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TERNARY ADDITIVE PROBLEMS OF WARING'S TYPE

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Abstract.

New upper bounds are obtained for the numbers of integers not exceeding X and not being the sum of a square, a cube and a kth power of natural numbers. An important ingredient is a certain fourth power moment estimate for a weighed cubic exponential sum.

1. Introduction.

In this paper we shall be concerned with representations of natural numbers as the sum of a square, a cube and a kth power of natural numbers. If we write $r_k(n)$ for the number of representations of an integer in the proposed manner, then one expects an asymptotic formula of the shape

$$r_k(n) \sim C_k \mathfrak{S}(n) n^{1/k-1/6}$$

to hold whenever $2 \leqq k \leqq 5$. Here C_k is a positive constant, and $\mathfrak{S}(n)$ is the standard singular series which, however, is more difficult than usual but can be shown to be $\gg n^{-\varepsilon}$. In particular it would follow that $r_k(n) > 0$ for all sufficiently large n.

Of course a proof of these asymptotic formulae is out of the scope of existing methods. But it can be shown that almost all natural numbers can be written as the sum of a square, a cube and a kth power. To be more precise let $E_k(X)$ be the number of all $n \leqq X$ which are not so representable. Then $E_k(X) = o(X)$ when $2 \leqq k \leqq 5$. This has been shown by various writers, see Vaughan [11], §8.1, and Hooley [7] for an account. More recently Vaughan [10] found $E_k(X) \ll X^{1-\delta}$ for some $\delta = \delta_k > 0$, and in chapter 4 of [3] the author obtained explicit values for δ_k, namely $\delta_2 = \frac{1}{3} - \varepsilon$, $\delta_3 = \frac{5}{42} - \varepsilon$, $\delta_4 = \frac{1}{18} - \varepsilon$, $\delta_5 = \frac{1}{42} - \varepsilon$.

Here we shall describe an approach which is rather different from [3], much simpler, and produces better results.

THEOREM 1. *Let $E_k(X)$ be the number of natural numbers not exceeding X and not being representable as the sum of a square, a cube and a kth power of integers. Then $E_3(X) \ll X^{6/7+\varepsilon}$, $E_4(X) \ll X^{13/14+\varepsilon}$, $E_5(X) \ll X^{29/30+\varepsilon}$.*

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The improvement comes from the new application of a Kloosterman refinement to a certain fourth moment of a cubic exponential sum. Since this mean value result might have other applications in the additive theory of numbers we shall now formulate it precisely. Introduce the weights

$$\gamma(t) = \exp(-1/(1-t^2)) \tag{1}$$

and $\Gamma(t) = \gamma(t-1)$. Then, using the abbreviation $e(\alpha) = \exp(2\pi i\alpha)$, we let

$$f(\alpha) = \sum_{x \leqq 2N} \Gamma\left(\frac{x}{N}\right) e(\alpha x^3). \tag{2}$$

Now let $1 \leqq P \leqq N^{3/2}$, and let $\mathfrak{M}(q,a)$ denote the interval $|q\alpha - a| \leqq P/N^3$. Write $\mathfrak{M}$ for the union of all $\mathfrak{M}(q,a)$ subject to $1 \leqq q \leqq P$ and $(a,q) = 1$. In this notation we can enunciate

THEOREM 2. *In the above notation,*

$$\int_{\mathfrak{M}} |f(\alpha)|^4 \, d\alpha \ll N^{\varepsilon}(N + P^{7/2}N^{-3} + P^2 N^{-1}).$$

It is easy to see that

$$\int_0^1 |f(\alpha)|^4 \, d\alpha = \sum_{\substack{0 \leqq x_1, \ldots, x_4 \leqq 2N \\ x_1^3 + x_2^3 = x_3^3 + x_4^3}} \prod_{i=1}^{4} \Gamma\left(\frac{x_i}{N}\right) \ll N^{2+\varepsilon}, \tag{3}$$

and with little more care it is possible to show that this integral is of order N^2. Therefore, Theorem 2 gives non-trivial results whenever $P \leqq N^{10/7}$. It is also not difficult to show that

$$\int_{\mathfrak{M}} |f(\alpha)|^4 \, d\alpha \gg N^{1-\varepsilon}.$$

Hence Theorem 2 is essentially best possible when $P \leqq N$. A further discussion of the Theorem, as well as an outline of the proof is postponed to the later sections.

2. A cubic exponential sum.

Our proof of Theorem 2 will follow the pattern established by Hooley [8]. The crucial aspect is that we are able to sum nontrivially the contribution arising from different $\mathfrak{M}(q,a)$ with q fixed. This approach nowadays is called a Kloosterman refinement, and we shall be able to give an unconditional treatment. In contrast, Hooley applies a double Kloosterman refinement, that is, summing nontrivially over q also, and it is here where Hooley assumes the truth of the Riemann hypothesis for Hasse-Weil L-functions of certain cubic threefolds.

At the very beginning we follow Hooley quite closely. By (1), (2) and the Poisson summation formula,

$$(4)\qquad f\left(\frac{a}{q}+\beta\right)=\sum_{r=1}^{q}\sum_{\substack{x\in Z\\ x\equiv r\,(\mathrm{mod}\,q)}} e(\beta x^3)e\left(\frac{ax^3}{q}\right)\Gamma\left(\frac{x}{N}\right)$$

$$=q^{-1}\sum_{m\in Z} S(q,a,m)\,J\left(\beta,\frac{m}{q}\right)$$

where

$$(5)\qquad S(q,a,b)=\sum_{x=1}^{q} e\left(\frac{ax^3-bx}{q}\right),$$

$$(6)\qquad J(\beta,\gamma)=\int_0^{2N}\Gamma\left(\frac{t}{N}\right)e(\beta t^3+\gamma t)\,dt.$$

For brevity we also write $S(q,a)=S(q,a,0)$, $J(\beta)=J(\beta,0)$, and define

$$(7)\qquad D(\alpha)=D(\alpha,q,a)=f(\alpha)-q^{-1}S(q,a)\,J\left(\alpha-\frac{a}{q}\right)$$

$$=q^{-1}\sum_{m\neq 0} S(q,a,m)\,J\left(\alpha-\frac{a}{q},\frac{m}{q}\right).$$

The final identity follows from (2). The difficult part of the paper is proving the following estimate.

LEMMA 1.

$$\int_{\mathfrak{M}}|D(\alpha)|^4\,d\alpha \ll P^{7/2}N^{\varepsilon-3}+P^2N^{\varepsilon-1}.$$

Most of the terms in (7) make a relatively small contribution to the sum over m. Let $|\beta/\gamma|\geqq 24N^2$. Then the proof of Lemma 1 of Hooley [8] is readily adopted to show that

$$(8)\qquad J(\beta,\gamma)\ll Ne^{-\delta(N|\gamma|)^{1/3}}$$

for some $\delta>0$. Now let W be a parameter given by

$$(9)\qquad W=W(q,\beta,N)=(\log N)^4\max(N^2q|\beta|,qN^{-1})$$

where $\beta = \alpha - \frac{a}{q}$, and split $D(\alpha)$ as

$$D(\alpha, q, a) = D_1(\alpha, q, a) + D_2(\alpha, q, a)$$

where D_1 is the part of the sum in (7) where $|m| > W$, and D_2 is the part with $0 < |m| \leqq W$. By (8), (9), the trivial bound for $S(q, a, m)$ and Lemma 4 of Hooley [8],

$$D_1(\alpha, q, a) \ll (N + q) \sum_{|m| > W} e^{-\delta(|m|N/q)^{1/3}} \ll 1. \tag{10}$$

The measure of $\mathfrak{M}$ is $\ll P^2/N^3$, so that

$$\int_{\mathfrak{M}} |D_1(\alpha)|^4 \, d\alpha \ll P^2 N^{-3} \tag{11}$$

which is acceptable. Note that if $P \leqq \frac{1}{2}N(\log N)^{-4}$ then $W < 1$ by (9). Hence we also have:

LEMMA 2. *Let* $P \leqq \frac{1}{2}N(\log N)^{-4}$ *and* $\alpha \in \mathfrak{M}$. *Then* $D(\alpha) \ll 1$.

The treatment of D_2 is more interesting. Here we have

$$\int_{\mathfrak{M}} |D_2(\alpha)|^4 \, d\alpha = \sum_{q \leqq P} q^{-4} \int_{-P/qN^3}^{P/qN^3} G(\beta, q) \, d\beta \tag{12}$$

where

$$G(\beta, q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left| \sum_{0 < |m| < W} S(q, a, m) J\left(\beta, \frac{m}{q}\right) \right|^4.$$

Note that W is independent of a. Since $S(q, a, b)$ is real (at once from (5)) we may rewrite this as

$$G(\beta, q) = \sum_{\substack{0 < |m_i| \leqq W \\ 1 \leqq i \leqq 4}} Q(\boldsymbol{m}, q) H(\beta, q^{-1}\boldsymbol{m}) \tag{13}$$

where $\boldsymbol{m} = (m_1, m_2, m_3, m_4)$, and

$$Q(\boldsymbol{m}, q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} S(q, a, m_1) \ldots S(q, a, m_4), \tag{14}$$

$$H(\beta, \boldsymbol{m}) = J(\beta, m_1) J(\beta, m_2) \bar{J}(\beta, m_3) \bar{J}(\beta, m_4). \tag{15}$$

Further progress on the mean value (12) will therefore depend on estimates for $Q(\boldsymbol{m}, q)$ and $H(\beta, \boldsymbol{m})$ which we shall deduce in the next two sections.

3. The properties of $Q(\boldsymbol{m}, q)$.

We shall first state a lemma giving bounds for $Q(\boldsymbol{m}, q)$ we can prove by traditional methods.

LEMMA 3. *As an arithmetical function of q, $Q(\boldsymbol{m}, q)$ is multiplicative. Let $\omega(q)$ denote the number of different prime divisors of q, and let $\tilde{\omega}(q)$ denote the multiplicative function defined by $\tilde{\omega}(p) = 1$, and $\tilde{\omega}(p^a) = p^{a/4}$ if $a > 1$. Then*

$$Q(\boldsymbol{m}, q) \ll A^{\omega(q)} q^3 \prod_{1 \leqq i \leqq 4} (q, m_i)^{1/4}$$

and

$$Q(\boldsymbol{m}, q) \ll A^{\omega(q)} q^3 \prod_{1 \leqq i \leqq 4} \tilde{\omega}(m_i)$$

where $A > 0$ is an absolute constant.

PROOF. See lemmata 5, 8, and 9 of Hooley [8].

We now introduce the cubic form

$$g(\boldsymbol{x}) = x_1^3 + x_2^3 + x_3^3 + x_4^3$$

and let $\nu(q)$ denote the number of incongruent solutions of the congruence of the congruence $g(\boldsymbol{x}) \equiv (\mathrm{mod}\, q)$. Furthermore, writing $\boldsymbol{mx}$ for the scalar product $m_1 x_1 + \ldots + m_4 x_4$, we let $\nu(q, \boldsymbol{m})$ denote the number of incongruent solutions of the simultaneous congruences $g(\boldsymbol{x}) \equiv \boldsymbol{mx} \equiv 0 \, (\mathrm{mod}\, q)$. We shall make use also of the discriminant

$$\Delta(\boldsymbol{m}) = 3 \prod (m_1^{3/2} \pm m_2^{3/2} \pm m_3^{3/2} \pm m_4^{3/2}) \tag{16}$$

where the product is over all choices of the ambigious signs.

LEMMA 4. *If $\Delta(\boldsymbol{m}) \not\equiv 0 \, (\mathrm{mod}\, p)$, then*

$$Q(\boldsymbol{m}, p) = \frac{p}{p-1} (p\nu(p, \boldsymbol{m}) - \nu(p))$$

and, whenever $a > 1$,

$$Q(\boldsymbol{m}, p^a) = 0.$$

PROOF. This again can be shown as lemmata 6 and 7 of Hooley [8].

We may use the first equality in Lemma 4 to apply the theory of local L-functions to the study of $Q(\boldsymbol{m}, p)$, at least when $\Delta(\boldsymbol{m}) \not\equiv 0 \pmod p$. Let $\mathscr{V}$ and $\mathscr{V}(\boldsymbol{m})$ denote the projective varieties over $\mathbb{Q}$, defined by $g(\xi) = 0$, and $g(\xi) = \boldsymbol{m}\xi = 0$, respectively. Here $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ is a point in threedimensional projective space over $\mathbb{Q}$. If $p \nmid \Delta(\boldsymbol{m})$, so that $p \neq 3$, we may interpret these equations as equations in the field F_p of p elements. This leads to the nonsingular varieties $\mathscr{V}(p)$ and $\mathscr{V}(\boldsymbol{m}, p)$ that are defined over F_p. Now $\mathscr{V}(p)$ is a surface, and $\mathscr{V}(\boldsymbol{m}, p)$ is an imbedding in three-space of a curve lying in the plane $\boldsymbol{m}\xi = 0$. We let $\varrho(p^r)$ and $\varrho(\boldsymbol{m}, p^r)$ be the number of points on $\mathscr{V}(p)$ and $\mathscr{V}(\boldsymbol{m}, p)$ respectively, having coordinates in F_{p^r}. Then

$$v(p) = (p-1)\varrho(p) + 1; \quad v(p, \boldsymbol{m}) = (p-1)\varrho(\boldsymbol{m}, p) + 1,$$

and by Lemma 4, $Q(\boldsymbol{m}, p) = p(p\varrho(\boldsymbol{m}, p) - \varrho(p) + 1)$. This we rewrite as

$$Q(\boldsymbol{m}, p) = p(pE(\boldsymbol{m}, p) - E(p)) \tag{17}$$

where

$$E(p^r) = \varrho(p^r) - \frac{p^{3r}-1}{p^r-1}; \quad E(\boldsymbol{m}, p^r) = \varrho(\boldsymbol{m}, p^r) - \frac{p^{2r}-1}{p^r-1}.$$

Next, we consider the L-functions

$$L(p; T) = \exp\left(- \sum_{r=1}^{\infty} \frac{E(p^r)}{r} T^r \right), \tag{18}$$

$$L(\boldsymbol{m}, p; T) = \exp\left(- \sum_{r=1}^{\infty} \frac{E(\boldsymbol{m}, p^r)}{r} T^r \right). \tag{19}$$

Here (18) is the quotient of the zeta functions of three-space and of $\mathscr{V}(p)$, and (19) is the quotient of the zeta functions of the projective plane, and of $\mathscr{V}(m, p)$. By Weil's theory ([12], [9], see also [6]), the Riemann hypothesis for the L-functions (19) holds, a fact which at once implies the important inequality

$$E(\boldsymbol{m}, p) \ll p^{1/2}.$$

Similarly, Weil's theory gives $E(p) \ll p^{3/2}$, a relatively weak bound which, however, suffices for this paper, and avoids reference to even deeper results in algebraic geometry. We now deduce from (17) the important

LEMMA 5. *If* $\Delta(\boldsymbol{m}) \not\equiv 0 \pmod p$ *then*

$$Q(\boldsymbol{m}, p) \ll p^{5/2}.$$

Given $\boldsymbol{m}$ with $\Delta(\boldsymbol{m}) \neq 0$, write $q = q_1 q_2$ where $(q_1, \Delta(\boldsymbol{m})) = 1$ and all prime factors of q_2 divide $\Delta(\boldsymbol{m})$. Then $(q_1, q_2) = 1$, and by Lemmas 3, 4, and 5

$$\sum_{q \leqq X} \frac{|Q(\boldsymbol{m}, q)|}{q^{5/2}} \ll X^{\varepsilon} \prod_{i=1}^{4} \tilde{\omega}(m_i) \sum_{q_1 q_2 \leqq X} q_2^{1/2} \ll X^{1+\varepsilon} \prod_{i=1}^{4} \tilde{\omega}(m_i) \sum_{q_2 \leqq X} q_2^{-1/2}.$$

Thus, supposing further that $\|\boldsymbol{m}\| \leqq W$, this estimation shows

LEMMA 6. *If* $\|\boldsymbol{m}\| \leqq W$ *and* $\Delta(\boldsymbol{m}) \neq 0$,

$$\sum_{q \leqq X} \frac{|Q(\boldsymbol{m}, q)|}{q^{5/2}} \ll W^{\varepsilon} X^{1+\varepsilon} \prod_{i=1}^{4} \tilde{\omega}(m_i).$$

4. The integrals $J(\beta, \gamma)$.

The object of this section is the following bound.

LEMMA 7. *Whenever* $\beta\gamma \neq 0$, *then* $J(\beta, \gamma) \ll |\beta\gamma|^{-1/4}$, *and* $J(\beta, 0) \ll |\beta|^{-1/3}$.

PROOF. This is by the same method as Lemma 2 of Hooley [8]. We first split the integral (6) as

$$J(\beta, \gamma) = \int_0^N \Gamma\left(\frac{t}{N}\right) e(\beta t^3 + \gamma t) dt + \int_N^{2N} \Gamma\left(\frac{t}{N}\right) e(\beta t^3 + \gamma t) dt$$
$$= J_1(\beta, \gamma) + J_2(\beta, \gamma), \text{ say.}$$

This has the advantage that $\Gamma(t/N)$ is monotone in the range of integration in both integrals. In view of the mean value theorem it is now advisable to consider the integrals

$$J(\beta, \gamma; \xi, \eta) = \int_{\xi}^{\eta} \cos(2\pi(\beta t^3 + \gamma t)) dt,$$

$$I(\beta, \gamma; \xi, \eta) = i \int_{\xi}^{\eta} \sin(2\pi(\beta t^3 + \gamma t)) dt$$

in the range $0 \leqq \xi < \eta$. Then, on pp. 57–58, Hooley [8] shows that

$$J(\beta, \gamma, \xi, \eta) \ll |\beta\gamma|^{-1/4}, \text{ and } J(\beta, 0, \xi, \eta) \ll |\beta|^{-1/3}$$

hold for any such choice of ξ, η, and remarks on p. 59 that the same bounds do hold as well for the integrals $I(\beta, \gamma; \xi, \eta)$. Now, by the second mean value theorem,

$$\operatorname{Re} J_1(\beta, \gamma) = \Gamma(N) J(\beta, \gamma; \vartheta, N);$$

for some ϑ, and similarly, $\operatorname{Im} J_1(\beta, \gamma)$ is reduced to $I(\beta, \gamma; \xi, \eta)$. Since $\Gamma(t)$ is bounded, this gives an acceptable bound for J_1, and J_2 can be treated in the same way. This proves the Lemma.

5. Completion of the proof of Lemma 1.

The results of the previous three sections are now put together to prove Lemma 1. Let $G_1(\beta, q)$ denote the sum in (13) subject to the additional constraint $\Delta(\boldsymbol{m}) \neq 0$, and let $G_2(\beta, q)$ be the sum in (13) restricted to the complementary condition $\Delta(\boldsymbol{m}) = 0$. For $1 \leqq R \leqq P$ let

$$\Theta_j(R) = \sum_{R<q\leqq 2R} q^{-4} \int_{-P/RN^3}^{P/RN^3} G_j(\beta, q)\, d\beta \tag{20}$$

Since $G(\beta, q) = G_1(\beta, q) + G_2(\beta, q)$ we find from (12) that

$$\int_{\mathfrak{M}} |D_2(\alpha)|^4 d\alpha \ll (\log P) \max_{1\leqq R\leqq P} (\Theta_1(R) + \Theta_2(R)) \tag{21}$$

Before we proceed further it is useful to introduce the notation

$$a(m) = a(m; \beta, R) = \begin{cases} N & \text{if } |\beta| \leqq N^{-3} \\ R^{1/4}|m\beta|^{-1/4} & \textit{if } |\beta| > N^{-3} \end{cases} \tag{22}$$

for any integer $m \neq 0$. By (15) and Lemma 7,

$$H(\beta, \boldsymbol{m}) \ll a(m_1)\, a(m_2)\, a(m_3)\, a(m_4) \tag{23}$$

Thus, by (20),

$$\begin{aligned} &\Theta_1(R) \\ &\ll \sum_{R<q\leqq 2R} q^{-4} \int_{-P/RN^3}^{P/RN^3} \sum_{\substack{0<\|\boldsymbol{m}\|<W \\ \Delta(\boldsymbol{m})\neq 0}} |Q(\boldsymbol{m}, q)|\, a(m_1)\, a(m_2)\, a(m_3)\, a(m_4)\, d\beta \\ &\ll R^{-\frac{3}{2}} \int_{-P/RN^3}^{P/RN^3} \sum_{\substack{0<\|\boldsymbol{m}\|\leqq W_0 \\ \Delta(\boldsymbol{m})\neq 0}} \sum_{q\leqq 2R} \frac{|Q(\boldsymbol{m}, q)|}{q^{5/2}}\, a(m_1)\, a(m_2)\, a(m_3)\, a(m_4)\, d\beta \end{aligned}$$

where $W_0 = \max W$ when q runs over $[R, 2R]$. By Lemma 6,

$$\Theta_1(R) \ll N^{\varepsilon} R^{-\frac{1}{2}} \int_{-P/RN^3}^{P/RN^3} \sum_{0<\|\boldsymbol{m}\|\leqq W_0} \prod_{j=1}^{4} \tilde{\omega}(m_j)\, a(m_j)\, d\beta.$$

By (22), Lemma 12 of Hooley [8], and (9), this is

$$\ll N^{\varepsilon} R^{-\frac{1}{2}} \left(\int_{0}^{N^{-3}} W_0^4 N^4 \, d\beta + \int_{N^{-3}}^{P/RN^3} R W_0^3 |\beta|^{-1} \, d\beta \right)$$

$$\ll N^{\varepsilon} R^{-\frac{1}{2}} \left(R^4 N^{-3} + \int_{0}^{P/RN^3} R\beta^{-1} (N^2 R\beta)^3 \, d\beta \right)$$

$$\ll N^{\varepsilon} R^{-\frac{1}{2}} (R^4 N^{-3} + P^3 R N^{-3})$$

so that if $R \leqq P$, it follows that

$$\Theta_1(R) \ll P^{7/2} N^{\varepsilon - 3}. \tag{24}$$

We now turn our attention to $\Theta_2(R)$. At the very beginning, the treatment is much the same as the one of $\Theta_1(R)$. By (20), (23) and Lemma 3,

$$\Theta_2(R) \ll R^{\varepsilon - 1} \int_{-P/RN^3}^{P/RN^3} \sum_{R < q \leqq 2R} \sum_{\substack{0 < \|\boldsymbol{m}\| \leqq W \\ \Delta(\boldsymbol{m}) = 0}} \prod_{1 \leqq j \leqq 4} (q, m_j)^{\frac{1}{4}} a(m_j) \, d\beta, \tag{25}$$

and further progress is dependent on a study of the equation $\Delta(\boldsymbol{m}) = 0$. We follow Hooley [8], p. 82, but the situation is somewhat simpler.

For any solution of $\Delta(\boldsymbol{m}) = 0$, let $m_j^3 = b_j c_j^2$ where b_j is squarefree and $c_j > 0$. We may suppose that $0 < m_j \leqq W$. By (16) we must have

$$c_1 \sqrt{b_1} \pm \ldots \pm c_4 \sqrt{b_4} = 0$$

for some choice of the ambigious signs. Let $d_1, \ldots, d_l$ be the distinct values of b_1, b_2, b_3, b_4. Then

$$e_1 \sqrt{d_1} + \ldots + e_l \sqrt{d_l} = 0$$

for some $e_j \in \mathbf{Z}$. Since the d_i are all distinct, the $\sqrt{d_i}$ are linearly independent over $\mathbf{Q}$. Thus $e_j = 0$ for $1 \leqq j \leqq l$; that is, a certain sum of the c_j has to vanish. This can only happen if and only if

$$b_1 = b_2 = b_3 = b_4 = b, \text{ say} \tag{26}$$

or

$$m_1 = m_2, \; m_2 = m_4 \tag{27}$$

after renumbering. In case (26), let $c = (c_1, c_2, c_3, c_4)$ and $m_j^3 = b_j c_j^2 = bc^2 \tilde{c}_j^2$ so that $(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \tilde{c}_4) = 1$. Hence

$$(m_1, m_2, m_3, m_4)^3 = bc^2 = \lambda^3; \text{ say.}$$

Therefore $\tilde{c}_j = \tilde{m}_j^3$ for some $\tilde{m}_j \in \mathbf{Z}$ which gives

$$\mathbf{m} = \lambda(\tilde{m}_1^2, \ldots, \tilde{m}_4^2). \tag{28}$$

Now we have

$$\begin{aligned}
&\sum_{\substack{0 < \|\mathbf{m}\| \leq W \\ \Delta(\mathbf{m}) = 0}} \prod_{1 \leq j \leq 4} (q, m_j)^{\frac14} a(m_j) \\
&\ll \sum_{\substack{0 < \|\mathbf{m}\| \leq W \\ \mathbf{m} = \lambda(\tilde{m}_1^2, \ldots, \tilde{m}_4^2)}} \prod_{1 \leq j \leq 4} (q, m_j)^{\frac14} a(m_j) \\
&+ \sum_{\substack{0 < \|\mathbf{m}\| \leq W \\ m_1 = m_2; m_3 = m_4}} \prod_{1 \leq j \leq 4} (q, m_j)^{\frac14} a(m_j).
\end{aligned} \tag{29}$$

First suppose that $|\beta| \leq N^{-3}$ so that $W = qN^{-1}(\log N)^4$. Then the first term on the right of (29) is, by (22) and [8], Lemma 13,

$$\begin{aligned}
&\ll N^4 \sum_{0 < \lambda \leq W} \Bigg(\sum_{0 < m \leq (W/\lambda)^{1/2}} (q, \lambda m^2)^{\frac14} \Bigg)^4 \\
&\ll N^4 \sum_{0 < \lambda \leq W} \lambda \Bigg(\sum_{0 < m \leq (W/\lambda)^{1/2}} (q, m^2)^{\frac14} \Bigg)^4 \\
&\ll N^{4+\varepsilon} W^2 \\
&\ll N^{2+2\varepsilon} q^2.
\end{aligned}$$

Similarly, the second term on the right of (29) is

$$\ll N^4 \Bigg(\sum_{0 < m \leq W} (q, m)^{\frac12} \Bigg)^2 \ll N^{2+\varepsilon} q^2.$$

Now suppose that $|\beta| > N^{-3}$ so that $W = (\log N)^4 q|\beta|$. In this case the first term on the right of (29) is estimated through the use of (22) and [8], Lemma 13, and is

$$\begin{aligned}
&\ll R|\beta|^{-1} \sum_{0 < \lambda \leq W} \Bigg(\sum_{0 < m \leq (W/\lambda)^{1/2}} \frac{(q, m)^{1/2}}{m^{1/2}} \Bigg)^4 \\
&\ll R|\beta|^{-1} q^{\varepsilon} W^{1+\varepsilon} \\
&\ll RN^{2+\varepsilon} q,
\end{aligned}$$

and the second term on the right of (29) contributes

$$\ll R|\beta|^{-1}\left(\sum_{0<m\leqq W}\frac{(q,m)^{1/2}}{m^{1/2}}\right)^2 \ll RN^{2+\varepsilon}q$$

by a similar estimation.

Collecting together we find *via* (25) and (29) that

$$\Theta_2(R) \ll R^{\varepsilon-1}\sum_{R<q\leqq 2R}\left(N^{2+2\varepsilon}q^2\int_0^{N^{-3}}d\beta + RN^{2+\varepsilon}q\int_{N^{-3}}^{P/RN^3}d\beta\right) \tag{30}$$
$$\ll PRN^{\varepsilon-1} \ll P^2N^{\varepsilon-1}$$

whenever $R \leqq P$. Lemma 1 now follows from (11), (21), (24) and (30).

Theorem 2 is now available. From Lemma 7, and Lemma 4.9 of Vaughan [11],

$$\sum_{q\leqq P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}(q,a)}\left|q^{-1}S(q,a)J\left(\alpha-\frac{a}{q}\right)\right|^4 d\alpha \ll N^{1+\varepsilon}. \tag{31}$$

Hence, Theorem 2 follows from (7), (31), and Lemma 1.

6. The approach to Theorem 1.

We shall concentrate on the case $k = 3$ in Theorem 1, that is, the exceptional set for sums of a square and two cubes. Later on we shall describe the modifications needed when $k = 4$ or 5.

Let $f(\alpha)$ be given by (2) where

$$N = X^{1/3},$$

and let

$$g_l(\alpha) = \sum_{x\leqq X^{1/l}} e(\alpha x^l). \tag{32}$$

For any measurable set $\mathscr{A} \subset [0,1]$ put

$$\varrho(n,X;\mathscr{A}) = \int_{\mathscr{A}} g_2(\alpha)f(\alpha)^2 e(-\alpha n)\,d\alpha. \tag{33}$$

If $n \leqq X$, then $\varrho(n,X;[0,1])$ equals the number of solutions of $n = x^2 + y^3 + z^3$ where any solution is counted with weight $\Gamma(y/N)\Gamma(z/N)$. In particular, $r_3(n) > 0$ if and only if $\varrho(n,X;[0,1]) \neq 0$.

The result on $E_3(X)$ is now deduced by a traditional method which goes back to Davenport and Heilbronn [4]. It is based on Bessel's inequality and a version of the Hardy-Littewood method. Let $\mathfrak{M} = \mathfrak{M}(P)$ be the set defined in the introduction. Now put

$$Y = Y_3 = N(\log N)^{-4} \tag{34}$$

and define $\mathfrak{m} = [0,1]\backslash\mathfrak{M}(Y) \pmod 1$. One key step is the estimate

$$\int_{\mathfrak{m}} |g_2(\alpha)f(\alpha)^2|^2\, d\alpha \ll X^{25/21+\varepsilon}, \tag{35}$$

the other one is hidden in

LEMMA 8. *For all but $O(X^{6/7+\varepsilon})$ values of $n \leqq X$, the estimate*

$$\varrho(n, X; \mathfrak{M}(Y)) > X^{\frac{1}{6}-\varepsilon}$$

holds.

The proof of Theorem 1 is now readily completed. We have

$$\varrho(n, X; [0,1]) = \varrho(n, X; \mathfrak{M}(Y)) + \varrho(n, X; \mathfrak{m}). \tag{36}$$

By Bessel's inequality, (33) and (35),

$$\sum_{n \leqq X} |\varrho(n, X; \mathfrak{m})|^2 \leqq \int_{\mathfrak{m}} |g_2(\alpha)f(\alpha)^2|^2\, d\alpha \ll X^{25/21+\varepsilon}.$$

Hence, the number of $n \leqq X$ for which $|\varrho(n, X; \mathfrak{m})| > X^{1/6-\varepsilon}$ is $\ll X^{6/7+4\varepsilon}$. Thus Theorem 1 in case $k = 3$ follows from Lemma 8 and (36).

We shall prove (35) and Lemma 8 in the final section, but shall now proceed to reduce the other cases to similar estimates. In these cases we consider

$$\varrho_k(n, X; \mathscr{A}) = \int_{\mathscr{A}} g_2(\alpha)g_k(\alpha)f(\alpha)e(-\alpha n)\, d\alpha \quad (k = 4, 5). \tag{37}$$

We now redefine

$$Y = Y_k = X^{1/k} \tag{38}$$

and put again $\mathfrak{m} = [0,1]\backslash\mathfrak{M}(Y_k) \pmod 1$. Then, we shall show that

$$\int_{\mathfrak{m}} |g_2(\alpha)g_k(\alpha)f(\alpha)|^2\, d\alpha \ll X^{\frac{2}{k}+\frac{2}{3}-\delta_k+\varepsilon} \tag{39}$$

where $\delta_4 = 1/14$, $\delta_5 = 1/30$. With the same values of δ_k we have:

LEMMA 9. *For all but $O(X^{1-\delta_k+\varepsilon})$ values of $n \leqq X$,*

$$\varrho_k(n, X; \mathfrak{M}(Y_k)) > X^{\frac{1}{k}-\frac{1}{6}-\varepsilon}.$$

A bound for $E_k(X)$ is then deduced from (39) and Lemma 9 in the same manner as a bound for $E_3(X)$ was deduced from (35) and Lemma 8.

7. The minor arc estimates.

We prove (35) first. Again let $\mathfrak{M} = \mathfrak{M}(P)$ be given as in Theorem 2, and $\mathfrak{N}(P) = \mathfrak{M}(2P)\backslash\mathfrak{M}(P)$. We note that $\mathfrak{M}(P^{3/2}) = \mathfrak{M}(X^{1/2}) = [0,1] \pmod 1$, and that therefore $\mathfrak{m}$ can be covered by $O(\log X)$ sets $\mathfrak{N}(P)$ with $Y < P \leqq X^{1/2}$. By Weyl's inequality ([11], Lemma 2.4),

$$\sup_{\alpha\in\mathfrak{N}(P)} |g_2(\alpha)| \ll X^{1/2+\varepsilon} P^{-1/2}. \tag{40}$$

Let $X^{10/21} < P \leqq X^{1/2}$. Then, by (4) and (40),

$$\int_{\mathfrak{N}(P)} |g_2(\alpha) g_k(\alpha) f(\alpha)|^2 \, d\alpha \ll (X^{1+\varepsilon}P^{-1})(X^{2/3+\varepsilon}) \ll X^{25/21+\varepsilon}. \tag{41}$$

Now let $X^{1/7} \leqq P \leqq X^{10/21}$. By (40) and Theorem 2,

$$\int_{\mathfrak{N}(P)} |g_2(\alpha)\, g_k(\alpha) f(\alpha)|^2 \, d\alpha \tag{42}$$
$$\ll (X^{1+\varepsilon}P^{-1})(X^{1/3} + P^{7/2}X^{-1} + P^2X^{-1/3}) \ll X^{25/21+\varepsilon}$$

This already proves (35) since $Y > X^{1/7}$.

We now prove (39). If $\alpha \in \mathfrak{M}(q,a)$ (in the notation of § 1) where $P \leqq N^{3/2}$, then by [1], Lemmas 8 and 9, and a partial integration,

$$g_2(\alpha) \ll q^{-\frac{1}{2}} X^{\frac{1}{2}+\varepsilon}\left(1 + X\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{2}}.$$

Hence, by Lemma 2 of Brüdern [2], when $k = 4$ or $k = 5$,

$$\int_{\mathfrak{N}(P)} |g_2(\alpha)^2 g_k(\alpha)^4| \, d\alpha \ll X^{\varepsilon}(PX^{\frac{2}{k}} + X^{\frac{4}{k}}). \tag{43}$$

The case $k = 4$ is easy. When $X^{1/4} \leqq P \leqq X^{1/2}$ the right hand side of (43) is $\ll X^{1+\varepsilon}$. Thus, by (41), (42), (43) and Cauchy's inequality,

$$\int_{\mathfrak{N}(P)} |g_2(\alpha)\, g_4(\alpha) f(\alpha)|^2 \, d\alpha \ll (X^{\frac{25}{21}+\varepsilon})^{\frac{1}{2}} (X^{1+\varepsilon})^{\frac{1}{2}} \ll X^{\frac{7}{6}-\delta_4+\varepsilon}.$$

Since $\mathfrak{m}$ is covered by $O(\log P)$ sets $\mathfrak{N}(P)$ where $Y_4 \leqq P \leqq X^{1/2}$, this proves (39) when $k = 4$.

The case $k = 5$ requires more care. Note that the first bound in (41) holds for any $P \geqq 1$. Hence, when $X^{2/5} \leqq P \leqq X^{1/2}$ we deduce from (41), (43) and Schwarz's inequality that

$$\int_{\mathfrak{N}(P)} |g_2(\alpha)\, g_5(\alpha) f(\alpha)|^2 \, d\alpha \ll X^{\varepsilon} (X^{\frac{5}{3}} P^{-1})^{\frac{1}{2}} (P X^{\frac{2}{5}})^{\frac{1}{2}} \ll X^{\frac{31}{30}+\varepsilon}.$$

But, when $X^{1/5} = Y_5 \leqq P \leqq X^{2/5}$, we find from (42), (43) and Schwarz's inequality that

$$\int_{\mathfrak{N}(P)} |g_2(\alpha)\, g_5(\alpha) f(\alpha)|^2 \, d\alpha \ll X^{\varepsilon} (X^{\frac{25}{21}})^{\frac{1}{2}} (X^{\frac{4}{5}})^{\frac{1}{2}} \ll X.$$

This proves (39) when $k = 5$.

8. The major arc estimates.

We prove Lemmas 8 and 9 along very traditional patterns. However, due to the relatively good error terms which are required here, some care is needed. Let J be given by (6), and put

$$J_l(\beta) = \int_0^{X^{1/l}} e(\alpha^l \beta)\, d\alpha. \tag{44}$$

Then, we may define

$$f^*(\alpha) = f^*(\alpha; q, a) = q^{-1} S(q, a)\, J\left(\alpha - \frac{a}{q}\right), \tag{45}$$

$$g^*(\alpha) = g_l^*(\alpha; q, a) = q^{-1} S(q, a)\, J_l\left(\alpha - \frac{a}{q}\right) \tag{46}$$

where

$$S_l(q,a) = \sum_{x \leqq q} e\left(\frac{ax^l}{q}\right). \tag{47}$$

When $\alpha \in \mathfrak{M}(Y)$ we have $f - f^* \ll 1$ by Lemma 2, and from Theorem 4.1 of Vaughan [11] we obtain $g_l - g_l^* \ll Y_k^{\frac{1}{2}+\varepsilon}$ whenever $l = 2$ and $\alpha \in \mathfrak{M}(Y)$, or $l = k$ and $\alpha \in \mathfrak{M}(Y_k)$. From Lemma 7 and [11], Lemma 2.8, we readily establish

$$f^*(\alpha) \ll q^{-\frac{1}{3}} X^{\frac{1}{3}} \left(1 + X\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{3}}, \tag{48}$$

$$g^*(\alpha) \ll q^{-\frac{1}{l}} X^{\frac{1}{l}} \left(1 + X\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{l}}. \tag{49}$$

The goal is now to approximate to $\varrho(n, X; \mathfrak{M}(Y))$ and $\varrho_k(n, X; \mathfrak{M}(Y_k))$ by numbers now to be defined, at least almost always. Let $\mathfrak{M}_0(Y)$ be the union of all intervals

$$\left\{\alpha : \left|\alpha - \frac{a}{q}\right| \leqq Y^{-2}\right\}$$

where $1 \leqq a \leqq q \leqq Y$, $(a,q) = 1$, and put

$$\varrho^*(n, \mathscr{A}) = \int_{\mathscr{A}} g_2^*(\alpha) f^*(\alpha)^2 e(-\alpha n)\, d\alpha, \tag{50}$$

$$\varrho_k^*(n, \mathscr{A}) = \int_{\mathscr{A}} g_2^*(\alpha)\, g_l^*(\alpha)^2 f^*(\alpha) e(-\alpha n)\, d\alpha, \tag{51}$$

where $\mathscr{A} \subset \mathfrak{M}(Y)$.

Note that $\varrho(n, X; \mathfrak{M}(Y)) - \varrho^*(n, \mathfrak{M}(Y))$ is the Fourier coeficient of the function which is $g_2 f^2 - g_2^* f^{*2}$ on $\mathfrak{M}(Y)$, and zero elsewhere. By Bessel's inequality,

$$\sum_{n \leqq X} |\varrho(n, X; \mathfrak{M}(Y)) - \varrho^*(n, \mathfrak{M}(Y))|^2 \leqq \int_{\mathfrak{M}(Y)} |g_2(\alpha) f(\alpha)^2 - g_2^*(\alpha) f^*(\alpha)^2|^2 d\alpha. \tag{52}$$

By (48), (49) and the remarks preceding these equations, we see that

$$|g_2(\alpha) f(\alpha)^2 - g_2^*(\alpha) f^*(\alpha)^2| \ll Y^{\frac{1}{2}+\varepsilon} (X^{\frac{2}{3}} q^{-\frac{2}{3}} + X^{\frac{5}{6}} q^{-\frac{5}{6}}) \left(1 + X\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{2}}.$$

Therefore,

$$(53)\qquad \int\limits_{\mathfrak{M}(Y)} |g_2(\alpha)f(\alpha)^2 - g_2^*(\alpha)f^*(\alpha)^2|^2\, d\alpha$$

$$\ll Y^{1+\varepsilon} \sum_{q \leqq Y} (X^{\frac{1}{3}}q^{-\frac{1}{3}} + X^{\frac{2}{3}}q^{-\frac{2}{3}}) \ll X^{\frac{10}{9}+\varepsilon}.$$

In much the same way as in (52),

$$(54)\qquad \sum_{n \leqq X} |\varrho^*(n, \mathfrak{M}(Y)) - \varrho^*(n, \mathfrak{M}_0(Y))|^2 \ll \int\limits_{\mathfrak{M}_0(Y)\setminus\mathfrak{M}(Y)} |g_2^*(\alpha)f^*(\alpha)^2|^2\, d\alpha,$$

and by (49) and an estimate very similar to (31),

$$\int\limits_{\mathfrak{M}_0(Y)\setminus\mathfrak{M}(Y)} |g_2^*(\alpha)f^*(\alpha)^2|^2\, d\alpha \ll \sup_{\alpha\in\mathfrak{M}_0(Y)\setminus\mathfrak{M}(Y)} |g_2^*(\alpha)|^2 \int\limits_{\mathfrak{M}_0(Y)} |f^*(\alpha)|^4\, d\alpha$$

$$\ll (XY^{-1})(X^{\frac{1}{3}+\varepsilon}) \ll X^{1+\varepsilon}.$$

This, when combined with (52), (53) and (54), gives

$$(55)\qquad \sum_{n \leqq X} |\varrho(n, X; \mathfrak{M}(Y)) - \varrho^*(n, \mathfrak{M}(Y))|^2 \ll X^{\frac{10}{9}+\varepsilon}.$$

Let $\mathscr{X}$ be set of all $n \leqq X$ for which

$$(56)\qquad |\varrho(n, X; \mathfrak{M}(Y)) - \varrho^*(n, \mathfrak{M}(Y))| < X^{1/7}$$

fails to hold. Then

$$(57)\qquad \sum_{n\in\mathscr{X}} 1 \leqq X^{-\frac{2}{7}} \sum_{n \leqq X} |\varrho(n, X; \mathfrak{M}(Y)) - \varrho^*(n, M(Y))|^2 \ll X^{\frac{5}{6}}.$$

We give a similar argument when a biquadrate or a fifth power is present. Imitating the procedure leading to (53) we see that

$$|g_2(\alpha)g_k(\alpha)f(\alpha) - g_2^*(\alpha)g_k^*(\alpha)f^*(\alpha)|$$

$$\ll Y^{\frac{1}{2}+\varepsilon}\left(\left(\frac{X}{q}\right)^{\frac{1}{2}+\frac{1}{3}} + \left(\frac{X}{q}\right)^{\frac{1}{2}+\frac{1}{k}} + \left(\frac{X}{q}\right)^{\frac{1}{3}+\frac{1}{k}}\right)\left(1 + X\left|\alpha - \frac{a}{q}\right|\right)^{-\frac{1}{2}}$$

which in turn implies

$$\int_{\mathfrak{M}(Y_k)} |g_2(\alpha)g_k(\alpha)f(\alpha) - g_2^*(\alpha)g_k^*(\alpha)f^*(\alpha)|^2 \, d\alpha$$

$$\ll Y_k^{1+\varepsilon} \sum_{q \leqq Y_k} \sum_{\substack{j,l \in \{2,3,k\} \\ j<l}} \left(\frac{X}{q}\right)^{\frac{2}{j}+\frac{2}{l}-1} \ll X^{1+\varepsilon}.$$

It is straightforward from (49) and a suitable analogue of (31) that

$$\int_{\mathfrak{M}_0(Y_k)\setminus\mathfrak{M}(Y_k)} |g_2(\alpha)g_k(\alpha)f(\alpha) - g_2^*(\alpha)g_k^*(\alpha)f^*(\alpha)|^2 \, d\alpha$$

$$\ll \sup_{\alpha \in \mathfrak{M}_0(Y_k)\setminus\mathfrak{M}(Y_k)} |g_k^*(\alpha)|^2 \left(\int_{\mathfrak{M}(Y_k)} |g_2^*(\alpha)|^4 \, d\alpha \right)^{\frac{1}{2}} \left(\int_{\mathfrak{M}(Y_k)} |f^*((\alpha)|^4 \, d\alpha \right)^{\frac{1}{2}} + X$$

$$\ll (X^{\frac{2}{k}} Y_k^{-\frac{2}{k}})(X^{1+\varepsilon})^{\frac{1}{2}}(X^{\frac{1}{3}+\varepsilon})^{\frac{1}{2}} \ll X^{\frac{2}{3}+\frac{2}{k}-\frac{2}{k^2}+\varepsilon}.$$

As before we deduce

$$\sum_{n \leqq X} |\varrho_k(n, X; \mathfrak{M}(Y_k)) - \varrho_k^*(n, \mathfrak{M}(Y_k))|^2 \ll \begin{cases} X^{25/24+\varepsilon} & (k=4), \\ X^{1+\varepsilon} & (k=5). \end{cases}$$

Let $\mathscr{X}_k$ be the set of all $n \leqq X$ for which

(58) $$|\varrho_k(n, X; \mathfrak{M}(Y_k)) - \varrho_k^*(n, M_0(Y_k))| < X^{\frac{1}{k}-\frac{1}{6}-\frac{1}{100}}$$

fails to hold. Then, by the argument used to establish (57), not more than $O(X^{1-\delta_k})$ numbers are in $\mathscr{X}_k$.

We may now concentrate on $\varrho^*(n, \mathfrak{M}_0(Y))$, and here we have of course that

$$\varrho^*(n, \mathfrak{M}_0(Y)) = \mathfrak{S}_3(n, Y)K_3(n)$$

and

$$\varrho_k^*(n, \mathfrak{M}_0(Y_k)) = \mathfrak{S}_k(n, Y_k)\, K_k(n) \quad (k = 4, 5)$$

where

(59) $$\mathfrak{S}_k(n, Z) = \sum_{q \leqq Z} q^{-3} S_2(q, a) S_3(q, a) S_k(q, a) e\left(-\frac{an}{q}\right)$$

and

$$K_3(n) = \int_{-Y^{-2}}^{Y^{-2}} J_2(\beta) J(\beta)^2 e(-\beta n)\, d\beta,$$

$$K_k(n) = \int_{-Y_k^{-2}}^{Y_k^{-2}} J_2(\beta) J_k(\beta)\, J(\beta) e(-\beta n)\, d\beta.$$

Now define $K_k^*(n)$ exactly as $K_k(n)$, but with integration taken over the whole real line. Then one has at once that

(60) $$K_k(n) - K_k^*(n) \ll X^{\frac{1}{k}-\frac{1}{6}-\frac{1}{100}} \quad (3 \leqq k \leqq 5).$$

A simple change of variable shows

$$J_2(\beta) J(\beta)^2 = \int_{-\infty}^{\infty} e(\beta v)\, V(v)\, dv$$

where

(61) $$V(v) = \frac{1}{18} \int_{-\infty}^{\infty} \int_{0}^{X} \vartheta_1^{-\frac{1}{2}} \vartheta_2^{-\frac{2}{3}} \sigma^{-\frac{2}{3}} \Gamma\left(\frac{\vartheta_2^{1/3}}{N}\right) \Gamma\left(\frac{\sigma^{1/3}}{N}\right) d\vartheta_1\, d\vartheta_2,$$

and where $\sigma = v - \vartheta_1 - \vartheta_2$. By Fourier's inversion theorem, $K_3(n) = V(n)$, so that (61) implies

$$0 \leqq K_3^*(n) \ll X^{1/6}.$$

Now let $\frac{1}{2}X \leqq n \leqq X$. When $\frac{1}{16}X \leqq \vartheta_i \leqq \frac{1}{8}X$ for $i = 1$ and $i = 2$, the integrand in (61) is $\ll X^{-11/6}$, and the set of all these $(\vartheta_1, \vartheta_2)$ has measure $\gg X^2$. Thus, for these n,

$$X^{1/6} \ll K_3^*(n) \ll X^{1/6}.$$

For the singular series we proved in [3], Lemma 4.5:

LEMMA 10. *Let $\mathfrak{S}_k(n, Z)$ be given by* (59) *where* $3 \leqq k \leqq 5$. *Then, for all but* $O(X^{\frac{7}{6}-\frac{1}{k}+\varepsilon})$ *integers in* $\frac{1}{2}X \leqq n \leqq X$ *we have* $\mathfrak{S}_k(n, X^{1/k}) \gg X^{-\varepsilon}$.

The proof of this lemma is based on the large sieve inequality, and follows in principle the pattern of Vaughan's argument in [10], but is a more delicate version thereof. For details the reader is referred to [3]. Lemma 8 now follows from (56), (57), (59), (62) and Lemma 10, and Lemma 9 is available from (58), (59), (62) and Lemma 10.

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