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THE EXISTENCE OF NONTRIVIAL SOLUTIONS OF VOLTERRA EQUATIONS

W. MYDLARCZYK

Abstract.

The paper is devoted to the study of the equation

$$u(x) = \int_0^x (x-s)^{\alpha-1} g(u(s))\,ds \quad (x \geqq 0, \alpha > 0).$$

Under weaker assumptions than in G. Gripenberg work [2] some necessary and sufficient conditions for the existence of nontrivial solutions u of this equation are given.

Introduction.

The following equation

$$u(x) = \int_0^x k(x-s) g(u(s))\,ds \quad (x \geqq 0), \tag{1.1}$$

where g is a nondecreasing function such that $g(0) = 0$, can be used for comparison purposes in the study of integral Volterra equations by monotonicity methods (see [2]).

Such equations arise also in some problems of mathematical physics. For example in the theory of nonlinear waves (see [3], [6]) (1.1) is studied with $k(x) = x^{\alpha-1}$ ($\alpha > 0$) and $g(u) = u^{1/\gamma}$ ($\gamma > 1$).

From a physical point of view only nonnegative nontrivial solutions of (1.1) are interesting. But let us note that $u \equiv 0$ *satisfies* (1.1). Under some assumptions (see [4]) the equation (1.1) has a continuous solution u such that $u(0) = 0$ and $u(x) > 0$ for $x > 0$.

The purpose of this paper is to study the existence of nonnegative continuous solutions of the equation

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$$(1.2) \qquad u(x) = \int_0^x (x-s)^{\alpha-1} g(u(s))\, ds \quad (x \geqq 0).$$

The main results of this work are inspired by G. Gripenberg paper [2]. To compare our results with [2] let us write g in the following form $g(u) = uh(u)$. In paper [2] it is assumed that

(1.3) $h: (0, +\infty) \to (0, +\infty)$ is continuous and nonincreasing on $(0, \delta)\,(\delta > 0)$,

(1.4) for each $p > 0$ the function $h^p(u)u$ is nondecreasing on $[0, \delta_p)(\delta_p > 0)$.

Under these assumptions in [2] it is shown that for $\alpha > 0$ the function $u \equiv 0$ is the unique continuous solution of (1.2) if and only if

$$\int_0^\delta [uh^{1/\alpha}(u)]^{-1}\, du = +\infty.$$

Here under weaker assumptions we get similar necessary and sufficient conditions for the existence of nonnegative nontrivial solutions of (1.2). Thus we can study a larger class of the equations (1.2), for example we can consider the case $g(u) = u^{1/\gamma}(\gamma > 1)$. Such a case would not be allowed in [2].

2. Assumptions and auxiliary theorems.

In this section we collect some useful facts concerning the equation (1.1).

We assume

(2.1) $k: \mathrm{R}_+ \to \mathrm{R}_+$ is a locally integrable function $(\mathrm{R}_+ = [0, \infty))$,

(2.2) the integral $K(x) = \int_0^x k(s)\, ds$ is positive for all $x > 0$,

(2.3) $g: \mathrm{R}_+ \to \mathrm{R}_+$ is a nondecreasing function,

(2.4) $g(0) = 0, g(x) > 0 \quad$ for $x > 0$.

Throughout the paper $\delta > 0$ always denotes a constant. We permit it to change its value from paragraph to paragraph.

REMARK 2.1. As it has been shown in [2], from (1.3), (1.4) it follows that g is absolutely continuous on $[0, \delta]$. Hence our assumptions on g are weaker than those in [2].

Let $M > 0$ be an arbitrary number. By (2.4) we can construct a continuous strictly increasing function

$$\Phi\colon [0, M] \to \mathsf{R}_+$$

such that

(2.5) $$\Phi(x) \leqq x/g(x) \quad \text{for } x \in (0, M] \text{ and } \Phi(0) = 0.$$

Now, let $\delta > 0$ be such a number that $K(\delta) < \Phi(M)$.

Denote

$$\phi(x) = \Phi^{-1}K(x) \quad \text{for } x \in [0, \delta].$$

LEMMA 2.1. *For any continuous nonnegative solution u of* (1.1) *we have*

(2.6) $$u(x) \leqq \phi(x), \quad \text{if } x \in [0, \delta_u]$$

PROOF. Taking $v(x) = \max_{s \in [0,x]} u(s)$, by (1.1) we get

(2.7) $$v(x) \leqq K(x)g(v(x)) \quad \text{for } x \in [0, \delta].$$

Since $v(0) = 0$, from the continuity of v and (2.5) it follows (2.6) and the proof is completed.

For any continuous function $w\colon \mathsf{R}_+ \to \mathsf{R}_+$ we define

$$T(w)(x) = \int_0^x k(x - s)g(w(s))\,ds.$$

The operator T is monotonuous and we have

LEMMA 2.2.

$$T(\phi)(x) \leqq \phi(x) \quad \text{for } x \in [0,\delta].$$

The proof of this lemma is very easy.

In much the same way as in [1], [5] we get

REMARK 2.2. The sequence

$$\phi_0 = \phi, \quad \phi_{n+1} = T(\phi_n) \quad \text{for } n \geqq 0$$

is the decreasing sequence convergent on $[0, \delta]$ to a nondecreasing solution $\bar{u}$ of (1.1).

By (2.6) this solution is maximal in the following sense

$$u(x) \leqq \bar{u}(x) \quad \text{for } x \in [0, \delta_u],$$

where u is an arbitrary solution of (1.1).

Integrating by parts in (1.1) we can represent the maximal solution $\bar{u}$ as the Lebesgue-Stieltjes integral in the following way

$$\bar{u}(x) = \int_0^x K(x-s)\,d(g\circ\bar{u})(s) \quad (x\in[0,\delta]). \tag{2.8}$$

Hence we get

THEOREM 2.1. *The maximal solution $\bar{u}$ of* (1.1) *is an absolutely continuous function on* $[0,\delta]$.

By the a priori estimate given in (2.6) the following theorem may be proved (see [1], [4], [5])

THEOREM 2.2. *The equation* (1.1) *has a nontrivial continuous solution if and only if there exists a continous function $F \not\equiv 0$ such that*

$$T(F)(x) \geqq F(x) \quad \text{for } x\in[0,\delta].$$

3. The main results.

In this section we consider the equation (1.2). Our aim is to prove the following theorem:

THEOREM 3.1. *Let $\alpha > 0$ and let* (2.3), (2.4) *be satisfied. Then* (1.2) *has a nontrivial continuous solution if and only if*

$$\int_0^\delta \frac{1}{s}[s/g(s)]^{1/\alpha}\,ds < +\infty \quad (\delta > 0). \tag{3.1}$$

PROOF. The constant $\gamma = \alpha^{-1/\alpha}$ will appear in many places of our proof.

First we prove that (3.1) is a necessary condition for the existence of a nontrivial solution of (1.2).

Assume that the maximal solution $\bar{u}$ of (1.2), which is a nondecreasing function, satisfies $\bar{u}(x) > 0$ for $x\in(0,\delta]$.

Denote

$$L(x) = \frac{1}{\bar{u}(x)}\int_0^x \bar{u}(s)\,ds \quad \text{for } x\in(0,\delta],$$

$$L(0) = \lim_{x\to 0+} L(x) = 0.$$

Representing $\bar{u}$ as in (2.8) and then applying the Jensen inequality we get

$$\bar{u}(x) \leqq \alpha^{-1} g(\bar{u}(x)) \left[\frac{1}{g(\bar{u}(x))} \int_0^x (x-s)^{\alpha+1} \, d(g\bar{u})(s) \right]^{\alpha/(\alpha+1)} \quad (x \in (0, \delta)).$$

Hence

$$[\bar{u}(x)/g(\bar{u}(x))]^{1/\alpha} \leqq \gamma(a+1)L(x) \quad (x \in (0, \delta)). \tag{3.2}$$

Since

$$L'(x) = 1 - \frac{\bar{u}'(x)}{\bar{u}(x)} L(x)$$

for almost every $x \in (0, \delta)$, by (3.2) we get

$$0 \leqq \frac{\bar{u}'(x)}{\bar{u}(x)} \left[\frac{\bar{u}(x)}{g(\bar{u}(x))} \right]^{1/\alpha} \leqq \gamma(a+1)(1 - L'(x)) \tag{3.3}$$

for almost every $x \in (0, \delta)$.

Note that by Theorem 2.1 $L(x)$ is absolutely continuous on every $[\delta_1, \delta]$, $0 < \delta_1 < \delta$. Hence the right side of (3.3) is integrable on $[0, \delta]$ and the substitution $\tau = \bar{u}(x)$ gives (3.1), which completes the proof of this part of the theorem.

Now we are going to prove that if (3.1) holds then equation (1.2) has a nontrivial solution.

Define

$$F_c^{-1}(x) = c \int_0^{x/2} \frac{1}{s} \left[\frac{s}{g(s)} \right]^{1/\alpha} ds \quad (x \in (0, \delta), c > 0).$$

First we establish some useful property of F_c^{-1}.

LEMMA 3.1. *There exists a constant c such that*

$$\int_0^y (F_c^{-1}(y) - F_c^{-1}(\tau))^\alpha \, dg(\tau) \geqq \alpha y \quad (y > 0).$$

PROOF. Note that

$$F_c^{-1}(y) - F_c^{-1}(y/2) \geqq cg(y/2)^{-1/\alpha} \int_{y/4}^{y/2} s^{1/\alpha - 1} \, ds =$$

$$c(2^{-1/\alpha} - 4^{-1/\alpha})\alpha y^{1/\alpha} g(y/2)^{-1/\alpha} \quad (y > 0)$$

and that

$$\int_0^y (F_c^{-1}(y) - F_c^{-1}(\tau))^\alpha \, dg(\tau) \geqq (F_c^{-1}(y) - F_c^{-1}(y/2))^\alpha g(y/2) \quad (y > 0).$$

Therefore it suffices to take c satisfying

$$c(2^{-1/\alpha} - 4^{-1/\alpha})\alpha = \alpha^{1/\alpha} \tag{3.4}$$

to get our assertion and the proof of the lemma is completed.

Now let F be the inverse function to F_c^{-1} with c as in (3.4). Substituting $s = F_c^{-1}(\tau)$ and then integrating by parts we get

$$T(F)(x) = \alpha^{-1} \int_0^{F(x)} (x - F_c^{-1}(\tau))^\alpha \, dg(\tau). \tag{3.5}$$

If we take $y = F(x)$ in Lemma 3.1, then from (3.5) we obtain

$$F(x) \leqq T(F(x)) \quad (x \in [0, \delta)),$$

which in view of Theorem 2.2 completes the proof of the theorem.

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MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW
POLAND