMA 2.3. *Let U be a pseudoconvex domain of a locally pseudoconvex space E. Let q be a continuous p-seminorm on E ($0 < p < 1$) such that $B_q(0, \delta) \subset U$ for $\delta > 0$, then*

$$U = U + \{x; q(x) = 0\}$$

The proof is essentially the same as for locally convex spaces with Schauder basis which was given by Dineen [11, lemma 1.1].

The following lemma has applications when the Levi problem is considered.

LEMMA 2.4. *Let U be a pseudoconvex domain of an open surjective limit $\varprojlim (E_i, \pi_i)$ of locally pseudoconvex spaces E_i $i \in A$. Then U is π_i-open for some $i \in A$.* (i.e. *for every $x \in U$, there exists V_x open in E_i such that $x \in \pi_i^{-1}(V_x) \subset U$*).

The proof is analogous to that for a locally convex space E given by Dineen [12, lemma 1.8] and using lemma 2.3 instead of the corresponding result in normed case.

3. Levi Problem In Separable Topological Vector Spaces.

Let E be a locally pseudoconvex topological vector space Lps. That is, the topology of E is defined by a fundamental family of absolutely pseudoconvex neighbourhoods of the origin, see Waelbroeck [27, Def. 7]. A subset A of a vector space E is *absolutely pseudoconvex* if it is absolutely p-convex for some $1 \geqq p > 0$ Here A is *absolutely p-convex* if $ax + by \in A$ whenever $x, y \in A$, $|a|^p + |b|^p \leqq 1$. (It was proved that a topological vector space is Lps if we can find a fundamental system (V_i) of neighbourhoods of the origin, and for each V_i some λ_i in such a way that $V_i + V_i \subset \lambda_i V_i$, see Waelbroeck [27, cor. 2]. For locally p-convex spaces we have $\lambda_i = 2^{1/p}$).

In fact, the topology of a Lps E can be determined by a family of p_i-seminorms $\|x\|_i$, $i \in A$. (Of course E is metrizable by the F-norm, $\|x\| = \sum_{i=1}^{\infty} 2^{-i} \frac{\|x\|_i}{1 + \|x\|_i}$ if the index set A is countable, cf. Waelbroeck [27, p. 2]).

In this section we solve the Levi problem for domains spread over Lps E with the b.a.p. and in particular over certain separable locally bounded spaces with the b.a.p. For metrizable Lps E the Levi problem is solved by the author in [2]. (Examples of these Levi spaces are given in section 4).

The method we are going to use here is different from ours in [1]. It depends on Hirschwitz's result [16] of the domain of holomorphy of $\mathbf{C}^N$, on a technique similar to that used by Dineen [11], and on the author's result [1].

3a. *The Levi in* Lps *with* b.a.p.

In this section we solve the Levi problem in Lps. This class of topological vector spaces contains non-metrizables non-locally convex ones. Thus we proved for instance that the class of holomorphic functions $H(E)$ has a certain type of richness. This will naturally clear many questionmarks concerning the holomorphic properties of this class of spaces.

THEOREM 3.1. *The collection of Hausdorff locally pseudoconvex spaces with the bounded approximation property in which the pseudoconvex domains are domains of holomorphy is closed under the open surjective limit. That is, an open surjective limit of Levi spaces with the b.a.p. is a Levi space.*

PROOF. Assume U is pseudoconvex domain of the surjective limit, $E = \varprojlim_{i \in A} E_i$. Then there exists an $i \in A$ such that $U = \pi_i^{-1}(\pi_i(U))$ and $\pi_i(U)$ is pseudoconvex domain in E_i (see Lemma 2.3).

Now $\pi_i(U)$ is a domain of holomorphy. If there exist open connected sets U_1, U_2 in E such that $U_2 \subset U$, $U \cap U_1 \supset U_2$ and for each $f \in H(U)$, there is an $f_1 \in H(U_1)$ with $f_{|U_2} = f_{1|U_2}$. Then, and as a consequence of that π_i is open, we get that $\pi_i(U_1) \subset \pi_i(U)$. Hence $U_1 \subset \pi_i^{-1}\pi_i(U) = U$ and consequently U is a domain of holomorphy. This completes the proof of theorem.

In what follows we give the solution of the Levi problem in Lps having the b.a.p. This extends the results of the locally convex spaces which were considered by [11, 14, 15, 25, 26].

COROLLARY 3.2. (Levi problem in Lps). *Let E be a Hausdorff locally pseudoconvex space with the bounded approximation property then every pseudoconvex domain U over E is a domain of holomorphy.*

PROOF. Every Lps E can be expressed as a surjective limit of p_α-normed spaces E_α with continuous p_α-norms q_α, see Example 2.1. Hence we apply theorem 3.1 to obtain the solution of the Levi problem.

3b. *The Levi problem in separable p-Banach spaces.*

In this section we solve the Levi problem in certain separable p-Banach spaces $(0 < p \leqq 1)$, i.e. in a complete locally bounded space with p-homogeneous norm, (the case $p = 1$ gives Banach spaces).

Since every separable Banach space E is isomorphic to a quotient space of $l_0 = \bigcap_{p>0} l_p$, (see Stiles [23]), and since l_0 is a Levi space (see corollary 3.2 or the author's result [1, Th. 3.1]), we can obtain the following interesting partial result for the Levi problem.

THEOREM 3.3. (Lêvi problem over certain separable Banach space). *Let U be a pseudoconvex domain spread over a Banach space E. Suppose that E is isomorphic to a quotient space of l_0 which has the bounded approximation property. Then U is a domain of holomorphy.*

PROOF. Since l_0 is a Fréchet space, its quotient space l_0/M is a Fréchet space. Now the space l_0/M is assumed to have the b.a.p. Hence a direct application of corollary 3.2 will imply the required result.

Let E_p be a p-Banach space. The fact that every separable *locally bounded* space is isomorphic to a quotient space of l_p $(0 < p \leqq 1)$ will help us to get the following consequence which is of special interest.

THEOREM 3.4. (Levi problem over separable p-Banach space). *Let E_p be a separable p-Banach space which is isomorphic to a quotient space l_p/M. Suppose that l_p/M has the bounded approximation property. Then every pseudoconvex domain over E_p is a domain of homorphy.*

PROOF. E_p is isomorphic to l_p/M for some closed subspace M of $l_p \cdot l_p/M$ is a Fréchet space, and by assumption it has the b.a.p. Then by applying Corollary 3.2, or [1, cor 3.2], we achieve the solution.

REMARK 3.5. The assumption that l_p/M has the b.a.p. can not be dropped in the above theorem. We note that $L^p[0, 1]$, $(0 < p < 1)$ does not admit holomorphic functions other than 0, and so each domain (pseudoconvex or not) is not a domain of holomorphy.

PROBLEM. What is the class of separable p-Banach spaces such that each of its elements is isomorphic to a quotient space l_p/M having the b.a.p.? Could Hardy space H^p $(0 < p < 1)$ be element of this class? We note that l_p $(0 < p < 1)$ is an element of this class since it has a basis and hence this class is not empty.

4. Examples.

In this section we give examples of non-locally convex spaces which have either a Schauder basis or the bounded approximation property. In fact, by the results of section 3 and of the author [1], all pseudoconvex domains in these spaces are domain of holomorphy, that is, they are *Levi spaces.*

4.1. *The inductive space* $\bigcup_{1>p>0} l_p$.

We define the q-topology on $\bigcup_{1>p>0} l_p$ to be the strongest vector topology such that each injection $i_p: l_p \to \bigcup l_p$ is continuous. The space $\bigcup l_p$ with this q-topology is complete separable non-locally convex Lps and has the unit vectors (e_n) as its symmetric Schauder basis. A sequence converges in $\bigcup l_p$ iff it is contained in and converges in some l_p. A set is compact in $\bigcup l_p$ iff the set is contained and compact in some l_p; no closed infinite dimensional subspace of $\bigcup l_p$ is contained in l_p; and no infinite dimensional subspace of $\bigcup l_p$ is metrizable. Hence $\bigcup l_p$ itself is *not metrizable*, see Stiles [23].

Since E is Lps with Schauder basis, then corollary 3.2 implies that E is a Levi space.

4.2. *The Hardy spaces* H^p $(1 > p > 0)$.

The Hardy space H^p of all analytic functions f on the unit disc of C is separable, locally bounded, non-locally convex space with respect to the p-norm, $\|f\| = \lim_{r\to 1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta, f \in H$. The Banach envelope $\tilde{H}^p$ of H^p is isomorphic to l_1, (see by Kalton [18]). This implies that $\tilde{H}^p$ has the b.a.p. Moreover H^p contains a non-locally convex closed subspace M of E isomorphic to l_p $(0 < p < 1)$, (cf. Shapiro [24]).

Now every pseudoconvex domain over this complemented subspace M of H^p, or over $\tilde{H}^p$ is a domain of holomorphy, by the author [1, Th. 3.1, cor. 3.2] or by corollary 3.2.

4.3. *The spaces of mappings with rapidly decreasing numbers s*(E).

Let E be a locally convex space and $L(E)$ be the space of all continuous linear mappings on E equipped with the linear mapping norm. For $T \in L(E)$ one defines the rth approximation numbers by

$$\alpha_r(T) = \inf\{\|T - S\|; S \in L(E), \dim S(E) \leqq r\}$$

If

$$l^p(E) = \{T \in L(E); \sum [\alpha_r(T)]^p < \infty\}$$

then on the intersection

$$s(E) = \bigcap_{p>0} l^p(E)$$

a locally pseudoconvex vector topology is generated by the sets

$$U_{p,\varepsilon}(T) = \{S \in S(E); \sum [\alpha_r(S - T)]^p < \varepsilon\}$$

We note that $s(E)$ is not a Fréchet space if E is not a Banach space. Now if $s(E)$ has the b.a.p. then the pseudoconvex domain U of E will be a domain of holmorphy by corollary 3.2.

REMARK 4.1. When E is a normed space, $s(E)$ will be a Fréchet space with the b.a.p., see Pietsch [21, p. 139], and the solution of the Levi problem for such a case has been giyen by the author [1, example 3].

4.4. *The spaces* l^Φ.

Let Φ be a continuous, unbounded, subadditive, increasing function on $[0, \infty]$ with $\Phi(t) = 0$ iff $t = 0$. We define l^Φ as

$$l^\Phi = \{x = (x_n); x_n \in \mathbb{C}, \sum_{n=1}^{\infty} \Phi(|x_n|) < \infty\}.$$

l^Φ is an F-space with basis in the norm $\|x\| = \sum_{n=1}^{\infty} \Phi(|x_n|)$, $x = (x_n) \in l^\Phi$.

Shapiro proved in [24] that if the function $f(t) = t^{-p}\Phi(t)$ is montone decreasing in $(0, \infty)$ for some $(0 < p < 1)$, then every closed, norm-bounded convex subset A of l^Φ is compact and hence is bounding. That is, $\|f\|_A < \infty$ for all $f \in H(l^\Phi)$ the space of holomorphic functions on l^Φ, see the author [3]. By a normed-bounded set we mean a set which is bounded with respect to the metric defined by the F-norm $\|.\|$, i.e. $\sup_{x \in A} \|x\| < \infty$.

Now and in connection with these topological properties, we can apply corollary 3.2 to obtain the interesting holomorphic property that l^Φ is a Levi space. We note that if Φ is restricted to be a positive homogeneous, that is, $\Phi(\alpha t) = \alpha\Phi(t)$, $t > 0$ then l^Φ becomes locally convex.

4.5. *The spaces* $l^+_{p_0}$ $(1 \geqq p_0 > 0)$.

The spaces $l^+_{p_0} = \bigcap_{p > p_0} l_p$ $(1 \geqq p_0 > 0)$ are Fréchet Lps whenever it takes the natural upper bounded topology defined by the F-norm $\|x\| = \sup_{p > p_0} \|x_p\|$ where $\|x\|_{p_0} = \sum_1^\infty |x_n|^p$. It is neither locally bounded nor locally convex. A sequence converges to zero in E iff it converges to zero in l^p for all $p > p_0$. A subset of $l^+_{p_0}$ is bounded iff it is bounded in l^p for each $p > p_0$. It was proved that every convex closed bounded subset of $l_{p_0}^+$ is compact. Hence no infinite-dimensional subspace is isomorphic to a normed space. However every infinite-dimensional subspace of $l_{p_0}^+$ contains a further infinite dimensional subspace which is locally convex, see Shapiro [24]. We note that the particular space $l_0^+ = \bigcap_{p>0} l_p$ is an interesting one since every separable Banach space is isomorphic to a quotient space of it, see Stiles [23].

Now, having the fact that $l_{p_0}^+$ are Fréchet Lps with basis we obtain that every pseudoconvex domain U spread over $l_{p_0}^+$ is a domain of holomorphy as an application of Corollary 3.2.

4.6. *The spaces* $l_{(p_n)}$ $(1 \geqq p_n > 0)$.

The spaces $l_{(p_n)} = \{x = (x_n);\ \sum |x_n|^{p_n} < \infty\}$, $(1 \geqq p_n > 0)$ with the F-norm $\|x\| = \sum_1^\infty |x_n|^{p_n}$ are F-spaces with basis. It was shown in [23, 24] that the linear and topological properties of $l_{(p_n)}$ depends on the sequence (p_n) we are going to choose. For example $l_{(p_n)}$ is locally bounded if $p_n \to 0$, *and* $l_{(p_n)}$ is (not L_{ps} if $P_n \nrightarrow 0$, see Rolewicz [21]. If $\overline{lim}\, p_n < 1$, then no infinite-dimensional subspace of $l_{(p_n)}$ is locally convex and this is equivalent to that every convex closed normbounded subset A of $l_{(p_n)}$ is) compact and hence it is bounding i.e. $\|f\|_A < \infty$ or all $f \in H(l_{(p_n)})$. However this is not the case if $p_n \to 1$, i.e. every ∞-dimensional subspaces of $l_{(p_n)}$ contains a further one, isomorphic to, a dense subspace of l_1, see Shapiro [24].

Now, regarding to the Levi problem, the solution does not depend on the choice of (p_n). In fact, The difference in linear and topological properties will not affect this holmorphic property that is, all $l_{(p_n)}$ are Levi spaces, by corollary 3.2, taking into account that all $l_{(p_n)}$ are F-spaces with bases.

REFERENCES

1. A. Bayoumi, *The Levi problem for domains spread over locally pseudoconvex Fréchet spaces with the bounded approximation property*, J. Complex Variables, Theory and Application 10 (1988), 141–151.
2. A. Bayoumi, *The Levi problem and the radius of convergence of holomorphic functions on metric vector spaces*, in *Advance in Functional Analysis Holomorphy and Approximation theory*, Lecture Notes in Math. 843, 9–32, Springer Verlag, 1981.
3. A. Bayoumi, *Bounding subsets of some metric vector spaces*, Ark. Mat. 18 (1980), 13–17.
4. A. Bayoumi, *The theory of bounding subsets of topological vector spaces without the convexity condition*, Portugal. Math. 47 (1990), 25–42.
5. A. Bayoumi, *On holomorphic Hahn-Banach extension theorem and properties of bounding and weakly bounding sets in some metric vector spaces*, Portugal Math. vol. (1989), 229–340.
6. J. Bochnak & J. Siciak, *Analytic Functions in topological vector spaces*, Studdia Math. (1970).
7. J. Bochnak & J. Siciak, *Polynomial and multilinear mappings in topological vector spaces*, Studia Math. (1971).

8. H. Bremermann, *Über die Äquivalenz der pseudokonvexen Gebiete und der Holomorphiegebiete im Raum von komplexen Veranderlichen*, Math. Ann. 120 (1954), 63–91.
9. J. F. Colombeaux & Mujica, *The Levi problem in nuclear Silva spaces*, Ark. Mat. 18 (1980).
10. S. Dineen, *Complex Analysis in Locally Convex Spaces*, Amsterdam, North-Holland 57, 1983.
11. S. Dineen, *Surjective limits of locally convex spaces and their applications to infinite dimensional holomorphy*, Bull. Soc. Math. France 103 (1975), 441–509.
12. S. Dineen, *Holomorphic functions on locally convex spaces*, II-pseudoconvex domains, Ann. Inst. Fourier (Grenoble), 23 (1973), 155–185.
13. S. Dineen, P. Novervaz et M. Schottenlooher, *Le problème de Lévi dans certains espaces vectoriels topologiques localement convexes*, Bull. Soc. Math. France 104 (1976), 87–97.
14. L. Gruman, *The Levi problem in certain infinite dimensional vector space*, Illinois J. Math. 18 (1972), 20–26.
15. L. Gruman et C. O. Kiselman, *Le problème de Levi dans les espaces de Banach à base*, C. R. Acad. Sci. Paris A. 274 (1972), 1296–1299.
16. A. Hirschowitz, *Remarques sur les ouverts d'holomorphi d'un produit denombrable de droites*, Ann. Inst. Fourier (Grenoble) 19 (1969), 219–229.
17. B. Josefson, *A counterexample in the Levi problem*, in *proceedings on infinite dimensional holomorphy*, Lecture Note in Mathematics 364, 168–177. Springer Verlag, 1974.
18. N. J. Kalton, Banach envelopes of non-locally convex spaces, Canad. J. Math. XXXVIII (1986), 65–86.
19. P. Lelong, *Recent results on analytic mapping and plurisubharmonic functions in topological linear spaces*, Lecture notes in Math. 185 (1971).
20. P. Lelong, *Theorème de Banach-Steinhaus pour les polynomies applications entières d'espaces vectoriels complexes*, Lecture notes in Math. 205, 87–112, 1971.
21. A. Pietsch. *Nuclear Locally Convex Spaces*, Ergebnisse der Math. 68, 1972.
22. S. Rolewicz, *Metric Linear Spaces*, Instytut Matematyczny Polskiej Akademi Nauk., Monografie Matermatyczne, 1972.
23. J. Stiles, *On the inductive Limit of* $\bigcup l_p$ $(0 < p < 1)$. Studia Math. LVI (1976).
24. J. Shapiro, *On convexity and compactness in* F-spaces with bases, Indiana Univ. Math. 21, (1971).
25. M. Schottenloher, *Holomorphe Funktionen auf Gebieten über Banach-Raümen zu vorgegebenen Konvergenzradian*, *Manuscripta Math.* 21 (1977), 315–327.
26. M. Shottenloher, *The Levi Problem for domains spread over locally convex spaces with finite dimensional Schauder decomposition*, Ann. Inst. Fourier (Grenoble) 26 (1976), 207–377.
27. L. Waelbroeck, *Topological Vector Spaces and Algebra*, Lecture Notes in Math. 230, Springer Verlag, 1971.

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