MATH. SCAND. 68 (1991), 59–82

ON THE POISSON EQUATION IN THE POTENTIAL THEORY OF A SINGLE KERNEL

E. NUMMELIN

Abstract.

We study the Poisson equation $f(x) = g(x) + Kf(x)$ for a non-negative transition kernel K on a measurable space $(E, \mathscr{E})$. The main results are concerned with the case where K is irreducible recurrent. It is shown that the general solution f is the sum of a recurrent potential, of a harmonic function (w.r.t. a "minorized" kernel) and of a harmonic function (w.r.t. K).

1. Introduction.

Let E be a set and $\mathscr{E}$ a countably generated σ-algebra of subsets of E. Let $K = (K(x, A);\, x \in E,\, A \in \mathscr{E})$ be a (non-negative) *kernel* on $(E, \mathscr{E})$; i.e., K is a map $K\colon E \times \mathscr{E} \to [0, \infty]$ such that
(i) for each $x \in E$, $K(x, \cdot)$ is a measure on $(E, \mathscr{E})$, and
(ii) for each $x \in \mathscr{E}$, $K(\cdot, A)$ is a measurable function on $(E, \mathscr{E})$.

Throughout this paper we assume that the kernel K is *σ-finite*, i.e., there exists a jointly measurable function $\bar{f}\colon E \times E \to (0, \infty)$ such that

(iii) $$\int_E K(x, dy)\bar{f}(x, y) < \infty \text{ for all } x \in E.$$

The kernel K acts as a linear operator on the cone $\mathscr{E}_+$ of non-negative measurable functions $f\colon E \to [0, \infty]$,

$$Kf(x) \overset{\text{def}}{=} \int_E K(x, dy) f(y), \qquad x \in E.$$

With the kernel K we can associate a *potential theory* by defining the family

$$\mathscr{H}_+ = \{h \in \mathscr{E}_+\colon h \not\equiv \infty, Kh \leqq h\}$$

as the class of *superharmonic functions.*

Received September 4, 1989

Our objective in this paper is to study the *Poisson equation* (abbrev. P.E.)

(1.1) $f(x) = g(x) + Kf(x)$ for $x \in D$ ($f(x)$ finite for some $x \in D$),

where g is a given measurable function: $E \to [-\infty, \infty]$, D is a given subset of E (often $D = E$), and f: $E \to [-\infty, \infty]$ is an unknown measurable function.

We will refer to the function g as the *charge* and to f as the *solution*. The set D is called the *domain*. So e.g. a superharmonic function h is a solution of the P.E. with domain $D = E$ and charge $g = h - Kh \in \mathscr{E}_+$. (We set $g = 0$ on $\{Kh = \infty\}$.)

REMARK 1.1. We do not demand that $Kf \stackrel{\text{def}}{=} Kf_+ - Kf_-$ or $g + Kf$ are finite everywhere on D but only that they are well defined (i.e. are neither of the form $\infty - \infty$ nor $-\infty + \infty$).

REMARK 1.2. The more general equation

$$f = g + rKf,$$

where $r > 0$ is a constant could be treated by considering the kernel rK instead of K.

Section 2 deals with preliminary general results some of which have independent interest, too. The main results are given in Section 3 and they are concerned with recurrent kernels. By introducing a new recurrent potential kernel (to be denoted by $G_{s,\nu}$, see Section 3) we are able to remove the usual assumption that the charge g is a special function. As an example we study the case of a Harris recurrent transition probability. In Section 4 we will briefly study the dual Poisson equation for measures.

Our approach is based on the probabilistic theory of positive kernels developed by Vere-Jones (1967, 1968), Tweedie (1974), Althreya & Ney (1982) and others. The notation and terminology follows Nummelin (1984) (abbrev. [N]).

There is an extensive literature on the potential theory of positive kernels. We refer the reader to the books by Kemeny, Snell & Knapp (1966), Constantinescu & Cornea (1972), Revuz (1975) and their bibliographies. The closest reference to us is the paper by Neveu (1972a). Neveu introduced the important concept of a special function and proved the existence and uniqueness of *bounded* solutions to the P.E. Our main contribution is to remove the assumption of specialty. This is done via a "reduced" kernel which allows the transformation of results from the transient case to the recurrent case. Due to the lack of the assumption of specialty we have first to investigate carefully the existence and finiteness of the solutions of the P.E. for transient kernels. This forms the topic of Section 2. Section 3 then deals with the main new results concerning with the P.E. for recurrent kernels.

Most articles deal with the case where K is a substochastic kernel, that means

$$K(x, E) \leqq 1 \text{ for all } x \in E.$$

But this is not an essential restriction, since by transforming the general kernel K via a superharmonic function h one gets the substochastic kernel

$$K_h(x, dy) = h(x)^{-1}K(x, dy)h(y),$$

and there is an obvious isomorphism between the solutions of the Poisson equations for K and K_h.

2. Preliminary results.

Let G be the *potential kernel of* K; that means,

$$G = \sum_{n=0}^{\infty} K^n.$$

(Note that G may be non-σ-finite. $G(x, A)$ may even attain only the values 0 and ∞; cf. Proposition 2.4 below.) A function $p \in \mathscr{E}_+$, $p \not\equiv \infty$, is called a *potential*, if

$$p = Gg \text{ for some } g \in \mathscr{E}_+.$$

Clearly every potential p satisfies the P.E. (on $D = E$) with charge g. Hence p is superharmonic.

Let φ be a measure on $(E, \mathscr{E})$ (that is, a countably additive set function: $\mathscr{E} \to [0, \infty]$). A set $A \in \mathscr{E}$ is said to *support* φ, if $\varphi(A^c) = 0$. The measure $\psi = \varphi G$ is called the *potential measure* (associated with φ). A set $A \in \mathscr{E}$ is called *φ-full*, if A supports the potential measure φG.

For any $x \in E$, $A \in \mathscr{E}$, we write $x \to A$ when $K^n(x, A) > 0$ for some $n = n(x, A) \geqq 1$. A non-empty set $F \in \mathscr{E}$ is called *closed* (for the kernel K), if F supports $K(x, \cdot)$ for all $x \in F$ (or, equivalently, $x \nrightarrow F^c$ for all $x \in F$).

We shall illustrate the concept of a φ-full set in three special cases:

EXAMPLE 2.1. If $\varphi = \varepsilon_\alpha$ (= the unit mass at α) for some element $\alpha \in E$, then A is ε_α-full if and only if $\alpha \in A$ and $\alpha \nrightarrow A^c$. In particular, every closed set $F \in \alpha$ is ε_α-full.

EXAMPLE 2.2. Let φ be a σ-finite measure on $(E, \mathscr{E})$. Assume that the kernel K is *φ-irreducible*; that means,

$$x \to A \text{ for all } x \in E, \text{ all } \varphi\text{-positive } A \in \mathscr{E}$$

(see e.g. [N], Section 2.2). Then any σ-finite measure ψ which is equivalent to the potential measure φG is also an *irreducibility measure* for K; ψ is even *maximal* in the sense that $\varphi' \ll \psi$ for all irreducibility measures φ'. By the uniqueness of the maximal irreducibility measure, if φ' is any irreducibility measure, its potential measure $\varphi' G$ is equivalent to ψ. Hence the classes of φ-full and φ'-full sets coincide (and are equal to the class $\{A \in \mathscr{E}: \psi(A^c) = 0\}$). In what follows, when dealing with an irreducible kernel K we call this unique class of φ-full sets simply the class of *full sets*. In this case also every closed set F is full ([N], Proposition 2.5). The elements of the class

$$\mathscr{E}^+ \stackrel{\text{def.}}{=} \{A \in \mathscr{E} : \psi(A) > 0\}$$

are called *positive sets*.

EXAMPLE 2.3. (Neveu (1972b)) Assume that K is *weakly φ-irreducible*; that means

$$x \to A \text{ for } \varphi\text{-a.e. } x \in E, \text{ all } \varphi\text{-positive } A \in \mathscr{E}.$$

(A standard example is the "deterministic" transition kernel $f \mapsto Kf = f \circ T$ associated with an ergodic measure-preserving transformation T on a probability space $(E, \mathscr{E}, \varphi)$.)

Then similarly as in the case of φ-irreducibility, any σ-finite measure which is equivalent to the potential measure φG is a maximal (weak) irreducibility measure. Again we call the unique class of φ-full sets the class of *full sets*, and the class of ψ-positive sets simply *positive sets*. Now every positive closed set is full.

It is easy to see that the classes of φ-full sets and of closed sets which support φ almost coincide:

PROPOSITION 2.1. (i) *A closed set $F \in \mathscr{E}$ is φ-full if and only if it supports φ.*
(ii) *If a set $A \in \mathscr{E}$ is φ-full, then there exists a closed φ-full set $F \subset A$.*

PROOF. (i) Obvious.
(ii) Set $F = \{x \in E : G(x, A^c) = 0\}$.

If the domain D is closed, then the values of the solution f outside D have no influence:

PROPOSITION 2.2. *Assume that f is a solution of the* P.E. *on a closed domain D. Let $\tilde{f}$ be any function such that $\tilde{f} = f$ on D. Then also $\tilde{f}$ is a solution of the* P.E. *on D.*

PROOF. Since D is closed, also $Kf = K\tilde{f}$ on D, from which the assertion follows.

REMARK 2.1. Proposition 2.2 allows us to apply any result proved for the case $D = E$ (= the whole space) to the case $D = F$ (= any closed set).

The following proposition gives a sufficient condition for the existence of a closed φ-full domain where f is finite:

PROPOSITION 2.3. *Assume that f is a φ-almost everywhere finite solution of the* P.E. *on a φ-full domain D'. Then there is a closed φ-full domain $D \subset D'$ where f is finite.*

PROOF. By Proposition 2.1 (ii) there is no loss of generality in assuming that D' is closed. Set

$$D = \{|f| < \infty\} \cap D'.$$

Then, by our hypotheses, D supports φ. Hence, by Proposition 2.1 (i), it suffices to show that D is closed. To this end, take an arbitrary element $x \in D$. Since $f(x)$ is finite and equal to the well defined (see Remark 1.1) quantity $g(x) + Kf(x)$, it follows that $Kf(x)$ is finite, too. Consequently, $K(x, \{|f| = \infty\}) = 0$, and therefore (recalling that D' is closed) $K(x, D^c) = 0$.

Proposition 2.3 has the following corollaries:

COROLLARY 2.1. (cf. Example 2.1) *Assume that f is a solution of the* P.E. *on a domain D', and that $\alpha \in D'$ is such that $f(\alpha)$ is finite and $\alpha \nrightarrow (D')^c$. Then there is a closed domain $D \ni \alpha$, $D \subset D'$ on which f is finite.*

COROLLARY 2.2. (cf. Examples 2.2 and 2.3) *Assume that K is weakly irreducible. Let f be a solution of the* P.E. *on a full domain D' and suppose that f is finite on a positive set. Then there exists a closed full domain $D \subset D'$ where f is finite.*

REMARK 2.2. Henceforth we will usually assume that the domain D is closed and that f is finite on D. Often we will even take $D = E$, since by Proposition 2.1 the Poisson equation can be properly restricted to any closed set.

The proof of Proposition 2.3 contains also the following result:

COROLLARY 2.3. *Suppose that the domain D is closed. Then, the intersection $D \bigcap \{|f| < \infty\}$ is closed. In particular, if $D = E$, then the set $\{|f| < \infty\}$ is closed.*

Next we will study the P.E. in the case where there exist finite potentials. Concerning the finiteness of potentials we have the following results:

Since a potential $p = Gg$ is a solution of the P.E., Corollary 2.3 implies that the set $\{p < \infty\}$ is closed. For irreducible and weakly irreducible kernels we can say more:

PROPOSITION 2.4. *Suppose that K is an irreducible kernel with maximal irreducibility measure ψ. Then either*

(i) *there is a closed full set F and strictly positive charge g_o, $g_o > 0$ everywhere, such that the potential $p_o = Gg_o$ is finite on F, or*

(ii) $Gg \equiv \infty$ *for all* $g \in \mathscr{E}_+$ *with* $\psi(g) > 0$.

In fact, Proposition 2.4 is a special case of Theorem 3.2 in [N]. Since the proof is a simple application of the previous results we present it here.

PROOF. If (ii) does not hold, then for some ψ-positive g_1, $p_1 = Gg_1$ is not identically infinite. By Corollary 2.3 the set $F = \{p_1 < \infty\}$ is closed (whence also full; see Example 2.2). Define a charge g_o by

$$g_o = \sum_{n=0}^{\infty} 2^{-(n+1)} K^n g_1.$$

By irreducibility $g_o > 0$ everywhere. Clearly

$$p_o = Gg_o \leqq p_1 < \infty \text{ on } F.$$

In the case (i) we call the irreducible kernel K transient. Similarly one can prove:

PROPOSITION 2.5. *Suppose that K is a weakly irreducible kernel with maximal weak irreducibility measure ψ. Then either*

(i) *there is a closed full set F and a charge $g_o, g_o > 0$ ψ-a.e., such that the potential $p_o = Gg_o$ is finite on F, or*

(ii) $Gg = \infty$ ψ-a.e. *for all $g \in \mathscr{E}_+$ with $\psi(g) > 0$.*

In the case (i) we again call K *transient.*

For the rest of this section we consider the P.E. in the case where the potential of the absolute value of the charge g exists:

$$G|g| \not\equiv \infty.$$

(In particular, if K is irreducible and $\psi|g| > 0$, this means that we are necessarily in the transient case; see Proposition 2.4.)

Let $g \in \mathscr{E}$ be a charge such that the potential $G|g| = Gg_+ + Gg_-$ is not identically infinite. (g_+ and g_- denote the usual positive and negative parts of the function g.) It follows that the function $p = Gg$ is well defined on the closed set

$$D_g \overset{\text{def.}}{=} \{Gg_+ < \infty\} \cup \{Gg_- < \infty\}$$

and finite on the closed set

$$\{G|g| < \infty\} = \{Gg_+ < \infty\} \cap \{Gg_- < \infty\}.$$

We call p the *signed potential with charge g (on the domain D_g).*

The following result is obvious:

PROPOSITION 2.6. *If $p = Gg$ is a signed potential, then p satisfies the* P.E. *with charge g on the domain D_g.*

We need also the concept of a signed harmonic function:

Let D be a closed set. A function $h \in \mathscr{E}$ is called *signed harmonic on D*, if $h(x)$ is finite for some $x \in D$ and

$$h = Kh \text{ on } D.$$

Thus a signed harmonic function h is a solution of the P.E. on D with zero charge. Hence all the previous results concerning the P.E. hold true for h. A non-negative signed harmonic function is simply called *harmonic*.

The classical Riesz decomposition result states that a superharmonic function can be written as the sum of a potential and of a harmonic function (see e.g. Doob (1959)). A similar decomposition is valid for the solution of the P.E.:

THEOREM 2.1. (i) *Suppose that is a closed domain D' such that $f \in \mathscr{E}$ satisfies the* P.E. *with charge g on D' and that the set $D' \cap \{|f| < \infty\} \cap \{G|g| < \infty\}$ is non-empty (whence closed). Define iteratively a sequence of functions (f_n) by setting*

$$f_o = f, f_n = Kf_{n-1} \text{ for } n \geqq 1.$$

Then, on the closed set $D = D' \cap \{G|g| < \infty\}$ we have:

The functions f_n are all well defined,

$$\lim f_n(x) = f_\infty(x)$$

exists, the limit function f_∞ is signed harmonic, and the solution f of the P.E. *can be written as th sum*

$$f = Gg + f_\infty.$$

Moreover, the functions f_n, f_∞ all are finite on the closed set $D \cap \{|f| < \infty\}$.

(ii) *Conversely, suppose that there is a closed set D and $g \in \mathscr{E}$ such that $G|g|$ is finite on D and h is a signed harmonic function on D. Then the sum*

$$f = Gg + h$$

satisfies the P.E. *with charge g on D. Moreover, the decomposition of f into a signed potential and a signed harmonic function is unique:*

$$f_\infty = \lim f_n = h \text{ on } D.$$

PROOF. The basic ideas of the proof are standard and follow those of the proof of the classical Riesz decomposition.

(i) On $D = D' \cap \{G|g| < \infty\}$ we have: Iteration of the equation

$$f - g = Kf = f_1$$

yields

$$f - \sum_0^{N-1} K^n g = f_N \text{ for all } N \geqq 1.$$

As $N \to \infty$ the left hand side tends to the limit $f - Gg$. Hence also $\lim f_n = f_\infty$ exists. That $f_\infty = f - Gg$ is signed harmonic follows from the equations

$$\begin{aligned} f - Gg &= f - g - KGg \\ &= Kf - KGg \\ &= K(f - Gg). \end{aligned}$$

(ii) On D we have:

$$\begin{aligned} f &= Gg + h \\ &= g + KGg + Kh \end{aligned}$$

by Proposition 2.6 and since h is singed harmonic,

$$= g + Kf.$$

By (i)

$$f = Gg + f_\infty,$$

and consequently,

$$h = f_\infty.$$

Note that in the iterative definition of the functions f_n we do not require that $K^n f \overset{\text{def}}{=} \int K^n(x, dy) f(y)$ should exist for $n \geqq 2$. Of course, if $K^n f$ exists (e.g., if f is non-negative) then $f_n = K^n f$ for such n.

According to Proposition 2.6 a signed potential is always a solution of the P.E. The following two corollaries of Theorem 2.1 give conditions which force the solution to be a signed potential. By Theorem 2.1 this is equivalent to requiring that

$$f_\infty = \lim f_n = 0.$$

COROLLARY 2.4. *Let f be a solution of the Poisson equation with charge g. If $|f| \leqq p_o$ for some potential $p_o = Gg_o \not\equiv \infty$, then $f = Gg$ on the closed set $\{p_o < \infty\}$. In particular, if $G|f| \not\equiv \infty$, then $f = Gg$ on the closed set $\{G|f| < \infty\}$.*

PROOF. Clearly $|f_\infty| \leqq (p_o)_\infty = 0$ on $\{p_o < \infty\}$.

EXAMPLE 2.4. An important special case is the case where K is a *substochastic kernel*, that means $K(x, E) \leqq 1$ for all $x \in E$, or in other words, the function 1, defined by $1(x) \equiv 1$, is superharmonic. In this case we use the symbol P instead of K. A substochastic kernel P always governs the transitions of a (possibly terminating) Markov chain $(X_n; n = 0, 1, \ldots)$ on $(E, \mathscr{E})$ (see e.g. [N], Section 1.2). Let L denote the *life time* of (X_n), that is the random time

$$L = \sup\{n \geqq 0 : X_n \in E\} \quad (\leqq \infty).$$

Note that in this case any signed potential $p = Gg$ has the probabilistic interpretation $Gg(x) = \mathsf{E}_x \sum_{n=0}^{L} g(X_n) (= \mathsf{E}_x \sum_{n=0}^{L} g_+(X_n) - \mathsf{E}_x \sum_{n=0}^{L} g_-(X_n))$. Clearly

$$\mathsf{P}_x\{L \geqq n\} = \mathsf{P}_x\{X_n \in E\} = P^n 1(x).$$

Hence the probability

$$\mathsf{P}_x \{\text{the Markov chain does not terminate}\} = \mathsf{P}_x\{L = \infty\}$$

is equal to the limit

$$\lim_{n \to \infty} P^n 1(x) = 1_\infty(x),$$

which is the harmonic part in the Riesz decomposition of the superharmonic function 1. The charge is

$$1 - P1(x) = \mathsf{P}_x\{L = 0\}.$$

Since clearly the set where a harmonic function equals zero is closed, Corollary 2.4 gives the following results:

COROLLARY 2.5. *Let $(X_n; n = 0, 1, \ldots)$ be a Markov chain with substochastic transition kernel P and life time L. Suppose that $\mathsf{P}_\alpha\{L < \infty\} = 1$ for some $\alpha \in E$. Let f be a bounded solution of the* P.E.

$$f = g + Pf.$$

Then

$$f = Gg = \sum_{n=0}^{\infty} P^n g = \mathsf{E}_{(\cdot)} \sum_{0}^{L} g(X_n)$$

on the closed set

$$\{1_\infty = 0\} = \{x \in E : \mathsf{P}_x\{L < \infty\} = 1\}.$$

3. Recurrent kernels.

Basic assumption. Throughout this section we assume that K is an irreducible kernel satisfying the following "limiting transience" conditions (cf. Proposition 2.4):

(i) For all $0 < r < 1$, the kernel rK is transient, i.e. for all such r there is a closed full set $F^{(r)}$ and strictly positive charge $g_0^{(r)}$ such that the potential

$$p_0^{(r)} = \sum_{n=0}^{\infty} r^n K^n g_0^{(r)}$$

is finite on $F^{(r)}$; and

(ii) the kernel K is not transient, i.e.

$$Gg \equiv \infty \text{ for all } g \in \mathscr{E}_+ \text{ with } \psi(g) > 0.$$

We call such a kernel K *recurrent*. (In Vere-Jones' (1967) and Tweedie's (1974) terminology K is 1-recurrent. See also [N], Section 3.2.) The important special case of a *Harris recurrent Markov chain* will be discussed later in this section.

From condition (ii) it follows that for charges $g \in \mathscr{E}$ with $\psi(|g|) > 0$ the signed potentials Gg appearing in the solutions given by Theorem 2.1 are either infinite or not well defined. However, it turns out that non-trivial solutions for the P.E. may still exist for such charges.

Our starting point is the following *minorization condition*, with holds true for any irreducible kernel K (see [N], Theorem 2.1):

There exist an integer $m_o \geqq 1$, a function $s \in \mathscr{E}_+$ with $\psi(s) > 0$ and non-zero measure ν on $(E, \mathscr{E})$ such that

$$K^{m_o}(x, A) \geqq s(x)\nu(A) \text{ for all } x \in E, A \in \mathscr{E}.$$

Using the notation $s \otimes \nu$ for the kernel $(s(x)\nu(A); x \in E, A \in \mathscr{E})$ we can write briefly

$$K^{m_o} \geqq s \otimes \nu.$$

Let us assume for a while that $m_o = 1$, i.e. *that* K *satisfies the minorization condition*

$$K \geqq s \otimes \nu. \tag{3.1}$$

Note that, in the special case where K has a *proper atom*, that is a point $\alpha \in E$ with $\{\alpha\} \in \mathscr{E}^+$, the minorization condition (3.1) is automatically satisfied. Namely, set $s(x) = K(x, \{\alpha\})$ for all x, $\nu = \varepsilon_\alpha =$ the unit mass at α (see also Example 3.1 below).

We denote by $G_{s,\nu}$ the potential kernel of the kernel $K - s \otimes \nu$,

$$G_{s,\nu} = \sum_{n=0}^{\infty} (K - s \otimes \nu)^n.$$

Note that

$$G_{s,\nu} = I + (K - s \otimes \nu)G_{s,\nu} \tag{3.2}$$

and $G_{s,\nu}$ is the minimal non-negative kernel G satisfying the inequality

$$G \geqq I + (K - s \otimes \nu)G. \tag{3.3}$$

By Theorem 5.1 of [N] (see also Tweedie (1974)) the potential

$$h_\nu \overset{\text{def}}{=} G_{s,\nu}s \tag{3.4}$$

is harmonic for K. Moreover, $h_\nu > 0$ everywhere, the set $\{h_\nu < \infty\}$ is closed and full, $\nu(h_\nu) = 1$, and h_ν is the unique minimal superharmonic function h with $\nu(h) = 1$ (that means, $h \geqq Kh$ implies $h \geqq ch_\nu$ everywhere and $h = ch_\nu$ ψ-almost everywhere, where $c = \nu(h)$ is a finite constant). Hence, in particular, there exist finite potentials for the minorized kernel $K - s \otimes \nu$.

Note that a set $F \in \mathscr{E}$ is closed (for K) if and only if F supports ν and F is closed for the kernel $K - s \otimes \nu$. From this and from Corollary 2.3. we immediately obtain the following lemma. Recall that any closed set F is also full (that means $\pi(F^c) = 0$, where π is the invariant measure of the recurrent kernel K).

LEMMA 3.1. *Let $g \in \mathscr{E}_+$. If $G_{s,\nu}g < \infty$ ν-a.e., then in fact the set $\{G_{s,\nu}g < \infty\}$ is closed.*

By Theorem 5.2 of [N] (see also Tweedie (1974)) the potential measure

$$\pi_s \stackrel{\text{def}}{=} \nu G_{s,\nu} \tag{3.5}$$

is invariant for K, that means

$$\pi_s K = \pi_s.$$

Moreover π_s is σ-finite and equivalent to ψ (hence is a maximal irreducibility measure), $\pi_s(s) = 1$ and π_s is the unique subinvariant measure π (that means $\pi \geqq \pi K$) satisfying $\pi(s) = 1$.

From Lemma 3.1 we get:

COROLLARY 3.1 (cf. Proposition 5.12 of [N]). *If $g \in \mathscr{E}_+$ is π-integrable, then set $\{G_{s,\nu}g < \infty\}$ is closed.*

Recall our basic assumption that K is an irreducible recurrent kernel satisfying the minorization condition (3.1). Our main result for such kernels is the following theorem:

THEOREM 3.1. (i) *Suppose that $f \in \mathscr{E}$ is a solution of the* P.E. *with charge $g \in \mathscr{E}$ on a closed domain D' satisfying $G_{s,\nu}|g| < \infty$ ν-almost everywhere. Define a sequence of functions (f_n) iteratively by setting*

$$f_o = f, f_n = (K - s \otimes \nu) f_{n-1} \text{ for } n = 1,2,\ldots. \tag{3.6}$$

Then, on the closed set $D \stackrel{\text{def}}{=} D' \cap \{G_{s,\nu}|g| < \infty\}$ we have:

The functions f_n $(n = 0, 1, \ldots)$ and $f_\infty \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n$ are all well defined, and they are finite on the closed set $D \cap \{|f| < \infty\}$. Moreover, f and the sum $G_{s,\nu}g + f_\infty$ are ν-integrable,

$$\nu(G_{s,\nu}g + f_\infty) = 0, \tag{3.7}$$

f can be written as the sum

$$f = G_{s,\nu}g + f_\infty + \nu(f)h_\nu,$$

and f_∞ is signed harmonic for the kernel $K - s \otimes \nu$:

$$f_\infty = (K - s \otimes \nu) f_\infty.$$

(ii) *Conversely, suppose that on some closed set D the potential $G_{s,\nu}|g|$ is finite, $e \in \mathscr{E}$ is finite and signed harmonic for the kernel $K - s \otimes \nu$ and that $\nu(G_{s,\nu}g + e) = 0$. Then for any constant $-\infty < c < \infty$; the function*

$$f = G_{s,\nu}g + e + ch_\nu$$

satisfies the P.E. *with charge g on the domain D. Moreover, this representation is unique; namely,*

$$e = \lim_{n \to \infty} f_n \text{ on } D$$

and $c = \nu(f)$.

REMARK 3.1. Note that the condition $G_{s,\nu}|g| < \infty$ ν-a.e. is automatically satisfied if g is π-integrable (see Corollary 3.1). Then also the condition (3.7) takes the form

$$\nu(f) = -\pi(g). \tag{3.8}$$

PROOF OF THEOREM 3.1. First observe that, if f is a solution of the P.E., then it is finite on a closed full set F. Since f satisfies the P.E., Kf (whence also $K|f|$) is finite on F. This and the minorization (3.1) imply that f is ν-integrable.

Secondly, note that f satisfies the P.E.

$$f = g + Kf,$$

if and only if it satisfies the "transient" P.E.

$$f = (g + \nu(f)s) + (K - s \otimes \nu)f. \tag{3.9}$$

Applying Theorem 3.1 to the P.E. (3.9) and recalling formula (3.4) proves the asserted results.

Using Corollary 2.4 we obtain a criterion for the vanishment of the signed harmonic part f_∞:

COROLLARY 3.2. *Suppose that $f \in \mathscr{E}$ is a solution of the* P.E. *with charge $g \in \mathscr{E}$, $G_{s,\nu}|g| < \infty$ ν-a.e. If $|f| \leqq G_{s,\nu}g_o$, $G_{s,\nu}g_o < \infty$ ν-a.e., for some $g_o \in \mathscr{E}_+$, then necessarily g is π-integrable, $\pi(g) = 0$ and*

$$f = G_{s,\nu}g + \nu(f)h_\nu \tag{3.10}$$

on the closed set $\{G_{s,\nu}g_o < \infty\}$.

In particular, if $G_{s,\nu}|f| < \infty$ ν-a.e. (e.g. if f is π-integrable), then (3.10) holds on the closed set $\{G_{s,\nu}|f| < \infty\}$.

Setting $g_o = s$ in the above corollary yields the following corollary. In what follows the symbol M denotes a constant $0 \leqq M < \infty$ whose value may vary in different contexts:

COROLLARY 3.3. *Suppose that* $f \in \mathscr{E}, |f| \leqq Mh_\nu$ *is a solution of the* P.E. *with charge* $g \in \mathscr{E}$, $G_{s,\nu}|g| < \infty$ ν-a.e. *Then* $\pi(g) = 0$ *and* (3.10) *holds on the closed set* $\{h_\nu < \infty\}$.

Specializing to the case $g \equiv 0$ gives the following uniqueness result for signed harmonic functions.

COROLLARY 3.4. *Suppose that* $h \in \mathscr{E}, |h| \leqq Mh_\nu$ *is signed harmonic. Then necessarily*

$$h = \nu(h)h_\nu.$$

In fact, it turns out that the assumptions of Corollary 3.3 can be weakened:

First recall from Section 5.7 of [N] (see Propositions 5.13 (iii) and 5.26 of [N]) the definition of a special function. A function $g \in \mathscr{E}$ is *special*, if

$$G_{s,\nu}|g| \leqq Mh_\nu.$$

(The concept of specialty was first introduced by Neveu (1972a) for Harris recurrent stochastic kernels $K = P$.)

Let us call function $g \in \mathscr{E}$ *special from below*, if

$$G_{s,\nu}g \geqq -Mh_\nu. \tag{3.11}$$

So, e.g. every non-negative function $g \in \mathscr{E}_+$ is special from below. Clearly, $g \in \mathscr{E}$ is special from below if its negative part g_- is special.

We have the following result:

THEOREM 3.2. (i) *Suppose that* $f \geqq -Mh_\nu$ *is a solution of the* P.E. *with charge* $g, \pi(g) = 0$. *Then necessarily, g is special from below,* $f_\infty = 0$ π-a.e., *and*

$$f = G_{s,\nu}g + \nu(f)h_\nu \qquad \pi\text{-a.e.}$$

(ii) *Conversely, suppose that* $g \in \mathscr{E}$, $\pi(g) = 0$, *is special from below. Then for any constant c, the function*

$$f = G_{s,\nu}g + ch_\nu$$

satisfies the P.E. *with charge g. Moreover,* $f \geqq -Mh_\nu$ *and* $c = \nu(f)$.

PROOF. (i) Since also $f' = f + Mh_\nu \geqq 0$ is a solution of the P.E. with the same charge g, we can assume without loss of generality that $f \geqq 0$. By (3.8)

$$\nu(f_\infty) = -\pi(g) = 0.$$

Hence, by Proposition 3.2 of [N], the set $\{f_\infty = 0\}$ is closed for $K - s \otimes \nu$. Since $f_\infty = 0$ ν-a.e., this set is closed (whence full) for K, too.

(ii) This is a direct consequence of Theorem 3.1.

The solution f with $f_\infty = 0$ has the following uniqueness property:

THEOREM 3.3. *The function* $f = G_{s,\nu}g + \nu(f)h_\nu$, $\pi(g) = 0$, *is the unique (modulo a* π*-null set and multiplication with a constant) solution of the* P.E., *which can be written in the form*

$$f = f_1 - f_2,$$

where for $i = 1,2$, $f_i \geqq 0$ *is a solution of the* P.E. *with charge* g_i, g_i *is special from below,* $\pi(g_i) = 0$ *and* $g = g_1 - g_2$. *We can take*

$$f_1 = G_{s,\nu}g_+ + c_1 h_\nu, \quad g_1 = g_+ - \pi_s(g_+)s,$$

$$f_2 = G_{s,\nu}g_- + c_2 h_\nu, \quad g_2 = g_- - \pi_s(g_-)s,$$

where $c_i \geqq 0$, $c_1 - c_2 = \nu(f)$.

PROOF. It is sufficient to prove the uniqueness only. So let $f = f_1 - f_2$ be a solution of the P.E. having the stated properties. Then by Theorem 3.2 (i)

$$f_i = G_{s,\nu}g_i + \nu(f_i)h_\nu,$$

from which it follows that

$$f = G_{s,\nu}g + \nu(f)h_\nu.$$

All the above results were based on the specific minorization condition (3.1) and the associated potential kernel $G_{s,\nu} = \sum (K - s \otimes \nu)^n$. One might ask how the potential kernel $G_{s',\nu'} = \sum (K - s' \otimes \nu')^n$ associated with a different minorization with quantities s' and ν' is related to the potential kernel $G_{s,\nu}$. The corresponding result in Neveu (1972a) is Proposition 5.7.

Let $E_h = \{h_\nu < \infty\}$ $(= \{h_{\nu'} < \infty\}$, by uniqueness).

PROPOSITION 3.2. *On* E_h:

$$\begin{aligned} G_{s',\nu'} &= G_{s,\nu} - G_{s,\nu}s' \otimes \pi_{s'} + h_{\nu'} \otimes \pi_{s'} - h_{\nu'} \otimes \nu' G_{s,\nu} + (\nu' G_{s,\nu} s')h_{\nu'} \otimes \pi_{s'} \\ &= (I - h_{\nu'} \otimes \nu')G_{s,\nu}(I - s' \otimes \pi_{s'}) + h_{\nu'} \otimes \pi_{s'}. \end{aligned}$$

REMARK. The corresponding result in Neveu (1972a) is Proposition 5.7.

PROOF. Note that for $x \in E_h$ $G_{s,\nu}(x,\cdot)$ and $G_{s',\nu'}(x,\cdot)$ are σ-finite measures. We define the kernels F and F' by

$$F = G_{s,\nu} + G_{s',\nu'}s \otimes \pi_s,$$

$$F' = G_{s',\nu'} + h_{\nu'} \otimes \nu' G_{s,\nu}.$$

We shall first prove that these two kernels are equal. To this end, from (3.2) and (3.3) it follows that

$$F + (K - s \otimes v)G_{s,v} = G_{s,v} + KG_{s,v} + (K - s' \otimes v')G_{s',v'}s \otimes \pi_s$$
$$= G_{s,v} + s' \otimes v'G_{s,v} + (K - s' \otimes v')F.$$

Hence

$$F = I + s' \otimes v'G_{s,v} + (K - s' \otimes v') \text{ on } E_h. \tag{3.12}$$

This implies (see (3.3)) that

$$F \geqq \text{the minimal solution of (3.12)}$$
$$= G_{s',v'}(I + s' \otimes v'G_{s,v})$$
$$= F'.$$

By symmetry, in fact $F = F'$ (on E_h). Let now $g \in \mathscr{E}_+$, $\pi(g) < \infty$, be arbitrary.
The identity

$$F(g - \pi_{s'}(g)s') = F'(g - \pi_{s'}(g)s')$$

leads to the identity

$$G_{s,v}g + \pi_s(g)G_{s',v'}s - \pi_{s'}(g)G_{s,v}s' - \pi_{s'}(g)\pi_s(s')G_{s',v'}s$$
$$= G_{s',v'}g + (v'G_{s,v}g)h_{v'} - \pi_{s'}(g)G_{s',v'}s' - \pi_{s'}(g)(v'G_{s,v}s')h_{v'}$$

from which the assertion follows.

We will now look at two examples. At first we consider the case where the kernel K is induced by a matrix:

EXAMPLE 3.1. Suppose that $E = \{0, 1, \ldots\}$, $\emptyset \neq F \subset E$ and let $K = (k(x, y); x, y \in E)$ be a Card_F-irreducible 1-recurrent matrix on E (see Examples 3.1 (a) and 4.1 (a) of [N]). Then any state $\alpha \in E$ is a proper atom, and letting K_α denote the matrix obtained by removing the αth column

$$k_\alpha(x, y) = \begin{cases} k(x, y) & \text{for } y \neq \alpha, \\ 0 & \text{for } y = \alpha, \end{cases}$$

$k_\alpha(x, y)$ can be written in the form

$$k_\alpha(x, y) = (k - s \otimes v)(x, y),$$

where $s(x) = k(x, \alpha)$ for all x, $v(y) = 0$ for $y \neq \alpha$, $v(\alpha) = 1$. Let

$$G_\alpha = \sum_0^\infty (K_\alpha)^n.$$

Then the column vector h_α with components

$$h_\alpha(x) = G_\alpha K(x, \alpha), \qquad x \in E,$$

is the unique (on F) harmonic vector satisfying $h_\alpha(\alpha) = 1$. The unique invariant row vector π_α satisfying $\pi_\alpha(\alpha) = 1$ is

$$\pi_\alpha(x) = G_\alpha(a, x), \qquad x \in E.$$

As a direct corollary of Theorem 3.1 we have:

COROLLARY 3.5. *Let K be a Card_F-irreducible 1-recurrent matrix on $E = \{0, 1, \ldots\}$. Let $\alpha \in F$ be arbitrary.*

(i) *Suppose that the column vector $f = (f(x); x \in E)$ is a solution of the* P.E. *on F with charge g, $\pi_\alpha|g| < \infty$. (Then in fact $G_\alpha|g|(x)$ for all $x \in F$.) Define the sequence (f_n) by* (3.6).

Then on F we have: f_n, $n = 0, 1, \ldots$, and f_∞ are finite, $f_\infty(\alpha) = -\pi_\alpha(g)$, and

$$f = G_\alpha g + f_\infty + f(\alpha)h_\alpha,$$

$$f_\infty = K_\alpha f_\infty.$$

(ii) *Conversely, suppose that g, $\pi_\alpha|g| < \infty$, is a column vector, e is a finite (on F) harmonic (for K_α on F) column vector with $e(\alpha) = -\pi_\alpha(g)$. Then for any constant c, the column vector*

$$f = G_\alpha g + e + ch_\alpha$$

satisfies the P.E. *with charge g on F. Moreover,*

$$e = f_\infty \text{ on } F$$

and $c = f(\alpha)$.

EXAMPLE 3.2. Suppose that $K = P$ is an irreducible recurrent transition kernel of a Markov chain (X_n) on $(E, \mathscr{E})$ satisfying the minorization condition (3.1). Then the harmonic function $h_\nu = 1$ ψ-a.e. (see [N], Proposition 3.13 and Corollary 5.1). (X_n) is called *Harris recurrent*, if

$$\mathsf{P}_x\{X_n \in A \text{ infinitely often}\} = 1 \text{ for all } x \in E, A \in \mathscr{E}^+.$$

By theorem 3.8 (ii) of [N] Harris recurrence is equivalent to the condition

$$h_\nu \equiv \text{a constant } (= (\nu(E))^{-1}, \text{since } \nu(h_\nu) = 1).$$

Usually we norm ν to a probability measure; hence in this case

$$h_\nu \equiv 1. \tag{3.13}$$

Since $h_\nu \equiv 1$ is harmonic, the kernel P is stochastic; that means

$$P(x, E) = 1.$$

This implies that the life time $L = \infty$ P_x-almost surely for all $x \in E$.

The minorization (3.1) has a probabilistic interpretation; namely there is a proper atom α in a suitably extended state space (see Athreya & Ney (1978) or [N], Section 4.4). With the atom α there is associated the hitting time T_α of the atom α. The potential $G_\alpha f \stackrel{\text{def}}{=} G_{s,\nu} f$ has the interpretation

$$G_\alpha f(x) = \mathsf{E}_x \sum_{n=0}^{T_\alpha} f(X_n). \tag{3.14}$$

In fact T_α is the life time of the Markov chain associated with the reduced kernel $P - s \otimes \nu$. It follows from the Harris recurrence that T_α is finite P_x-a.s. for all $x \in E$, i.e. (X'_n) terminates (a.s.).

From the expression (3.4) we also see that $h_\nu \equiv 1$ is the potential with charge s for the kernel $P - s \otimes \nu$:

$$1 = G_{s,\nu} s.$$

Theorem 3.1 and Corollary 3.2 could easily be restated for the Harris recurrent kernel P. Let us however only remark that the general solution of the Poisson equation has the form

$$f = G_{s,\nu} g + f_\infty + c,$$

where the constant $c = \nu(f)$. Corollary 3.3 takes the following form.

COROLLARY 3.6. *Let (X_n) be a Harris recurrent Markov chain with transition kernel P satisfying* (3.1). *Let f be a bounded solution of the Poisson equation with charge g, $G_{s,\nu}|g| < \infty$ ν-a.e. Then $\pi(g) = 0$ and*

$$f = G_{s,\nu} g + \nu(f).$$

In fact this corollary could be obtained also by applying Corollary 2.5 to the ter terminating Markov chain (X'_n) introduced above.

Note that, conversely, if $|g|$ is special then $f = G_\alpha g$ is a bounded solution of the P.E. (cf. (1972a), Proposition 6.1).

So far we have assumed that K satisfies the minorization condition (3.1). As noted there, this assumption is not very restrictive, since for an irreducible kernel K on a countably generated state space there always exists an integer m_o such that K^{m_o} satisfies the minorization condition, i.e.

$$K^{m_o} \geqq s \otimes \nu \tag{3.15}$$

for some $s \in \mathscr{E}_+$ with $\psi(s) > 0$ and non-zero measure ν on $(E, \mathscr{E})$. In this section we will briefly discuss this general case. *So we will assume for the rest of this section that K is a recurrent kernel satisfying the minorization condition* (3.15).

We denote by $G_{m_o,s,\nu}$ the potential kernel of $K^{m_o} - s \otimes \nu$, i.e.

$$G_{m_o,s,\nu} = \sum_{n=0}^{\infty} (K^{m_o} - s \otimes \nu)^n.$$

By Corollary 5.1 of [N] the function

$$h_\nu \overset{\text{def}}{=} G_{m_o,s,\nu} s \tag{3.16}$$

is harmonic for K, and it is the minimal superharmonic function h satisfying $\nu(h) = 1$. It follows that $G_{m_o,s,\nu}$ satisfies the identities

$$G_{m_o,}{}^{s}{}_{,\nu} = I + K^{m_o} G_{m_o,s,\nu} - s \otimes \pi_s = I + G_{m_o,s,\nu} K^{m_o} - h_\nu \otimes \nu. \tag{3.17}$$

(The invariant measure π_s satisfying $\pi_s(s) = 1$ is given by the formula

$$\pi = \pi_s = \nu G_{m_o,s,\nu}; \tag{3.18}$$

see Theorem 5.2 of [N].)

We need the following lemma:

LEMMA 3.2. *Suppose that $g \in \mathscr{E}$ is π-integrable. Then the set*

$$F = \bigcap_{n=0}^{\infty} \{K^n |g| < \infty\}$$

is closed (*whence* $\pi(F^c) = 0$).

PROOF. Clearly, $\pi(F^c) = 0$; hence F is not empty. If $x \in F$, then $K(x, \{K^n|g| = \infty\}) = 0$ for all n, which proves the assertion.

For any π-integrable g we write $\bar{g}$ for the function $\bar{g} = g + Kg + \ldots + K^{m_o - 1} g$. $\bar{g}$ is well defined at least on the set F of the above lemma. We say that g is *special from belowe*, if $\bar{g}$ is special from below for the iterated kernel K^{m_o}, i.e.

$$G_{m_o,s,\nu} \bar{g} \geqq -M h_\nu,$$

(see (3.11)).

Theorems 3.1 (i) and 3.2 (i) extend to the following form:

THEOREM 3.4. (i) *Suppose that $f \in \mathscr{E}$ satisfies the* P.E. *with charge $g \in \mathscr{E}$, $\pi(g) = 0$, on a closed domain D'. Then there is a closed set $D \subset D'$ such that f is finite on D and*

$$f = G_{m_o,s,\nu} \bar{g} + f_\infty + \nu(f) h_\nu \quad on \quad D,$$

where

$$f_\infty = \lim_{n \to \infty} f_n,$$

$$f_o = f,\ f_n = (K^{m_o} - s \otimes \nu) f_{n-1} \quad for \quad n \geqq 1$$

all are finite on D. Moreover $\nu(f_\infty) = 0$.

(ii) *If, in addition, g is special from below, then* $f_\infty = 0$ π-a.e.

PROOF. (i) Iterating the Poisson equation we see that f satisfies also the Poisson equation for the kernel K^{m_o} and with charge $\bar{g}$ on the closed set F of Lemma 3.2. Then applying Theorem 3.1 (i) we see that f has the form

$$f = G_{m_o, s, \nu}\bar{g} + f_\infty + \nu(f)h_\nu.$$

Since $\nu G_{m_o,s,\nu}\bar{g} = \pi(g) = 0$ by (3.18), it follows that $\nu(f_\infty) = 0$.

By Theorem 3.2 (i) $f_\infty = 0$ if $\bar{g}$ is special from below for K^{m_o}. This proves the second part.

In the converse direction we are able to prove the following result (cf. Theorems 3.1 (ii) and 5.2 (ii)).

THEOREMS 3.5. *Suppose that* $g \in \mathscr{E}$ *is* π*-integrable with* $\pi(g) = 0$. *Then there is a closed domain D such that for any constant c, the function*

$$f = G_{m_o, s, \nu}\bar{g} + ch_\nu$$

satisfies the P.E. *with charge g on D. Necessarily* $c = \nu(f)$. *If, in addition, g is special from below then* $f \geqq -Mh_\nu$ *for some constant* $M < \infty$.

PROOF. Clearly, it suffices to consider the case $c = 0$. Suppose first that $g \in \mathscr{E}$, $\pi(g) = 0$, is special from below, i.e.

$$f = G_{m_o, s, \nu}\bar{g} \geqq -Mh_\nu.$$

Let $f' = f + Mh_\nu = G_{m_o,s,\nu}(\bar{g} + Ms) \geqq 0$ (see (3.16)). Since $\nu(f) = \pi(\bar{g}) = 0$, $\nu(f') = M$. By (3.17)

$$f' = \bar{g} + Ms + (K^{m_o} - s \otimes \nu)f' = \bar{g} + K^{m_o}f', \tag{3.19}$$

i.e. f' satisfies the P.E. for the kernel K^{m_o} and with charge $\bar{g}$. Since f' is a potential (for the kernel $K^{m_o} \otimes \nu$), $f'_\infty = 0$. Further, by "multiplying" the P.E. (3.19) with K, we see that Kf' satisfies the P.E.

$$Kf' = K\bar{g} + K^{m_o}(Kf').$$

Applying Theorem 3.2 (i) to the kernel K^{m_o}, we see that $(Kf')_\infty = 0$ π-a.e. and

$$Kf' = G_{m_o,s,\nu}K\bar{g} + \nu(Kf')h_\nu \qquad \pi\text{-a.e.}$$

For the first term on the right hand side we have

$$\begin{aligned} G_{m_o,s,\nu}K\bar{g} &= G_{m_o,s,}{}^{\nu}(\bar{g} + K^{m_o}g - g) \\ &= G_{m_o,s,\nu}\bar{g} - g + \nu(g)h_\nu \text{ by (3.17)} \\ &= f' - g + (\nu(g) - M)h_\nu. \end{aligned}$$

Consequently,

$$Kf' = f' - g + c'h_\nu \qquad \pi\text{-a.e.,} \tag{3.20}$$

where c' is the constant $c' = \nu(g) - M + \nu(Kf))$. Iteration of the above P.E. gives

$$K^{m_o}f' = f' - \bar{g} + c'm_o h_\nu, \qquad \pi\text{-a.e.}$$

Comparison with (3.19) yields $c' = 0$. Hence by (3.20) f' satisfies the P.E. with charge g. But then also $f = f' - Mh_\nu$ satisfies the same P.E. So the assertion is proved in the case where g is special from below.

In the general case the function $f = G_{m_o,s,\nu}\bar{g}$ is first decomposed into two parts, $f = f_1 - f_2$, where

$$f_i = G_{m_o,s,\nu}\bar{g}_i \text{ with}$$

$$g_1 = g_+ - \pi_s(g_+)s, \quad g_2 = g_- - \pi_s(g_-)s.$$

Clearly $\pi(g_1) = \pi(g_2) = 0$.

Since s is a special function (see Propositions 5.13 (iii) and 5.26 of [N]), g_1 and g_2 are special from below. Now the assertion follows from the first part of the proof and from Proposition 2.1 (ii).

4. On the Poisson equation for measures.

Let λ be a signed measure on $(E, \mathscr{E})$. We write

$$\lambda K(A) = \int \lambda(dx)K(x, A) \stackrel{\text{def}}{=} \int \lambda_+(d)K(x, A) - \int \lambda_-(d)K(x, A)$$

(whenever well defined).

In this section we shall briefly discuss the "dual" P.E.

$$\lambda = \mu + \lambda K \tag{4.1}$$

where λ and μ are two σ-finite signed measures on $(E, \mathscr{E})$. The given signed measure μ is called the *charge* and the unknown signed measure λ a *solution* of the P.E.

REMARK 1. By a σ-finite signed measure we mean a (partially defined) set function from $\mathscr{E}$ into the extended real line $[-\infty, \infty]$ such that it acts as a bounded signed measure on some sets $E_i \in \mathscr{E}$, $i = 0,1,\ldots$, covering E. Note that the assumption of the σ-finiteness of λ and μ in (4.1) forces λK to be σ-finite, too. The convergence of a sequence (λ_n) of signed measures towards a signed measure λ always means setwise convergence:

'lim $\lambda_n = \lambda_\infty$ if and only if there are sets

$E_i \in \mathscr{E}$, $i = 0,1,\ldots, \cup E_i = E$, such that

$$\lim_{n\to\infty} \lambda_n(A) = \lambda_\infty(A) \text{ for all } A \subset E_i, \text{ all } i.$$

The absolute value $|\lambda|$ of a signed measure is defined in the usual manner as the sum of the positive and negative parts of λ:

$$|\lambda| = \lambda_+ + \lambda_-.$$

Note that λ is σ-finite if and only if so is $|\lambda|$.

We have again first to treat the transient case. Our main new result is Theorem 4.2 where we can remove the assumption that the charge be a special measure (cf. Neveu (1972a), Proposition 6.7).

The following theorem is the dual result to Theorem 2.1:

THEOREM 4.1. *Suppose that the potential measure* $|\mu|\, G = \sum_0^\infty |\mu|\, K^n$ *is σ-finite.*

(i) *If* λ *satisfies the* P.E. *with charge* μ, *then an induction argument based on the inequality* $|\lambda|\,(K) \leqq |\mu| + |\lambda|$ *shows that* $|\lambda|\, K^n$ *is σ-finite for all n, and hence the signed measures* λ_n, $n = 0, 1, \ldots$, *defined iteratively by*

$$\lambda_o = \lambda, \lambda_n = \lambda_{n-1} K \text{ for } n = 1, 2, \ldots,$$

are σ-finite. The limit $\lim \lambda_n = \lambda_\infty$ *exists, is σ-finite and invariant for* K:

$$\lambda_\infty = \lambda_\infty K.$$

The solution λ *can be written as the sum*

$$\lambda = \mu G + \lambda_\infty.$$

(ii) *Conversely, suppose that*

$$\lambda = \mu G + \rho$$

where ρ *is a σ-finite invariant measure. Then* λ *satisfies the* P.E. *with charge* μ *and necessarily*

$$\rho = \lambda_\infty = \lim \lambda_n.$$

PROOF. The basic idea is again well known. The only thing one has to worry about is whether the quantifies involved are all well defined.

Let (E_i) be a countable partition of E such that

$$|\mu|\, G(E_i) < \infty \text{ for all } i.$$

By iterating the P.E. we get

$$\lambda - \sum_0^{N-1} \mu K^n = \lambda_N \text{ for all } N \geqq 1. \tag{4.2}$$

Letting $N \to \infty$ we have for all $j, A \subset E_j$,

$$\lim_{N\to\infty} \sum_0^N \mu K^n(A) = \lim_{N\to\infty} \uparrow \sum_0^N \mu_+ K^n(A) - \lim_{N\to\infty} \uparrow \sum_0^N \mu_- K^n(A)$$
$$= \mu_+ G(A) - \mu_- G(A)$$
$$= \mu G(A) \in (-\infty, \infty).$$

Consequently,

$$\lim_{N\to\infty} \sum_0^N \mu K^n = \mu G,$$

and hence by (4.2), $\lim_{n\to\infty} \lambda_n = \lambda_\infty$ exists, and

$$\lambda - \mu G = \lambda_\infty.$$

The rest of the proof is analogous to that of Theorem 2.1.

EXAMPLE 4.1. Let (X_n) be a Markov chain on $(E, \mathscr{E})$ with transition kernel P and life time L (see Example 2.4). As a corollary of Theorem 4.1 we have:

COROLLARY 4.1. *Let λ be a finite signed measure such that*

$$\mathsf{P}_x\{L < \infty\} = 1 \text{ for } |\lambda|\text{-almost all } x \in E.$$

Then necessarily

$$(\lambda - \lambda P)G = \lambda.$$

PROOF. Let $\mu = \lambda - \lambda P$. It is easy to see that the hypotheses imply that the potential measure $|\mu|\, G$ is σ-finite. By the monotone convergence theorem

$$\limsup_{n\to\infty} |\lambda P^n(A)| \leqq \lim_{n\to\infty} \downarrow |\lambda|\, P^n 1 = 0 \text{ for all } A \in \mathscr{E}.$$

Hence $\lambda_\infty = 0$.

For recurrent kernels satisfying the minorization condition (3.1) we have the following result (cf. Theorem 3.1).

THEOREM 4.2. *Suppose that K is a recurrent kernel satisfying the minorization condition* (3.1) *and suppose that μ is a signed measure such that the potential measure $|\mu|\, G_{s,\nu} = \sum_{n=0}^{\infty} |\mu|\,(K - s \otimes \nu)^n$ is σ-finite.* (*This is automatically true if μ is finite.*)

(i) *Suppose that* λ *is a solution of the* P.E. *with charge* μ. *Define a sequence of signed measures iteratively by setting*

$$\lambda_o = \lambda, \lambda_n = \lambda_{n-1}(K - s \otimes \nu) \text{ for } n = 1, 2, \ldots.$$

Then the signed measures λ_n $(n = 0, 1, \ldots)$ *and* $\lambda_\infty \stackrel{\text{def}}{=} \lim \lambda_n$ *are all well defined and* σ*-finite. Moreover,* $\lambda(s)$ *and* $\lambda_\infty(s)$ *are finite, and in the case where* μ *is finite we have*

$$\mu(E) = -\lambda_\infty(s).$$

Moreover, λ *can be written as the sum*

$$\lambda = \mu G_{s,\nu} + \lambda_\infty + \lambda(s)\pi_s$$

and λ_∞ *is invariant for the kernel* $K - s \otimes \nu$:

$$\lambda_\infty = \lambda_\infty(K - s \otimes \nu).$$

(ii) *Conversely, suppose that* ρ *is a* σ*-finite signed measure, invariant for* $K - s \otimes \nu$,

$$\rho = \rho(K - s \otimes \nu),$$

and that $(\mu G_{s,\nu} + \rho)s = 0$. *Then for any constant* $-\infty < c < \infty$, *the signed measure*

$$\lambda = \mu G_{s,\nu} + \rho + c\pi_s$$

satisfies the P.E. *with charge* μ. *Moreover, this representation is unique; namely,*

$$\rho = \lim \lambda_n = \lambda_\infty,$$

and $c = \lambda(s)$.

PROOF. The proof is analogous to that of Theorem 3.1, and it is based on the application of Theorem 4.1 to the P.E.

$$\lambda = (\mu + \lambda(s)\nu) + \lambda(K - s \otimes \nu).$$

We record the following corollary giving a criterion for the vanishing of λ_∞:

COROLLARY 4.2. *In addition to the assumptions of Theorem* 4.2 *assume that* λ *is a solution of the* P.E. *satisfying* $|\lambda|(h_\nu) < \infty$. *Then necessarily*

$$\lambda = \mu G_{s,\nu} + \lambda(s)\pi_s \text{ on } \{h_\nu < \infty\}.$$

PROOF. In the proof of Theorem 6.7 of [N] it is noted that

$$\lim_{n\to\infty} \downarrow |\lambda|(K - s \otimes \nu)^n h_\nu = 0$$

implying $\lambda_n \to 0$.

Specializing to the case $\mu = 0$ we get:

COROLLARY 4.3. *Let K be as in Theorem 4.2. Let π be a signed invariant measure for K satisfying $|\pi|(h_v) < \infty$. Then necessarily*

$$\pi = \pi(s)\pi_s.$$

Note that for Harris recurrent Markov chains $h_v \equiv 1$, whence the above requirement $|\lambda|(h_v) < \infty$ means that λ is a finite measure.

ACKNOWLEDGEMENT. It turned out that the original manuscript contained quite many little errors or unclear formulations. I would like to thank the referee for a very careful reading of the manuscript. His comments greatly improved the quality of the paper.

REFERENCES

K. B. Athreya & P. Ney, *A new approach to the limit theory of recurrent Markov chains*, Trans. Amer. Math. Soc. 245 (1978), 493–501.

K. B. Athreya & P. Ney, *A renewal approach to the Perron-Frobenius theory of non-negative kernels on general state spaces*, Math. Z., 179 (1982), 507–529.

C. Constantinescu & A. Cornea, *Potential Theory on Harmonic Spaces*, Springer, Berlin, 1972.

J. G. Kemeny, J. L. Snell & A. W. Knapp, *Denumerable Markov Chains*, Van Nostrand, Princeton, 1966.

J. Neveu, *Potentiel markovien récurrent des chaînes de Harris*, Ann. Inst. Fourier 22 (1972), 85–130.

J. Neveu, *Sur l'irréductibilité des chaînes de Markov*, Ann. Inst. H. Poincaré B, 8 (1972), 249–254.

E. Nummelin, *General Irreducible Markov Chains and Non-negative Operators*, Cambridge Tracts in Math. 83, Cambridge Univ. Press, 1984.

D. Revuz, *Markov Chains*, North-Holland, Amsterdam, 1975.

D. Revuz, *Remarks on the filling scheme for recurrent Markov chains*, Duke Math. J., 45 (1978), 681–689.

H. Rost, *Markoff-Ketten bei sich füllenden Löchern im Zustandraum*, Ann. Inst. Fourier 21 (1971), 253–270.

R. L. Tweedie, *R-theory for Markov chains on a general state space* I, II. Ann. Probab. 2 (1974), I: 840–864, II: 865–872.

D. Vere-Jones, Ergodic properties of non-negative matrices I, II. Pacific J. Math., I: 22, (1968), 361–385, II: 26 (1968), 601–620.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
00100 HELSINKI
FINLAND