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EQUIVARIANT EILENBERG-MACLANE SPACES OF TYPE 1*

JAN-ALVE SVENSSON

§0. Introduction.

As realized by Bredon, the way to extend non equivariant algebraic topology to the equivariant setting is through functors with source the orbit category of the group. In particular this applies to the fundamental groupoid of a space. Thus if X is a G-space $\pi^G X$ is the contra-variant functor $G/H \mapsto \pi(X^H)$, $O(G)^{\mathrm{OP}} \mapsto$ Groupoids. As is usual in topology, it then becomes important to associate a classifying G-space to any such functor $\mathscr{G}$. The associated G-space is denoted $K(\mathscr{G}, 1)$. It is important to know the relation between $\mathscr{G}$ and $\pi^G K(\mathscr{G}, 1)$.

The spaces $K(\mathscr{G}, 1)$ are considered in [3] where some theorems are proved assuming the existence of a certain map μ: $\pi^G K(\mathscr{G}, 1) \to \mathscr{G}$. The existence of μ is supposed to follow from [2], but this is not the case.

I prove the existence of a weak equivalence μ: $\pi^G K(\mathscr{G}, 1) \to \mathscr{G}$ uniquely determined up to homotopy by a certain property (Theorem 2.7). The homotopy type of $\pi^G K(\mathscr{G}, 1)$ is also determined in terms of a simple endo-functor L on the category of $O(G)$-groupoids. In fact, there is a cannonical homotopy class $\pi^G(K(\mathscr{G}, 1) \to \mathrm{L}\mathscr{G}$ which is a homotopy equivalence (Corollary 2.8).

For recent applications to equivariant surgery theory, see [4] where the map μ plays a vital role.

A slightly different approach to groupoids in equivariant topology is taken in [1]. I discuss its relation to $O(G)$-groupoids in (1.10).

For further properties of the $K(\mathscr{G}, 1)$ I refer to [3].

§1. The algebra

Let $\mathscr{C}$ be a small category an GP the category of small groupoids. A category is a groupoid if all its morphisms are isomorphisms. Consider the functor category $\mathrm{GP}^{\mathscr{C}^{\mathrm{OP}}}$, $\mathscr{C}$-GP for short. Its objects will be called *$\mathscr{C}$-groupoids*. Let J denote the groupoid with two elements and two non-trivial morphisms (i.e. E(Z/2)).

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A *homotopy* between two maps (i.e. natural transformations) $\phi, \psi\colon \mathscr{G}_0 \to \mathscr{G}_1$ of $\mathscr{C}$-groupoids is a map $\theta\colon \mathscr{G}_0 \times J \to \mathscr{G}_1$ such that composition with the two obvious maps $i_0, i_1\colon \mathscr{G}_0 \to \mathscr{G}_0 \times J$ is ϕ and ψ respectively. This is an equivalence relation and defines the *homotopy category of $\mathscr{C}$-groupoids* denoted $h\mathscr{C}$-GP.

The only application in sight is for $\mathscr{C} = O(G)$, the orbit category of a discrete group G. $O(G)$ has objects G/H and a map $G/H \to G/K$ is a G-equivariant map of G-sets. The use of an abstract category $\mathscr{C}$ will shorten notation considerably.

The main source (but by far not the only one) is the following. Let X be a G-space. Let $\pi^G X$ be the $O(G)$-groupoid which to G/H associates $\pi(X^H) = \pi(\mathrm{Map}_G(G/H, X))$, the fundamental groupoid of X^H. $\pi^G X$ acts on maps G/H → G/K in the obvious way.

A map $\phi\colon \mathscr{G}_0 \to \mathscr{G}_1$ in $\mathscr{C}$-GP is a *weak equivalence* if $\phi(c)\colon \mathscr{G}_0(c) \to \mathscr{G}_1(c)$ is an equivalence of categories for each object c if $\mathscr{C}$ (this amounts to $\phi(c)$ being a bijection on iso-classes of objects and on hom-sets).

Let $\mathscr{G}$ be a $\mathscr{C}$-groupoid, c an object of $\mathscr{C}$ and $x \in \mathrm{ob}\,\mathscr{G}(c)$. *A pair* (f, y) consisting of $f\colon c \to d$ (in $\mathscr{C}$) and $y \in \mathrm{ob}\,\mathscr{G}(d)$ *is a generator* for x if $\mathscr{G}(f)y = x$. A generator (f, y) for x is *universal* if for any other generator $(f'\colon c \to d', y')$ for x there is a unique $g\colon d' \to d$ such that $\mathscr{G}(g)y = y'$ and for this g we also have $gf' = f$.

Any x has a generator but in general not a universal one. If $(f\colon c \to d, y)$ and $(f'\colon \mathrm{c} \to \mathrm{d}')$ are universal for x then there is an isomorphism $g\colon d' \to d$ such that $\mathscr{G}(g)y = y'$.

A $\mathscr{C}$-groupoid $\mathscr{G}$ is *geometric* if for each $c \in \mathrm{ob}\,\mathscr{C}$ and $x \in \mathrm{ob}\,\mathscr{G}(c)$ there is a universal generator for x. The examples $\pi^G X$ are geometric.

Let $g\mathscr{C}$-GP be the full subcategory of $\mathscr{C}$-GP consisting of geometric objects. $hg\mathscr{C}$-GP is the associated homotopy category. I am going to construct a functor, called *geometrisation*

$$L\colon \mathscr{C}\text{-GP} \to g\mathscr{C}\text{-GP}$$

such that L is right adjoint to the forgetful functor (ℓ) on homotopy categories.

Let $\mathscr{G}$ be in $\mathscr{C}$-GP. Define $L\mathscr{G}$ as follows. For $c \in \mathrm{ob}\,\mathscr{C}$ set

$$\mathrm{ob}\, L\mathscr{G}(c) = \coprod_{d \in \mathrm{ob}\,\mathscr{C}} \mathscr{C}(c, d) \times \mathrm{ob}\,\mathscr{G}(d).$$

A morphism ω in $L\mathscr{G}(c)$

$$\mathscr{C}(c, d) \times \mathrm{ob}\,\mathscr{G}(d) \ni (f, x) \xrightarrow{\ \omega\ } (g, y) \in \mathscr{C}(c, d') \times \mathrm{ob}\,\mathscr{G}(d')$$

is a map $\omega\colon \mathscr{G}(f)x \to \mathscr{G}(g)y$ in $\mathscr{G}(c)$. $L\mathscr{G}(c)$ is a groupoid. The functor $L\mathscr{G}(h)\colon L\mathscr{G}(c) \to L\mathscr{G}(c')$ associated to a $h\colon c' \to c$ (in $\mathscr{C}$) is

$$L\mathscr{G}(h)(\omega\colon (f, x) \to (g, y)) = (\mathscr{G}(h)\omega\colon (fh, x) \to (gh, y)).$$

$L\mathscr{G}$ is geometric since $(f, (1_d, x))$ is a universal generator for $(f: c \to d, x)$. For $\phi: \mathscr{G}_0 \to \mathscr{G}_1$ and $c \in \mathrm{ob}\,\mathscr{C}$ define $L\phi(c): L\mathscr{G}_0(c) \to L\mathscr{G}_1(c)$ by

$$L\phi(c)(\omega: (f, x) \to (g, y)) = (\phi(c)\omega: (f, \phi(d)x) \to (g, \phi(d')y)).$$

It is straight forward to check that L is a functor.

There is a natural transformation $\varepsilon: L \to \mathrm{id}$ given for $\mathscr{G}$ and $c \in \mathrm{ob}\,\mathscr{C}$ by

$$\varepsilon_{\mathscr{G}}(c): L\mathscr{G}(c) \to \mathscr{G}(c)$$

$$\varepsilon_{\mathscr{G}}(c)(\omega: (f, x)) \to (g, y)) = (\omega: \mathscr{G}(f)x \to \mathscr{G}(g)y)$$

and $L\varepsilon_{\mathscr{G}} \simeq \varepsilon_{L\mathscr{G}}: LL\mathscr{G} \to L\mathscr{G}$.

PROPOSITION 1.1 *Any $\mathscr{C}$-groupoid $\mathscr{G}$ is weakly equivalent to a geometric $\mathscr{C}$-groupoid. In fact, $\varepsilon_{\mathscr{G}}: L\mathscr{G} \to \mathscr{G}$ is a weak equivalence.*

PROOF. A homotopy inverse to $\varepsilon_{\mathscr{G}}(c)$ is given by $(\omega: x \to y) \mapsto (\omega: (1_c, x) \to (1_c, y))$.

To prove that $\varepsilon_{\mathscr{G}}$ is a homotopy equivalence when $\mathscr{G}$ is geometric I will have to put some restrains on the indexing category $\mathscr{C}$.

DEFINITION 1.2. Let $\mathscr{G}$ be geometric. A choice of universal generators $\{(f_x, z_x)\}_{x \in \mathscr{G}(c), c \in \mathscr{C}}$ is *coherent* if $(f_x f, z_x) = (f_{\mathscr{G}(f)x}, z_{\mathscr{G}(f)x})$ for any $f: d \to d'$ in $\mathscr{C}$ and $x \in \mathrm{ob}\,\mathscr{G}(d')$.

DEFINITION 1.3. The indexing category $\mathscr{C}$ is *amenable* if any geometric $\mathscr{C}$-groupoid has a conherent choice of universal generators.

LEMMA 1.4. *$O(G)$ is amenable.*

PROOF. Let $q_H: G/1 \to G/H$ be $q_H(g) = gH$. For each $g \in G$ and $H \leqq G$ there is a G-map $c_g: G/H \to G/H^{g^{-1}}$, $c_g(aH) = agH^{g^{-1}}$. In particular G acts on the objects of $\mathscr{G}(G/1)$ by $gx = \mathscr{G}(c_g)x$. Choose an object in each orbit of $\mathrm{ob}\,\mathscr{G}(G/1)$ under this action, i.e. a function σ: $\mathrm{ob}\,\mathscr{G}(G/1) \to \mathrm{ob}\,\mathscr{G}(G/1)$ such that $\sigma(gx) = \sigma(x)$ and for each x there is $g \in G$ such that $x = g\sigma(x)$. Choose for each x $g(x) \in G$ such that $x = g(x)\sigma(x)$ and $g(\sigma(x)) = 1$. Choose a universal generator $(f_{\sigma(x)}, z_{\sigma(x)})$ for each $\sigma(x)$. Then $(f_{\sigma(x)} c_{g(x)}, z_{\sigma(x)})$ is universal for x. Let $y \in \mathscr{G}(G/H)$. Then $\mathscr{G}(q_H)y = g(\mathscr{G}(q_H)y)\sigma(\mathscr{G}(q_H)y)$. Let (f, z) be short for

$$(f_{\sigma(\mathscr{G}(q_H)y)} \circ c_{g(\mathscr{G}(q_H)y)}, z_{\sigma(\mathscr{G}(q_H)y)}),$$

the universal generator for $\mathscr{G}(q_H)y$ already chosen. In particular $\mathscr{G}(f)z = \mathscr{G}(q_H)y$. Hence there is a unique map a_y: $G/H \to G/H_z$ ($z \in \mathrm{ob}\,\mathscr{G}(G/H_z)$) such that

$\mathscr{G}(a_y)z = y$. I also have $f = a_y q_H$. A straight forward verification shows that (a_y, z) is a universal generator for y. I claim that

$$\{(a_y, z_{\sigma(\mathscr{G}(q_H)y)})\}$$

is a coherent choice. Suppose f: $G/K \to G/L$ and $y \in \mathscr{G}(G/L)$. Notice that there is a $g \in G$ such that

$$\begin{array}{ccc} G/1 & \xrightarrow{c_g} & G/1 \\ {\scriptstyle q_K}\downarrow & & \downarrow{\scriptstyle q_L} \\ G/K & \xrightarrow{f} & G/L \end{array}$$

commutes. Thus $\sigma(\mathscr{G}(q_K)\mathscr{G}(f)y) = \sigma(g\mathscr{G}(q_L)y) = \sigma(\mathscr{G}(q_L)y)$. Let σ be the common value. Since $(a_{\mathscr{G}(f)y}, z_\sigma)$ is universal, there is a unique map b: $G/L \to G/H_z$ such that $\mathscr{G}(b)z_\sigma = y$ and furthermore $bf = a_{\mathscr{G}(f)y}$. But $\mathscr{G}(a_y)z_\sigma = y$. Thus $b = a_y$ and $a_y f = a_{\mathscr{G}(f)y}$.

PROPOSITION 1.5. *If $\mathscr{C}$ is amenable and $\mathscr{G}$ a geometric $\mathscr{C}$-groupoid, then*

$$\varepsilon = \varepsilon_{\mathscr{G}}: L\mathscr{G} \to \mathscr{G}$$

is a homotopy equivalence.

PROOF. Let $\{(f_x, z_x)\}$ be a coherent choice of universal generators for $\mathscr{G}$. Define η: $\mathscr{G} \to L\mathscr{G}$ for $c \in \operatorname{ob}\mathscr{C}$ by

$$\eta(c): \mathscr{G}(c) \to L\mathscr{G}(c)$$

$$\eta(c)(\omega: x \to y) = (\omega: (f_x, z_x) \to (f_y, z_y)).$$

Using coherence, it is easy to check that η is well-defined. A natural equivalence $\eta\varepsilon \to 1_{L\mathscr{G}}$, is given by $1_{\mathscr{G}(f)x}$, $(f, x) \in \operatorname{ob} L\mathscr{G}(c)$.

Let ℓ: $hg\mathscr{C}$-GP $\to h\mathscr{C}$-GP be the forgetful functor and let $\varepsilon_{\mathscr{G}}^{-1}$ denote a homotopy inverse to $\varepsilon_{\mathscr{G}}$ if $\mathscr{G}$ is geometric and $\mathscr{C}$ amenable. Notice that the η constructed above is not natural, however its homotopy class is since ε is. Also, observe that L preserves the homotopy relation.

Theorem 1.6. *If $\mathscr{C}$ is amenable then L: $h\mathscr{C}$-GP $\to hg\mathscr{C}$-GP is a right adjoint to ℓ. That is*

$$[\mathscr{G}_0, L\mathscr{G}_1]_{g\mathscr{C}} \cong [\ell\mathscr{G}_0, \mathscr{G}_1]_{\mathscr{C}}$$

and the bijection is natural.

PROOF. By (1.1), $[\varepsilon_{\mathscr{G}}^{-1}]$ and $[\varepsilon_{\mathscr{G}}]$ is the unit and counit respectively.

PROPOSITION. *Suppose $\mathscr{C}$ amenable and that ϕ: $\mathscr{G}_0 \to \mathscr{G}_1$ is a weak equivalence. Then $L\phi$ is a homotopy equivalence.*

PROOF. For each $d \in \mathrm{ob}\,\mathscr{C}$ choose $\psi(d)\colon \mathscr{G}_1(d) \to \mathscr{G}_0(d)$ and $\eta(d)\colon 1_{\mathscr{G}_1(d)} \to \phi\psi(d)$. (Such choices exist by assumption). Define $\hat{\psi}.\ L\mathscr{G}_1 \to L\mathscr{G}_0$ as follows. For $c \in \mathrm{ob}\,\mathscr{C}$ and $(\omega\colon (f, x) \to (g, y))$ in $L\mathscr{G}_1(c)$ there is a diagram

$$\begin{array}{ccc} \mathscr{G}_1(f)x & \xrightarrow{\mathscr{G}_1(f)\eta(d_x)} & \mathscr{G}_1(f)\phi\psi(d_x)x = \phi(c)\mathscr{G}_0(f)\psi(d_x)x \\ \omega\downarrow & & \downarrow\vartheta \\ \mathscr{G}_1(g)y & \xrightarrow{\mathscr{G}_1(g)\eta(d_y)} & \mathscr{G}_1(g)\phi\psi(d_y)y = \phi(c)\mathscr{G}_0(g)\psi(d_y)y \end{array}$$

(all maps are isomorphisms and ϑ is the unique map determined by commutativity). Notice that the source and target of ϑ are in the image of $\phi(c)$. Hence there is a unique $\omega'\colon \mathscr{G}_0(f)\psi(d_x)x \to \mathscr{G}_0(g)\psi(d_y)y$ such that $\vartheta = \phi(c)\omega'$. Let

$$\hat{\psi}(c)(\omega\colon (f, x) \to (g, y)) = (\omega'\colon (f, \psi(d_x)x) \to (g, \psi(d_y)y)).$$

$\hat{\psi}(c)$ is a functor $L\mathscr{G}_1(c) \to L\mathscr{G}_0(c)$. It is easy to see that $\hat{\psi}$ is in fact a map $L\mathscr{G}_1 \to L\mathscr{G}_0$. It remains to find $\nu\colon L\phi \circ \hat{\psi} \to 1$ and $\mu\colon 1 \to \hat{\psi} \circ L\phi$. ν is given by $\nu_{(f,x)} = \mathscr{G}_1(f)\eta(d_x)$. To construct μ, consider

$$\mathscr{G}_1(f)\eta(d_x)\colon \mathscr{G}_1(f)\eta(d_x)\colon \mathscr{G}_1(f)(\phi(d_x)x) \to \mathscr{G}_1(f)\phi\psi(\phi(d_x)x).$$

Rewriting gives

$$\mathscr{G}_1(f)\eta(d_x)\colon \phi(c)(\mathscr{G}_0(f)x) \to \phi(c)(\mathscr{G}_0(f)\psi\phi(d_x)x).$$

Since the source and target are in the image of $\phi(c)$, there is a unique map

$$\mu_{(f,x)}\colon \mathscr{G}_0(f) \to \mathscr{G}_0(f)\psi\phi(d_x)x.$$

COROLLARY 1.8. *Let $\mathscr{C}$ be amenable. If $\phi\colon \mathscr{G}_0 \to \mathscr{G}_1$ is a weak equivalence between geometric objects, then ϕ is in fact a homotopy equivalence.*

PROOF. Consider

$$\begin{array}{ccc} L\mathscr{G}_0 & \xrightarrow{L\phi} & L\mathscr{G}_1 \\ \varepsilon_0\downarrow & & \downarrow\varepsilon_1 \\ \mathscr{G}_0 & \xrightarrow{\phi} & \mathscr{G}_1 \end{array}$$

By (1.5) and (1.7) ε_0, ε_1 and $L\phi$ are homotopy equivalences. Hence so is ϕ.

COROLLARY 1.9. *Let $\mathscr{C}$ be amenable. A map $\phi\colon \mathscr{G}_0 \to \mathscr{G}_1$ is a weak equivalence if and only if*

$$\phi_*\colon [\mathscr{G}, \mathscr{G}_0] \to [\mathscr{G}, \mathscr{G}_1]$$

is a bijection for all geometric $\mathscr{C}$-groupoids $\mathscr{G}$.

PROOF. Suppose ϕ is a weak equivalence. By (1.7) $L\phi$ is a homotopy equivalence, by (1.6) the following diagram commutes

$$\begin{array}{ccc} [\mathcal{G}, L\mathcal{G}_0 & \xrightarrow{L\phi_*} & [\mathcal{G}, L\mathcal{G}_1] \\ \cong\downarrow & & \downarrow\cong \\ [\mathcal{G}, G_0 & \xrightarrow{\phi_*} & [\mathcal{G}, \mathcal{G}_1] \end{array}$$

Suppose on the other hand that ϕ_* is a bijection. Setting $\mathcal{G} = L\mathcal{G}_1$, there is a map $\psi. L\mathcal{G}_1 \to \mathcal{G}_0$ such that $\phi\psi \simeq \varepsilon_{\mathcal{G}_1}$. By (1.1) $\varepsilon_{\mathcal{G}_1}(c)$ is bijective on iso-classes objects and hom-sets. Hence $\psi(c)$ is injective in the same way. Letting $\mathcal{G} = L\mathcal{G}_0$, I get $\phi\varepsilon_{\mathcal{G}_0} = \varepsilon_{\mathcal{G}_1}$: $L\phi \simeq \phi\psi L\phi$. Hence since ϕ_* is injective $\varepsilon_{\mathcal{G}_0} \simeq \psi L\phi$. Thus $\psi(c)$ is surjective on iso-classes of objects and hom-sets. From $\phi\psi \simeq \varepsilon_{\mathcal{G}_1}$ I deduce that $\phi(c)$ is bijective on iso-classes of objects and hom-sets since ψ and ε are so.

REMARK 1.10. Some authors (e.g. [1]) have considered the notion of *groupoids over* $\mathcal{C}$. These are closely related to $\mathcal{C}$-groupoids. Following [1] p.9 a groupoid over $\mathcal{C}$ is a functor $\phi: \mathcal{D} \to \mathcal{C}$ such that

i) for each $b \in \mathrm{ob}\,\mathcal{C}$, the (honest) fibre $\mathcal{D}(b)$ over b is a groupoid.

ii) for $y \in \mathcal{D}$ and $f: c \to \phi(y)$ in $\mathcal{C}$, there is a $\omega: x \to y$ in $\mathcal{D}$ such that $\phi(\omega) = f$

iii) for $\omega: x \to y$ and $\omega': x' \to y$ in $\mathcal{D}$ and $f: \phi(x) \to \phi(x')$ in $\mathcal{C}$ there is a unique $\delta: x \to x'$ in $\mathcal{D}$ such that $\phi(\delta) = f$ and $\omega'\delta = \omega$.

Now, iii) is stronger than the axiom used in [1] (the weaker version of iii) relates to the study of actions of compact Lie-groups). Groupoids over $\mathcal{C}$ and $\mathcal{C}$-groupoids are related as follows. Let $\phi: \mathcal{D} \to \mathcal{C}$ be a groupoid over $\mathcal{C}$. Define a $\mathcal{C}$-groupoid $d\phi \in \mathcal{C}$-GP:

An object of $d\phi(c)$ is a functor $\eta: \mathcal{C}/c \to \mathcal{D}$ such that

$$\begin{array}{ccc} \mathcal{C}/c & \xrightarrow{\eta} & \mathcal{D} \\ & \searrow & \downarrow\phi \\ & & \mathcal{C} \end{array}$$

commutes. A morphism $\eta \to \eta_1$ is a natural transformation $\omega: \eta \to \eta_1$ such that $\phi\omega = \mathrm{id}$. It is easy to complete the definition.

Given $\mathcal{G} \in \mathcal{C}$-GP let $\int \mathcal{G}$ be the category with objects all pairs (c, x) where $c \in \mathcal{C}$ and $x \in \mathcal{G}(c)$. A morphism $(c, x) \to (d, y)$ is a pair (f, ω) where $f: c \to d$ and $\omega: \mathcal{G}(f)y \to x$. There is a functor $p: \int \mathcal{G} \to \mathcal{C}$ defined as $p((f, \omega): (c, x) \to (d, y)) = (f: c \to d)$.

Now there is a weak equivalence of $\mathcal{C}$-groupoids $\mathcal{G} \to d\int \mathcal{G}$ and an equivalence of groupoids over $\mathcal{C}$, $\int d\phi \simeq \phi$.

§2. The topology.

Recall that there is a functor B: Cat $\to$ Spaces, which to a small category $\mathcal{D}$ associates its classifying space $B\mathcal{D}$. Objects of $\mathcal{D}$ determine points in $B\mathcal{D}$ and morphisms determine paths in $\mathcal{D}$. In fact this determines a natural functor

$$\alpha\colon \mathcal{D} \to \pi B\mathcal{D} \tag{2.1}$$

from $\mathcal{D}$ to the fundamental groupoid of $B\mathcal{D}$.

LEMMA 2.2. *If $\mathcal{D}$ is a groupoid, then α is an equivalence of categories.*

PROOF. Since $\pi(\ )$ and $B(\ )$ preserves disjoint unions on reduces to the case when $\mathcal{D}$ has only one iso-class of objects, but then $\mathcal{D}$ is equivalent to the automorphism group of any object of $\mathcal{D}$. Thus it suffices to prove 2.2 for $\mathcal{D}$ a group. This is well known.

For an $O(G)$-groupoid $\mathcal{G}$, let $B\mathcal{G}$ denote the functor $O(G)^{\mathrm{OP}} \to$ Groupoids $\subset$ Cat $\to$ Spaces. Thus $(B\mathcal{G})(G/H)$ is the classifying space for $\mathcal{G}(G/H)$. Notice that $B\mathcal{G}$ is an $O(G)$-Space, i.e. a functor $O(G)^{\mathrm{OP}} \to$ Spaces. To any such functor T, there is an $O(G)$-groupoid πT defined by

$$(\pi T)(G/H \xrightarrow{f} G/K) = (\pi(T(G/H)) \xleftarrow{TF_*} \pi(T(G/K))).$$

Since the map in (2.1) is natural, I have a map of $O(G)$-groupoids

$$\mathcal{G} \xrightarrow{\alpha} \pi B\mathcal{G} \tag{2.3}$$

which is a weak equivalence by (2.2). Recall [2], [5] that for a OG-space T there is an associated G-space $B(\iota, \mathrm{OG}, T)$ where ι: OG $\hookrightarrow$ G-Top and a map (natural transformation) of OG-spaces β: $B(\iota, \mathrm{OG}, T)^{(-)} \to T$ which is a pointwise homotopy equivalence. Hence, $\pi B(\iota\, \mathrm{OG}, T)^{(-)} \to \pi T$ is a weak equivalence of OG-groupoids. For an OG-groupoid $\mathcal{G}$, $K(\mathcal{G}, 1)$ is defined als $B(\iota, \mathrm{OG}, B\mathcal{G})$. (Alternatively, $K(\mathcal{G}, 1) \simeq G \operatorname{hocolim}_{\int \mathcal{G}} \iota p$ (cf. 1.10)).

THEOREM 2.7. *There is a weak equivalence of $O(G)$-groupoids*

$$\mu\colon \pi^G K(\mathcal{G}, 1) \to \mathcal{G}$$

determined up to homotopy by $\alpha\mu \simeq \pi\beta$.

PROOF. By (2.3) there is a $\alpha\colon \mathcal{G} \to \pi B\mathcal{G}$. By (1.9) it induces bijection

$$\alpha_*\colon [\pi^G K(\mathcal{G}, 1), \mathcal{G}] \xrightarrow{\cong} [\pi^G K(\mathcal{G}, 1), \pi B\mathcal{G}]$$

since $\pi^G X$ is geometric for any G-space. Hence there is a unique homotopy class $[\mu]$ $\mu\colon \pi^G K(\mathcal{G}, 1) \to \mathcal{G}$ such that $\alpha\mu \simeq \pi\beta$. By (2.2) and (2.5) α and $\pi\beta$, respectively, are weak equivalences. Thus μ is a weak equivalence.

COROLLARY 2.8. *$\pi^G K(\mathscr{G}, 1)$ is homotopy equivalent to $L\mathscr{G}$.*

PROOF. By (1.1) there is a weak equivalence $\varepsilon: L\mathscr{G} \to \mathscr{G}$. By (1.9) the map μ of (2.7) factors up to homotopy $\mu \simeq \varepsilon\alpha$, some weak equivalence $\alpha: \pi^G K(\mathscr{G}, 1) \to L\mathscr{G}$. By (1.8) α is a homotopy equivalence.

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DEPT. OF MATHEMATICS
UNIVERSITY OF GÖTEBORG
S-412 96 GÖTEBORG
SWEDEN