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TRACES OF COMMUTATORS OF MÖBIUS TRANSFORMATIONS

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1. Introduction.

A Möbius transformation g mapping the upper-halfplane U onto itself is of the form

$$g(z) = \frac{az+b}{cz+d}, \quad \text{with } ad - bc = +1.$$

Therefore we can associate a matrix

$$\tilde{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

in SL (2, R) to each Möbius transformation g: U → U. This is, of course, standard but not quite unique. Both matrices $\tilde{g}$ and $-\tilde{g}$ correspond to the same Möbius transformation.

Let g and h be two Möbius transformations in PSL (2, R). Let $\tilde{g}$ and $\tilde{h}$ be the associated matrices. Then $\tilde{c} = \tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}\tilde{g}$ is a matrix corresponding to the commutator $c = h \circ g^{-1} \circ h^{-1} \circ g$ of h and g. Observe that even though the matrices $\tilde{h}$ and $\tilde{g}$ corresponding to the mappings h and g are determined only up to sign, the matrix $\tilde{c}$ depends neither on the choice of the sign of $\tilde{h}$ nor on that of $\tilde{g}$. So we may speak of *the matrix* of the commutator c.

In this paper we study traces of matrices of the commutators c of hyperbolic Möbius transformations h and g whose axes intersect. We show that for hyperbolic commutators c the trace of the corresponding matrix $\tilde{c}$ is always negative.

Let $G \subset$ PSL (2, R) be a Fuchsian group corresponding to a compact Riemann surface of genus p and let $g_1, \ldots, g_{2p}$, be the standard set of generators for G. Let $c_j = [g_{2j-1}, g_{2j}] = g_{2j} \circ g_{2j-1}^{-1} \circ g_{2j}^{-1} \circ g_{2j-1}$ be the commutator of g_{2j-1} and g_{2j} and let $\tilde{c}_j$ be its matrix. In our main theorems we show that the traces of the

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matrices

$$\tilde{c}_j \tilde{c}_{j-1} \dots \tilde{c}_1$$

are all negative for $j < p$ and equal to $+2$ for $j = p$.

A corollary of this main result is that a Fuchsian subgroup of PSL (2, R) can always be lifted to a subgroup of SL (2, R). This is well known but not obvious since the exact sequence

$$1 \to \{\pm I\} \to \mathrm{SL}(2, \mathrm{R}) \xrightarrow{\pi} \mathrm{PSL}(2, \mathrm{R}) \to 1 \tag{2}$$

does not split. The fact that Fuchsian groups lift has actually been proven several times by many methods and authors. An interesting survey of the history of the problem together with an elegant proof for a generalization of this result can be found in Irwin Kra's paper [2].

Our methods are elementary. For our considerations it is enough to know how matrices are multiplied. Therefore we feel justified to add to the literature still another proof for this classical result.

2. Pairs of Möbius transformations.

Let (g, h) be a pair of hyperbolic transformations mapping the upper half-plane U onto itself. Assume that g and h do not have common fixed-points. Let $k_1 = k(g)$, $k_2 = k(h)$ and $k_3 = k(g \circ h)$ be the multipliers of the corresponding mappings.

Let $t = (r(g), r(h), a(h), a(g))$ be the cross ratio of the *repelling* and *attracting* fixed-points of g and h. To abbreviate notation define $f(k) = \sqrt{k} + 1/\sqrt{k}$, for $k > 0$. Denote $f(k(g)) = f(g)$.

Recall the classification of pairs of Möbius transformations into three disjoint classes $\mathscr{H}$, $\mathscr{P}$ and $\mathscr{E}$ as presented in [3].

$$\begin{aligned}
(g,h) \in \mathscr{H} &\Leftrightarrow f(g \circ h) = t f(k_1 k_2) + (1-t) f(k_1/k_2) \geq 2 \\
&\Leftrightarrow t \geq t_1 = \frac{2 - f(k_1/k_2)}{f(k_1 k_2) - f(k_1/k_2)}, \\
(g,h) \in \mathscr{P} &\Leftrightarrow f(g \circ h) = -t f(k_1 k_2) - (1-t) f(k_1/k_2) \geq 2 \\
&\Leftrightarrow t \leq t_2 = \frac{-2 - f(k_1/k_2)}{f(k_1 k_2) - f(k_1/k_2)} \\
(g,h) \in \mathscr{E} &\Leftrightarrow t_2 < t < t_1.
\end{aligned}$$

Here $\mathscr{H}$ stands for "handle", $\mathscr{P}$ for "pants" and $\mathscr{E}$ for "elliptic". We say that (g, h) belongs to Int $\mathscr{P}$ if $t < t_2$ and to Int $\mathscr{H}$ if $t > t_1$.

Note that $g \circ h$ is hyperbolic if and only if $(g, h) \in \operatorname{Int} \mathscr{P} \cup \operatorname{Int} \mathscr{H}$. Moreover, if $(g, h) \in \operatorname{Int} \mathscr{P}$, then $(g, h^{-1}) \in \operatorname{Int} \mathscr{H}$.

3. Traces of commutators.

Let g and h be Möbius transformations fixing the upper half-plane U, and let $\tilde{g}$ and $\tilde{h}$ be some matrices in SL (2, R) corresponding to g and h. Then the matrix product $\tilde{g}\tilde{h}$ is a representation of $g \circ h$. Note that $\chi(\tilde{g}\tilde{h}) = \chi((-\tilde{g})(-\tilde{h}))$. The following result is well known.

LEMMA 3.1. $\chi(\tilde{g}) = \chi(\tilde{h}\tilde{g}\tilde{h}^{-1})$.

Suppose that g and h are hyperbolic transformations without common fixed points.

LEMMA 3.2. *If* $(g, h) \in \operatorname{Int} \mathscr{P}$ *and* $\chi(\tilde{g})\chi(\tilde{h}) > 0$, *then* $\chi(\tilde{g}\tilde{h}) < 0$.

PROOF. By Lemma 3.1, we may suppose that $a(h) = 0$, $r(h) = 1$ and $a(g) = \infty$. Then

$$t = r(g) < t_2 = \frac{-2 - f(k_1/k_2)}{f(k_1k_2) - f(k_1/k_2)}$$

and

$$g(z) = k_1 z - t(k_1 - 1), \qquad k_1 = k(g) > 1,$$

$$h(z) = \frac{z}{(1 - k_2)z + k_2}, \qquad k_2 = k(h) > 1.$$

Hence

$$\tilde{g} = \begin{pmatrix} \sqrt{k_1} & -t\left(\sqrt{k_1} - \dfrac{1}{\sqrt{k_1}}\right) \\ 0 & \dfrac{1}{\sqrt{k_1}} \end{pmatrix}, \qquad \chi(\tilde{g}) > 0,$$

$$\tilde{h} = \begin{pmatrix} \dfrac{1}{\sqrt{k_2}} & 0 \\ -\left(\sqrt{k_2} - \dfrac{1}{\sqrt{k_2}}\right) & \sqrt{k_2} \end{pmatrix}, \qquad \chi(\tilde{h}) > 0.$$

It follows that

$$\begin{aligned} \chi(\tilde{g}\tilde{h}) &= \frac{\sqrt{k_1}}{\sqrt{2}} + t\left(\sqrt{k_1} - \frac{1}{\sqrt{k_1}}\right)\left(\sqrt{k_2} - \frac{1}{\sqrt{k_2}}\right) + \frac{\sqrt{k_2}}{\sqrt{k_1}} \\ &= tf(k_1/k_2) + (1 - t)f\left(\frac{k_1}{k_2}\right) < -2. \end{aligned}$$

Suppose now that g and h have intersecting axes.

LEMMA 3.3. *The commutator*

$$c = [g,h] = h \circ g^{-1} \circ h^{-1} \circ g$$

is hyperbolic if and only if $(h, g^{-1} \circ h^{-1} \circ g) \in \operatorname{Int} \mathscr{P}$.

PROOF. Denote $h' = g^{-1} \circ h^{-1} \circ g$. To consider the class of the pair (h, h'), let $t = (r(h), r(h'), a(h'), a(h))$ and $k = k(h) = k(h')$. Recall that here $r(h)$, $r(h')$ are the repelling fixed points and $a(h)$, $a(h')$ are the attracting fixed points while k is the common multiplier of h and h'.

Then

$$t_1 = \frac{2 - f(k/k)}{f(k^2) - f(k/k)} = 0,$$

$$t_2 = \frac{-2 - f(k/k)}{f(k^2) - f(k/k)} = \frac{-4}{f(k^2) - 2} < 0.$$

By Lemma 3.1, we may suppose that $a(h) = \infty$, $a(h') = 0$, $r(h') = 1$ and $r(h) = t$. Since

$$g^{-1}(r(h)) = a(h') = 0$$

$$g^{-1}(a(h)) = r(h') = 1,$$

we have $g(0) = t$ and $g(1) = \infty$. On the other hand, the axis of g intersects with the axis of h and that of h'. From $g(1) = \infty$ it then follows that

$$a(g) < t < 0 < r(g) < 1.$$

The commutator $c = h \circ h'$ is hyperbolic if and only if $(h, h') \in \operatorname{Int} \mathscr{P} \cup \operatorname{Int} \mathscr{H}$, i.e., if and only if $t < t_2$ or $t_1 < t$. Since $t_1 = 0$ and $t < 0$, the assertion follows.

LEMMA 3.4. *If* $c = [g,h]$ *is hyperbolic and* $\tilde{c} = \tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}\tilde{g}$, *then* $\chi(\tilde{c}) < 0$.

PROOF. Since $\chi(\tilde{h}) = \chi(\tilde{h}^{-1})$, we have $\chi(\tilde{h})\chi(\tilde{g}^{-1}\tilde{h}^{-1}\tilde{g}) > 0$ by Lemma 3.1. By Lemma 3.3, $(h, g^{-1} \circ h^{-1} \circ g) \in \operatorname{Int} \mathscr{P}$. Hence the pair $(h, g^{-1} \circ h^{-1} \circ g)$ fulfills the assumptions of Lemma 3.2 for any choice of the representations $\tilde{g}$ and $\tilde{h}$ of g and h, and the assertion follows.

Let G be a Fuchsian group acting in U. Suppose that U/G is a compact Riemann surface of genus p. Suppose that $g \in G$ and $h \in G$ correspond to simple closed geodesics α and β of U/G, respectively. If α and β do not intersect, then either $(g, h) \in \operatorname{Int} \mathscr{P}$ or $(g, h^{-1}) \in \operatorname{Int} \mathscr{P}$ depending on the cyclic order of the fixed points of g and h. (See Theorem 2.1 in [4]).

Let $g_1, g_2, \ldots, g_{2p}$ be a canonical set of generators of G. Let $\tilde{g}_1, \tilde{g}_2, \ldots, \tilde{g}_{2p}$ be representations of the generating transformations in SL(2, R). Then $\tilde{c}_j = [\tilde{g}_{2j-1}, \tilde{g}_{2j}]$ is a representation of c_j. Since U/G is compact, G contains, besides the identity, only hyperbolic elements. Hence by Lemma 3.4, $\chi(\tilde{c}_j) < 0$.

For any $j = 1, \ldots, p-2$, the transformations

$$c_j \circ c_{j-1} \circ \ldots \circ c_1, c_{j+1}, \ldots, c_p$$

correspond to simple closed geodesics on U/G. Moreover, all these geodesics are pairwise disjoint and $(c_{j+1}, c_j \circ \ldots \circ c_1) \in \operatorname{Int} \mathscr{P}$.

THEOREM 3.5. $\chi(\tilde{c}_j \tilde{c}_{j-1} \ldots \tilde{c}_1) < 0$ *for* $j = 1, \ldots, p-1$.

PROOF. For $j = 1$ the assertion holds by Lemma 3.4. Suppose that $\chi(\tilde{c}_{j-1} \ldots \tilde{c}_1) < 0$. Since $\chi(\tilde{c}_j) < 0$ and $(c_j, c_{j-1} \circ \ldots \circ c_1) \in \operatorname{Int} \mathscr{P}$, we have, by Lemma 3.2, $\chi(\tilde{c}_j, \tilde{c}_{j-1} \ldots \tilde{c}_1) < 0$.

If we choose $j = p-1$, then we have

$$\chi(\tilde{c}_{p-1} \tilde{c}_{p-2} \ldots \tilde{c}_1) < 0,$$

$$\chi(\tilde{c}_p) < 0,$$

$$(c_{p-1}, c_{p-2} \circ \ldots \circ c_1) \in \operatorname{Int} \mathscr{P},$$

$$c_p = (c_{p-1} \circ \ldots \circ c_1)^{-1}.$$

THEOREM 3.6. $\chi(\tilde{c}_p \tilde{c}_{p-1} \ldots \tilde{c}_1) = 2$.

PROOF. Since $c_p \circ c_{p-1} \circ \ldots \circ c_1 = \mathrm{id}$, we have $\tilde{c}_p \tilde{c}_{p-1} \ldots \tilde{c}_1 = \pm$ Identity. By Lemmata 3.1 and 3.4, we may suppose that

$$\tilde{c}_p = \begin{pmatrix} -a & 0 \\ 0 & -a^{-1} \end{pmatrix}, \quad a > 1.$$

Then, by Theorem 3.5, $\tilde{c}_{p-1} \tilde{c}_{p-2} \ldots \tilde{c}_1 = \tilde{c}_p^{-1}$. Hence we have in fact

$$\tilde{c}_p \tilde{c}_{p-1} \ldots \tilde{c}_1 = \text{Identity}. \tag{3}$$

4. Liftings of Fuchsian groups.

Let $G \subset \mathrm{PSL}(2, \mathrm{R})$ be a group of Möbius transformations. Consider the exact sequence (2). We say that a subgroup $G \subset \mathrm{PSL}(2, \mathrm{R})$ *can be lifted to* SL(2, R) if there exists a subgroup $\Gamma \subset \mathrm{SL}(2, \mathrm{R})$ such that $\pi: \Gamma \to G$ is an isomorphism.

Let G be the group generated by the elliptic Möbius transformation $g(z) = -1/z$ of order two. It is immediate that this group cannot be lifted to SL(2, R) because the matrix of g,

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is of order 4 while g is of order 2. In [2] Kra proves the more general result stating that a subgroup $G \subset \mathrm{PSL}(2, \mathsf{R})$ can be lifted to $\mathrm{SL}(2, \mathsf{R})$ if and only if G does not have any elements of order 2 ([2, Theorem p. 181]).

Kra's proof is based on the existence of square roots of the canonical bundle of a compact Riemann surface. He treats the general case with Maskit's combination theorem.

Let $g_1, g_2, \ldots, g_{2p}$ be standard generators for a Fuchsian group G as before. Then all generators g_j and commutators c_j are hyperbolic Möbius transformations. Let $\tilde{g}_j$ be *any* matrix corresponding to the Möbius transformation g_j. Let $\tilde{c}_j = [\tilde{g}_{2j-1}, \tilde{g}_{2j}], j = 1, 2, \ldots, p$.

Consider the group $\Gamma = \langle \tilde{g}_1, \ldots, \tilde{g}_p \rangle$. By Theorem 3.6 the matrices $\tilde{c}_j$ satisfy the relation 3 for any choice of the matrices $\tilde{g}_j$. It is also obvious that there cannot be any other relations among the generators $\tilde{g}_j$ of Γ because any such relation would imply a new relation among the generators g_j of G. We conclude that the groups Γ and G are isomorphic and that the restriction of the projection $\pi: \mathrm{SL}(2, \mathsf{R}) \to \mathrm{PSL}(2, \mathsf{R})$ to Γ is an isomorphism. Therefore we have:

THEOREM 4.1. *Genus $p, p > 1$, Fuchsian subgroup of* PSL (2, R) *have 2^{2p} different liftings to* SL (2, R).

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