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# RELATIVE COMMUTANT ALGEBRAS OF POWERS' BINARY SHIFTS ON THE HYPERFINITE $II_1$ FACTOR

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**Abstract.**

In this paper, we determine the structure of relative commutant algebras for binary shifts whose signature sequences have finite supports and we see that most binary shifts are classified by relative commutant algebras up to outer conjugacy.

Also, we show that there exist two different signature sequences $a_1$ and $a_2$ such that the Bratteli diagrams of the relative commutant algebras for binary shifts obtained from $a_1$ and $a_2$ respectively are the same.

## 1. Introduction.

V. Jones initiated his famous index theory in [8] and M. Pimsner and S. Popa studied it deeply in [10]. Subsequently A. Ocneanu developed a classification theory for embedding subfactors in the hyperfinite $II_1$ factor $R$ by studying relative commutant algebras. The aim of this paper is to give the structure of the relative commutant algebras associated with a shift considered by Powers. Recall that R. T. Powers [11] studied certain *-endomorphisms on the hyperfinite $II_1$ factor using Jones' index. He defined a shift $\sigma$ on the hyperfinite $II_1$ factor $R$ to be an identity preserving *-endomorphism of $R$ such that $\cap_{k=0}^{\infty} \sigma^k(R) = \mathbf{C}I$ and called the index of $\sigma$ the Jones index $[R:\sigma(R)]$. Several other authors have also studied shifts on the hyperfinite $II_1$ factor $R$ ([2], [4], [5], [12]). A shift $\sigma$ is called a Powers' binary shift if there is a sequence $\{u_i; i = 0,1,2,\ldots\}$ of self-adjoint unitaries which pairwise either commute or anticommute and generate $R$ such that $\sigma(u_i) = u_{i+1}$. G. Price [12] determined a condition that the von Neumann algebra generated by $\{u_i\}$ becomes a factor. Shifts $\alpha$ and $\beta$ are called *conjugate* if there exists an automorphism $\theta$ on $R$ such that $\beta = \theta\alpha\theta^{-1}$. They are *outer conjugate* if there is an automorphism $\theta$ on $R$ and a unitary $u$ in $R$ such that $\beta = \mathrm{Ad}u\theta\alpha\theta^{-1}$. R. T. Powers [11] classified binary shifts completely up to conjugacy. He also considered an outer conjugacy invariant

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$$q(\sigma) = \min\{k \in \mathrm{N};\ \sigma^k(R)' \cap R \neq \mathrm{C}I\}$$

for a binary shift $\sigma$ and he raised the problem whether or not the numbers $q(\sigma)$ are a complete outer conjugacy invariant for binary shifts $\sigma$. In our previous paper [5], we gave a negative answer to the problem by giving examples of suitable relative commutant algebras $C_k(\sigma) = \sigma^k(R)' \cap R$ with generators $\{u_i\}$ for $k = 0, 1, 2, \ldots$.

In this paper we shall determine the structure of the relative commutant algebras $C_k(\sigma)$ for binary shifts whose signature sequences have finite supports. In particular, we find the following interesting results:

(1) The relative commutant algebras have a *finite depth*, i.e. the sequence $\{\dim(\text{the center of } C_k(\sigma))\}_{k \in \mathrm{N}}$ is bounded.

(2) The sequence of inclusion matrices of the Bratteli diagram [1] of relative commutant algebras is periodic.

(3) Each relative commutant algebra is of the form $M_{2^p} \otimes \mathrm{C}^{2^q}$ for some natural numbers $p$ and $q$.

(4) Most binary shifts are classified by relative commutant algebras up to outer conjugacy.

On the other hand D. Bures and H. S. Yin [3] independently succeeded in classifying certain shifts up to outer conjugacy using the notion of derived shifts. In particular, they get the following beautiful result as a corollary:

THEOREM (Bures and Yin). *Let $\alpha$ and $\beta$ be binary shifts on $R$ whose signature sequences have finite support. Then $\alpha$ and $\beta$ are outer conjugate if and only if they are conjugate.*

Combining their results with ours, we get the following peculiar example:

*There exist two binary shifts which are not outer conjugate but their relative commutant algebras are all isomorphic with the same Bratteli diagrams.*

Finally we shall determine a condition for the relative commutant algebras $C_k(\sigma)$ to be trivial for all $k = 0, 1, 2, \ldots$.

## 2. Relative commutant algebras.

Let $R$ be the hyperfinite factor of type $\mathrm{II}_1$. A shift $\sigma$ on $R$ is called a Powers' binary shift if there is a sequence $\{u_n; n = 0, 1, 2, \ldots\}$ of unitaries satisfying the requirements:

(1) $u_n^2 = I$,

(2) $u_n u_m = (-1)^{a(n-m)} u_m u_n$,

(3) $R = \{u_0, u_1, u_2, \ldots\}''$,

(4) $\sigma(u_n) = u_{n+1}$

for $n, m = 0, 1, 2, \ldots$, where $a$ is a function $\mathbf{Z}$ to the set $\{0, 1\}$. Powers' binary shifts are realized as follows [5]. Let $\mathbf{Z}_2 = \{0, 1\}$ be the group of order 2 and $G = \coprod_{i=0}^{\infty} G_i$ be the restricted direct product of $G_i$ where each $G_i$ is isomorphic to $\mathbf{Z}_2$. A function $a: \mathbf{Z} \to \mathbf{Z}_2$ is called a *signature sequence if* $a(0) = 0$ and $a(n) = a(-n)$ for $n \in \mathbf{Z}$. Define a multiplier $m = m_a \in Z^2(G, \mathrm{T})$ by

$$m(x, y) = (-1)^{\Sigma_{i>j} a(i-j) x(i) y(j)},$$

for $x = (x(i)), y = (y(i)) \in G$. We define unitary operators $\lambda_m(x)$ on $l^2(G)$ by

$$(\lambda_m(x)\xi)(y) = m(x, x^{-1} y)\xi(x^{-1} y)$$

for $x, y \in G$ and $\xi \in l^2(G)$. Let $R_m(G)$ be the von Neumann algebra generated by $\{\lambda_m(x); x \in G\}$. A signature sequence $a$ is periodic if there exists a positive integer $k$ such that $a(k + n) = a(n)$ for $n \in \mathbf{Z}$. G. Price ([12], [13]) showed that $R_m(G)$ is a factor if and only if $a$ is not periodic. Define the canonical shift $\sigma$ on the group $G$ as follows:

$$(\sigma(x))(i) = x(i-1) \text{ for } i \geqq 1 \text{ and } (\sigma(x))(0) = 0 \text{ for } x = (x(i)) \in G.$$

Then the shift $\sigma$ on $G$ can be extended to a shift $\sigma$ on $R_m(G)$, which is exactly a Powers' binary shift with the signature sequence $a$. In fact put $e_0 = (1, 0, 0, 0, \ldots) \in G$ and $e_n = \sigma^n(e_0) \in G$. Let $u_n = \lambda_m(e_n)$. Then $R_m(G) = \{u_n; n = 0, 1, 2, \ldots\}''$.

Define the $k$th relative commutant algebra $C_k(\sigma)$ by $\sigma^k(R)' \cap R$. Then we see that (the isomorphism classes of) relative commutant algebras with inclusion matrices are an outer conjugacy invariant, which was independently observed by D. Bures and H. S. Yin [3]. We described the relative commutant algebras $C_k(\sigma)$ in terms of generators $u_0, u_1, u_2, \ldots$ as follows:

THEOREM ([5]). *Let a be a signature whose support* $\{i \in \mathbf{Z}; a(i) \neq 0\}$ *is finite. Put* $d = \max\{i \in \mathbf{N}; a(i) \neq 0\}$. *Let* $\sigma$ *be the Powers' binary shift with a signature sequence a. Then*

$$C_k(\sigma) = \begin{cases} \mathbf{C}I & \text{if } 0 \leqq k \leqq d, \\ \{u_i; 0 \leqq i \leqq k - d - 1\}'' & \text{if } d + 1 \leqq k. \end{cases}$$

In the following we shall determine the structure of the $C^*$-algebras $P_n$ generated by unitaries $\{u_i; 0 \leqq i \leqq n\}$ associated with a fixed signature sequence $a$. The $C^*$-algebra $P_n$ is the universal $C^*$-algebra generated by unitaries $u_0, u_1, u_2, \ldots, u_n$ with the following relations:

(1) $u_i^2 = I$,

(2) $u_i u_j = (-1)^{a(i-j)} u_j u_i$ for $i, j = 0, 1, 2, \ldots, n$.

From now on, we assume that the support of a signature function $a$ is finite and we put the degree $d = \max\{i \in \mathbf{N}; a(i) \neq 0\} < \infty$. We denote the center of $P_n$ by $Q_n$ and define the $(n+1) \times (n+1)$ matrix $A(n)$ by $A(n)_{ij} = a(i-j)$, that is,

$$A(n) = \begin{pmatrix} a(0) & a(1) & a(2) & \dots & a(n) \\ a(1) & a(0) & a(1) & \dots & a(n-1) \\ a(2) & a(1) & a(0) & \dots & a(n-2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a(n) & a(n-1) & a(n-2) & \dots & a(0) \end{pmatrix}.$$

LEMMA 2.1.

$$\dim(Q_n) = 2^{\dim(Ker\ A(n))}.$$

PROOF. Note that $Q_n$ is generated by elements of the form $u_0^{x(0)} u_1^{x(1)} \dots u_n^{x(n)}$ in $Q_n$.
However

$$u_0^{x(0)} u_1^{x(1)} \dots u_n^{x(n)} \in Q_n,$$

$$\text{if and only if } \sum_{k=0}^{n} a(i-k)x(k) = 0 \ \text{ for } 0 \leqq i \leqq n,$$

$$\text{if and only if } A(n)X = 0 \text{ over the finite field } F_2 = \{0, 1\},$$

where $X = {}^t(x(0), x(1), \dots, x(n))$ and $x(i) \in F_2$ for $i = 0, 1, 2, \dots$. Therefore the number of elements $u_0^{x(0)} u_1^{x(1)} \dots u_n^{x(n)}$ in $Q_n$ equals the number of solutions $X$ of the equation $A(n)X = 0$ over $F_2$. Hence $\dim(Q_n)$ is the number of the solutions of the equation $A(n)X = 0$ over $F_2$. Thus $\dim(Q_n) = 2^{\dim(Ker A(n))}$.

PROPOSITION 2.2. *The $C^*$-algebra $P_n$ is isomorphic to $M_{2^k} \otimes \mathbf{C}^{2^l}$ for some $k, l \geqq 0$.*

PROOF. We shall prove this by induction on $n$. We assume that $P_n \cong M_{2^k} \otimes \mathbf{C}^{2^l}$ for some $k, l \geqq 0$. Let $\alpha_{n+1}$ be an automorphism of $P_n$ such that $\alpha_{n+1} = \mathrm{Ad}\, u_{n+1}$. Then $P_{n+1} \cong P_n \rtimes_{\alpha_{n+1}} \mathbf{Z}_2$. Let $E$ be the set of central minimal projections $e$ in $P_n$ such that $\alpha_{n+1}(e) = e$. Put $e_0 = \sum_{e \in E} e$ and $s = {}^{\#}E$, then $0 \leqq s \leqq 2^l$. Using this projection $e_0$, we have $P_n = P_n e_0 \oplus P_n e_0^{\perp}$ and

$$P_n \rtimes_{\alpha_{n+1}} \mathbf{Z}_2 = (P_n e_0 \rtimes_{\alpha_{n+1}} \mathbf{Z}_2) \oplus (P_n e_0^{\perp} \rtimes_{\alpha_{n+1}} \mathbf{Z}_2).$$

Since $\alpha_{n+1}$ is inner on $P_n e_0$, we have

$$P_n e_0 \rtimes_{\alpha_{n+1}} \mathbf{Z}_2 \cong P_n e_0 \otimes \mathbf{C}^2.$$

If a minimal central projection $f$ is not in $E$, then we obtain $\alpha(f) \notin E$ and $f \neq \alpha(f)$.

So we can decompose $P_n e_0^\perp$ as follows:

$$P_n e_0^\perp = \sum_i \oplus (P_n f_i \oplus P_n \alpha_{n+1}(f_i)).$$

Hence

$$P_n e_0^\perp \rtimes_{\alpha_{n+1}} \mathbf{Z}_2 = \sum_i (P_n f_i \oplus P_n \alpha_{n+1}(f_i)) \rtimes_{\alpha_{n+1}} \mathbf{Z}_2.$$

By the assumption, we obtain

$$P_n f_i \cong P_n \alpha_{n+1}(f_i) \cong M_{2^k}$$

and

$$(P_n f_i \oplus P_n \alpha_{n+1}(f_i)) \rtimes_{\alpha_{n+1}} \mathbf{Z}_2 \cong M_{2^{k+1}}.$$

Hence we have

$$(*) \qquad \dim(Q_{n+1}) = 2s + \frac{(2^l - s)}{2}.$$

So

$$2^{l-1} \leqq \dim(Q_{n+1}) \leqq 2^{l+1}.$$

By Lemma 2.1, $\dim(Q_{n+1}) = 2^m$ for some $m \in \mathbf{N}$. This implies

$$\dim(Q_{n+1}) = 2^{l-1}, 2^l \text{ or } 2^{l+1}.$$

If $\dim(Q_{n+1}) = 2^l$ then we have $2^l = 2s + \frac{(2^l - s)}{2}$ by (*). This is a contradiction since $3s = 2^l$. Thus $\dim(Q_{n+1}) = 2^{l \pm 1}$. In the case that $\dim(Q_{n+1}) = 2^{l-1}$, by the above argument, we have

$$P_{n+1} \cong M_{2^{k+1}} \otimes \mathbf{C}^{2^{l-1}}.$$

Furthermore, in the case that $\dim(Q_{n+1}) = 2^{l+1}$, we have

$$P_{n+1} \cong M_{2^k} \otimes \mathbf{C}^{2^{l+1}}.$$

So by induction on $n$, we get the proposition.

LEMMA 2.3. *Let $t$ be an indefinite element satisfying the identity*

$$t^n + x(n-1)t^{n-1} + \ldots + x(1)t + 1 = 0,$$

*where the coefficients $x(1), \ldots, x(n-1)$ are elements of $F_2$. Then there exists a positive integer $m$ such that $t^m + 1 = 0$.*

PROOF. We put

$$p_0(t) = t^n + x(n-1)t^{n-1} + \ldots + x(1)t + 1.$$

If one of the $x(i)$ is 1, then we put $i_1 = \min\{i; x(i) = 1\}$ and

$$p_1(t) = t^{i_1}p_0(t) + p_0(t) = t^{n+i_1} + x_1(n-1)t^{n+i_1-1} + \ldots + x_1(1)t^{i_1+1} + 1.$$

If one of the $x_1(i)$ is 1, then we repeat the same procedure, that is, we put $i_2 = \min\{i; x_1(i) = 1\}$ and

$$p_2(t) = t^{i_1+i_2}p_0(t) + p_1(t) = t^{n+i_1+i_2} + x_2(n-1)t^{n+i_1+i_2-1} + \ldots + x_2(1)t^{i_1+i_2+1} + 1.$$

If we can repeat this procedure infinitely, then we can find two positive integers $k$ and $l(k < l)$ such that

$$x_k(i) = x_l(i) \quad (i = 1, \ldots, n-1).$$

Then we get $p_l(t) + p_k(t)t^{l-k} = t^{l-k} + 1$. Therefore $t^{l-k} + 1 = 0$. If $x_m(i) = 0$ for all $i$, after $m$ times repetitions, then

$$p_m(t) = (t^{i_1+i_2+\ldots+i_m} + \ldots + t^{i_1+i_2} + t^{i_1} + 1)p_0(t) = t^{n+i_1+\ldots+i_m} + 1 = 0.$$

By the method of the above proof, we can find a positive integer $m$ such that $t^m + 1 = 0$, and we set $D = \min\{m; t^m + 1 = 0\}$. We call the number $D$ the *order* of the sequence $\{1, x(1), \ldots, x(n-1), 1\}$. Then we can get the following lemma by the same method as in the proof of Lemma 2.3.

LEMMA 2.4. *Let $\{y(i); i = 0, \ldots, D\}$ be a sequence in $F_2$ which satisfies the following equations,*

$$y(j) + x(1)y(j+1) + \ldots + x(n-1)y(j+n-1) + y(j+n) = 0 \quad (j = 0, \ldots, D-n),$$

*where $D$ is the order of $\{1, \alpha(1), \ldots, x(n-1), 1\}$. Then $y(0) = y(D)$.*

For the determination of the structure of $Q_n$, we introduce some notation. For any vector $X = {}^t(x(0), \ldots, x(n))$ in $(F_2)^{n+1}$ and any positive integer $k$,

$$u(X) = u(x(0), \ldots, x(n)) = u_0^{x(0)} \ldots u_n^{x(n)},$$

$$v(X) = v(x(0), \ldots, x(n)) = \begin{cases} u(X) & \text{if } u(X)^2 = 1, \\ \sqrt{-1}\,u(X) & \text{otherwise,} \end{cases}$$

$$v_k(X) = v(0, \ldots, 0, x(0), \ldots, x(n)) \text{ in } P_{n+k},$$

and

$$v_{-k}(X) = \begin{cases} v(x(k), x(k+1), \ldots, x(n)) \text{ in } P_{n-k} & \text{if } n \geqq k, \\ 1 & \text{if } n < k. \end{cases}$$

Then $v(X)$ is a self-adjoint unitary operator. We have seen in the above that the algebra $Q_n$ is the linear span of $\{v(X); A(n)X = 0\}$.

PROPOSITION 2.5. *Let $D$ be the order of the sequence*

$$\{a(d), a(d-1), \ldots, a(1), a(0), a(1), \ldots, a(d-1), a(d)\}$$

*and $k$ $(0 \leqq k < D)$ and $l$ be non-negative integers. Then $Q_{k+lD}$ is isomorphic to $Q_k$. Moreover the isomorphism $\beta_{k+lD,k}$ of $Q_{k+lD}$ to $Q_k$ is given by the following relation,*

$$\beta_{k+lD,k}(v(X)) = v_{-lD}(X),$$

*where $v(X)$ belongs to $Q_{k+lD}$.*

PROOF. At first, we consider the case $d \leqq k \leqq D$. Let $X = {}^t(x(0), x(1), \ldots, x(k))$ and $Y = {}^t(y(0), y(1), \ldots, y(k + lD))$ be vectors in $(F_2)^{k+1}$ and $(F_2)^{k+lD+1}$ respectively. Then $v(X)$ belongs to $Q_k$ (i.e. $A(k)X = 0$) if and only if $x(0), x(1), \ldots, x(k)$ satisfy the following relation $[R;j]$, for any $j(-d \leqq j \leqq k - d)$,

$$[R;j] \qquad \sum_{s=-d}^{d} a(|s|)x(j + d + s) = 0,$$

where we put

$$x(-d) = x(-d+1) = \ldots = x(-1) = 0,$$

$$x(k+1) = x(k+2) = \ldots = x(k+d) = 0.$$

Relations $[R; -d], \ldots, [R; k - 2d]$ mean that variables $x(d), x(d+1), \ldots, x(k)$ can be represent as linear combinations of variables $x(0), x(1), \ldots, x(d-1)$, that is, there exist linear functions $\{f_j; d \leqq j \leqq k\}$ such that

$$x(j) = f_j(x(0), x(1), \ldots, x(d-1)) \quad \text{for any } j \ (d \leqq j \leqq k).$$

Therefore we can regard the relations $[R; k - 2d + 1], \ldots, [R; k - d]$ as binding conditioning on the variables $x(0), x(1), \ldots, x(d-1)$.

In a similar way, $v(Y)$ belongs to $Q_{k+lD}$ (i.e. $A(k + lD)Y = 0$) if and only if $y(0), y(1), \ldots, y(k + lD)$ satisfy the following same relation $[R;j]$, for any $j(-d \leqq j \leqq k + lD - d)$,

$$[R;j] \qquad \sum_{s=-d}^{d} a(|s|)y(j + d + s) = 0,$$

where we put

$$y(-d) = y(-d+1) = \ldots = y(-1) = 0,$$

$$y(k + lD + 1) = y(k + lD + 2) = \ldots = y(k + lD + d) = 0.$$

Relations $[R; -d], \ldots, [R; k-2d+lD]$ mean that variables $y(d), y(d+1), \ldots, y(k+lD)$ can be represented as linear combinations of variables $y(0), y(1), \ldots, y(d-1)$, that is, there exist linear functions $\{g_j; d \leqq j \leqq k+lD\}$ such that

$$y(j) = g_j(y(0), y(1), \ldots, y(d-1)) \text{ for any } j \ (d \leqq j \leqq k+lD),$$

where the functions $g_j$ and $f_j$ are the same for any $j$ ($d \leqq j \leqq k$). Therefore we can regard the relations $[R; k+lD-2d+1], \ldots, [R; k+lD-d]$ as binding conditions on the variables $y(0), y(1), \ldots, y(d-1)$. By Lemma 2.4, we have

$$y(j) = y(j+D) \quad \text{for any } -d \leqq j \leqq k+(l-1)D+d.$$

Then the conditions $[R; k+lD-2d+1], \ldots, [R; k+lD-d]$ for variables $y(0), y(1), \ldots, y(d-1)$ are identical to the conditions $[R; k-2d+1]$ for variables $x(0), x(1), \ldots, x(d-1)$.

For a vector $X = {}^t(x(0), x(1), \ldots, x(k))$, we define vectors

$$\tilde{X} = {}^t(x(0), x(1), \ldots, x(D-1)),$$
$$\hat{X} = {}^t(x(0), x(1), \ldots, x(k+lD)),$$

by the following relations,

$$\begin{cases} x(j) = g_j(x(0), \ldots, x(d-1)) & \text{for any } k < j < D, \\ x(j) = x(j+D) & \text{for any } 0 \leqq j \leqq k+(l-1)D. \end{cases}$$

By the above observation, the correspondence of $X$ and $\hat{X}$ induces a bijection from the set $\{X; A(k)X = 0\}$ to the set $\{Y; A(k+lD)Y = 0\}$. So we can construct isomorphisms $\alpha_{k,k+lD}$ from $Q_k$ to $Q_{k+lD}$ and $\beta_{k+lD,k}$ from $Q_{k+lD}$ to $Q_k$ by the following relations,

$$\alpha_{k,k+lD}(v(X)) = v(\tilde{X})\, v_D(\tilde{X}) \ldots v_{(l-1)D}(\tilde{X})\, v_{lD}(X) = v(\hat{X}),$$
$$\beta_{k+lD,k}(v(Y)) = v_{-lD}(Y),$$

where $v(X), v(Y)$ belong to $Q_k$, $Q_{k+lD}$ respectively.

In the case $0 \leqq k < d$, we treat relations $[R; -d], \ldots, [R; k-d]$ as binding conditions on the variables $x(0), \ldots, x(d-1)$. By the same argument as above, we can construct the isomorphism $\beta_{k+lD}$ to $Q_k$.

COROLLARY 2.6. *We use the same notation as in Proposition 2.5. Then,*

(1) *the sequence* $\{\dim Q_n; n = 0, 1, \ldots\}$ *has a period* $D$,

(2) $\dim Q_n = 2^d$ *if* $n = lD - d - 1$,

(3) $\max\{\dim Q_n; n \in \mathrm{N}\} = 2^d$.

PROOF. (1) This follows immediately from Proposition 2.5.

(2) We set $x(-d) = x(-d+1) = \ldots = x(-1) = 0$. For any elements $x(0)$, $x(1), \ldots, x(d-1)$ in $F_2$, we inductively define $x(d), x(d+1), \ldots, x(lD-1)$ by the

following relation,

$$[R;j] \qquad \sum_{s=-d}^{d} a(|s|)x(j+d+s) = 0,$$

for any $j(-d \leqq j \leqq lD - 2d - 1)$. Then we have

$$x(-d) = x(lD-d) = 0, \ldots, x(-1) = x(lD-1) = 0.$$

The vector $X = {}^t(x(0), x(1), \ldots, x(lD-d-1))$ becomes a solution of the equation $A(lD-d-1)X = 0$. Therefore $\dim Q_{lD-d-1} = 2^d$.

(3) By the proof of Proposition 2.5, it follows that $\dim Q_n \leqq 2^d$.

By this corollary, the sequence $\{\dim Q_n\}_n$ is bounded so we can conclude that the relative commutant algebras have a finite depth.

For an increasing sequence $\{i_1, \ldots, i_k\}$ of non-negative integers, we define a vector $X = {}^t(x(0), \ldots, x(i_k))$ in $(F_2)^{i_k+1}$ by

$$\begin{cases} x(j) = 1 & \text{if } j = i_l \text{ for some } 1 \leqq l \leqq k, \\ x(j) = 0 & \text{otherwise,} \end{cases}$$

and we denote $v(X)$ by $(i_1, \ldots, i_k)$ or $(X)$. We decompose $(X)$ into the difference of two projections, and we write,

$$(X) = (X)^+ - (X)^- \ (\text{or } (i_1, \ldots, i_k) = (i_1, \ldots, i_k)^+ - (i_1, \ldots, i_k)^-),$$

that is,

$$(X)^+ = \frac{1+(X)}{2}, \ (X)^- = \frac{1-(X)}{2}.$$

We remark that the center $Q_n$ of $P_n$ is the linear span of $\{v(X);\ A(n)X = 0\}$. If $\{v(X_1), \ldots, v(X_K)\}$ $(K = \dim(\operatorname{Ker} A(n)))$ is a generator of $\{v(X);\ A(n)X = 0\}$, then $Q_n$ is the linear span of

$$\{(X_1)^{\delta(1)}(X_2)^{\delta(2)} \ldots (X_K)^{\delta(K)};\ \delta(1), \delta(2), \ldots, \delta(K) = + \text{ or } -\},$$

and $(X_1)^{\delta(1)}(X_2) \ldots (X_K)^{\delta(K)}$ is a minimal projection of $Q_n$.

THEOREM 2.7. *Let $D$ be the order of the sequence*

$$\{a(d), a(d-1), \ldots, a(1), a(0), a(1), \ldots, a(d-1), a(d)\}.$$

*Then the inclusion matrix from $P_n$ to $P_{n+1}$ is equal to the inclusion matrix from $P_{n+lD}$ to $P_{n+1+lD}$ for any non-negative integers $l$ and $n$.*

PROOF. By Proposition 2.5, there exists an isomorphism $\beta_{n+lD,n}$ from $Q_{n+lD}$ to $Q_n$. Then a minimal projection $(X_1)^{\delta(1)} \ldots (X_K)^{\delta(K)}$ in $Q_{n+lD}$ is mapped to a minimal projection $(v_{-lD}(X_1))^{\delta(1)} \ldots (v_{-lD}(X_K))^{\delta(K)}$ in $Q_n$ by this isomorphism $\beta_{n+lD,n}$, where $K$ is the corank of $A(n)$.

The inclusion matrix from $P_n$ to $P_{n+1}$ is determined by the orthogonality of a minimal projection in $Q_n$ and a minimal projection in $Q_{n+1}$. By Proposition 2.2, the algebra $Q_n$ contains the algebra $Q_{n+1}$ or is contained in the algebra $Q_{n+1}$. If $Q_n \supset Q_{n+1}$ (resp. $Q_n \subset Q_{n+1}$), then

$$\beta_{n+lD,n|Q_{n+1+lD}} = \beta_{n+1+lD,n+1} \text{ (resp. } \beta_{n+1+lD,n+1|Q_{n+lD}} = \beta_{n+1D,n}).$$

Therefore the inclusion matrix from $Q_n$ to $Q_{n+1}$ is equal to the inclusion matrix from $Q_{n+lD}$ to $Q_{n+1+lD}$ with respect to the correspondence of a minimal projection in $Q_n$ (resp. $Q_{n+1}$) and $Q_{n+lD}$ (resp. $Q_{n+1+lD}$) by the isomorphism $\beta_{n+lD,n}$ (resp. $\beta_{n+1+lD,n+1}$).

By this theorem, we can see that the sequence of inclusion matrices of the Bratteli diagram of relative commutant algebras is periodic.

As we have seen previously, the dimension of the center $Q_n$ of relative commutant algebras equals to $2^{\dim(\ker A(n))}$ (see Lemma 2.1) and the sequence $\{\dim Q_n\}$ is periodic (see Corollary 2.6). So we can determine the sequence $\{\dim Q_n\}$ by calculating finitely many $\dim Q_n$'s.

In the case $d \leqq 6$, using a computer, we can find that all the sequences $\{\dim Q_n\}$ except six sequences in the following examples are different.

EXAMPLE 1. ($d = 4$)

$$\begin{cases} a_1 = (0,0,1,0,1,0,\ldots), \\ a_2 = (0,0,1,1,1,0,\ldots). \end{cases}$$

The period is 12. The dimension of $Q_n$ is as follows,

$$\begin{array}{llll} \dim(Q_0) = 2, & \dim(Q_1) = 4, & \dim(Q_2) = 2, & \dim(Q_3) = 1, \\ \dim(Q_4) = 2, & \dim(Q_5) = 4, & \dim(Q_6) = 8, & \dim(Q_7) = 16, \\ \dim(Q_8) = 8, & \dim(Q_9) = 4, & \dim(Q_{10}) = 2, & \dim(Q_{11}) = 1. \end{array}$$

EXAMPLE 3. ($d = 6$)

$$\begin{cases} a_1 = (0,1,0,1,0,1,1,0,\ldots), \\ a_2 = (0,1,1,0,1,1,1,0,\ldots). \end{cases}$$

The period is 24. The dimension of $Q_n$ is as follows,

$$\begin{array}{llll} \dim(Q_0) = 2, & \dim(Q_1) = 1, & \dim(Q_2) = 2, & \dim(Q_3) = 4, \\ \dim(Q_4) = 8, & \dim(Q_5) = 16, & \dim(Q_6) = 8, & \dim(Q_7) = 4, \\ \dim(Q_8) = 2, & \dim(Q_9) = 1, & \dim(Q_{10}) = 2, & \dim(Q_{11}) = 1, \\ \dim(Q_{12}) = 2, & \dim(Q_{13}) = 4, & \dim(Q_{14}) = 8, & \dim(Q_{15}) = 16, \end{array}$$

$$\dim(Q_{16}) = 32, \quad \dim(Q_{17}) = 64, \quad \dim(Q_{18}) = 32, \quad \dim(Q_{19}) = 16,$$
$$\dim(Q_{20}) = 8, \quad \dim(Q_{21}) = 4, \quad \dim(Q_{22}) = 2, \quad \dim(Q_{23}) = 1.$$

By the results of D. Bures and H. S. Yin [3], the two binary shifts in each example above are not outer conjugate.

In the rest of this section, we determine the inclusion matrices of the sequences of algebras for $a_1$ and $a_2$ in Example 1. As the result of this argument, we get the following theorem.

THEOREM 2.8. *There exist two different sequences* $\{a_1(n)\}$ *and* $\{a_2(n)\}$ *such that the Bratteli diagram of* $\overline{\cup_n P_n}$ *for* $a_1$ *coincides the Bratteli diagram of* $\overline{\cup_n P_n}$ *for* $a_2$.

PROOF. At first, we shall determine the Bratteli diagram for $\{0, 0, 1, 0, 1, 0, \ldots\}$. Each of algebras $Q_0, Q_1, \ldots, Q_{11}$ has minimal projections, which are expressed in the following form.

| | | | |
|---|---|---|---|
| $Q_0$ | $(0)^{\pm}$, | $Q_1$ | $(0)^{\pm}(1)^{\pm}$, |
| $Q_2$ | $(1)^{\pm}$, | $Q_3$ | 1, |
| $Q_4$ | $(024)^{\pm}$, | $Q_5$ | $(024)^{\pm}(135)^{\pm}$, |
| $Q_6$ | $(06)^{\pm}(135)^{\pm}(246)^{\pm}$, | $Q_7$ | $(06)^{\pm}(17)^{\pm}(246)^{\pm}(357)^{\pm}$, |
| $Q_8$ | $(17)^{\pm}(246)^{\pm}(357)^{\pm}$, | $Q_9$ | $(246)^{\pm}(357)^{\pm}$, |
| $Q_{10}$ | $(357)^{\pm}$, | $Q_{11}$ | 1. |

We calculate the orthogonality of the above minimal projections in the following way,

$$\begin{aligned}
(024)^{+} &= \frac{1}{2}\{1 + (024)\} \\
&= \frac{1}{2}\{1 - (06) + (06) + (06)(246)\} \\
&= (06)^{-} + (06)(246)^{+} \\
&= (06)^{+}(246)^{+} + (06)^{-}(246)^{-}.
\end{aligned}$$

By these calculations, we get the following formulae

$$\begin{cases}
(014)^{+} = (06)^{+}(246)^{+} + (06)^{-}(246)^{-}, \\
(024)^{-} = (06)^{+}(246)^{-} + (06)^{-}(246)^{+}, \\
(135)^{+} = (17)^{+}(357)^{+} + (17)^{-}(357)^{-}, \\
(135)^{-} = (17)^{+}(357)^{-} + (17)^{-}(357)^{+},
\end{cases}$$

Therefore we get the relation of the orthogonality of minimal projections as Figure 1 and Figure 2, and the Bratteli diagram as Figure 4.

Next, we determine the Bratteli diagram for $\{0, 0, 1, 1, 1, 0, \ldots\}$. Each of algebras $Q_0, Q_1, \ldots, Q_{11}$ has minimal projections, which are expressed in the following form.

| | | | |
|---|---|---|---|
| $Q_0$ | $(0)^{\pm}$, | $Q_1$ | $(0)^{\pm}(1)^{\pm}$, |
| $Q_2$ | $(1)^{\pm}$, | $Q_3$ | 1, |
| $Q_4$ | $(0134)^{\pm}$, | $Q_5$ | $(0134)^{\pm}(1245)^{\pm}$, |
| $Q_6$ | $(06)^{\pm}(1245)^{\pm}(2356)^{\pm}$, | $Q_7$ | $(06)^{\pm}(17)^{\pm}(2457)^{\pm}(3467)^{\pm}$, |
| $Q_8$ | $(17)^{\pm}(2457)^{\pm}(3467)^{\pm}$, | $Q_9$ | $(2457)^{\pm}(3467)^{\pm}$, |
| $Q_{10}$ | $(3467)^{\pm}$, | $Q_{11}$ | 1. |

We calculate the orthogonality of the above minimal projections by the following way,

$$
\begin{aligned}
(0134)^{+} &= \frac{1}{2}\{1 + (0134)\} \\
&= \frac{1}{2}\{1 - (06) + (06) - (06)(1245) + (06)(1245) - (06)(1245)(2356)\} \\
&= (06)^{-} + (06)(1245)^{-} + (06)(1245)(2356)^{-} \\
&= (06)^{+}(1245)^{+}(2356)^{-} + (06)^{+}(1245)^{-}(2356)^{+} \\
&\quad + (06)^{-}(1245)^{+}(2356)^{+} + (06)^{-}(1245)^{-}(2356)^{-}
\end{aligned}
$$

By these calculations, we get the following formulae

$$
\begin{cases}
(0134)^{+} = (06)^{+}(1245)^{+}(2356)^{-} + (06)^{+}(1245)^{-}(2356)^{+} \\
\qquad\qquad + (06)^{-}(1245)^{+}(2356)^{+} + (06)^{-}(1245)^{-}(2356)^{-} \\
(0134)^{+} = (06)^{+}(1245)^{+}(2356)^{+} + (06)^{+}(1245)^{-}(2356)^{-} \\
\qquad\qquad + (06)^{-}(1245)^{+}(2356)^{-} + (06)^{-}(1245)^{-}(2356)^{+} \\
(1245)^{+} = (17)^{+}(2457)^{+} + (17)^{-}(2457)^{-} \\
(1245)^{-} = (17)^{+}(2457)^{-} + (17)^{-}(2457)^{+} \\
(2356)^{+} = (2457)^{+}(3467)^{-} + (2457)^{-}(3467)^{+} \\
(2356)^{-} = (2457)^{+}(3467)^{+} + (2457)^{-}(3467)^{-}.
\end{cases}
$$

Therefore we get the relation of the orthogonality of minimal projections as in Figure 1 and Figure 3, and the Bratteli diagram as in Figure 4. So, we get the desired conclusion.

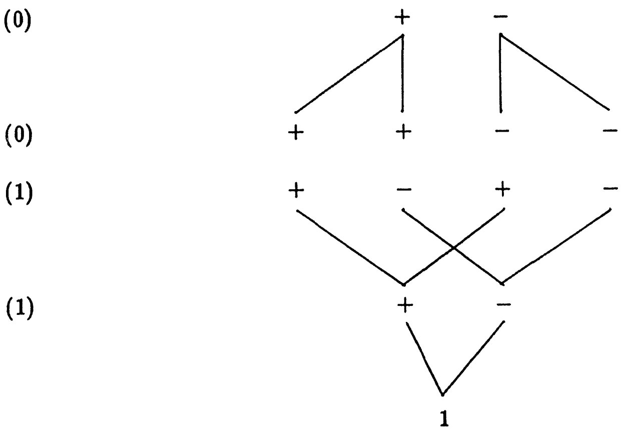


Figure 1. The orthogonality of minimal projections in $Q_0$, $Q_1$, $Q_2$, $Q_3$ for $\{0,0,1,0,1,0,\ldots\}$ and $\{0,0,1,1,1,0,\ldots\}$.

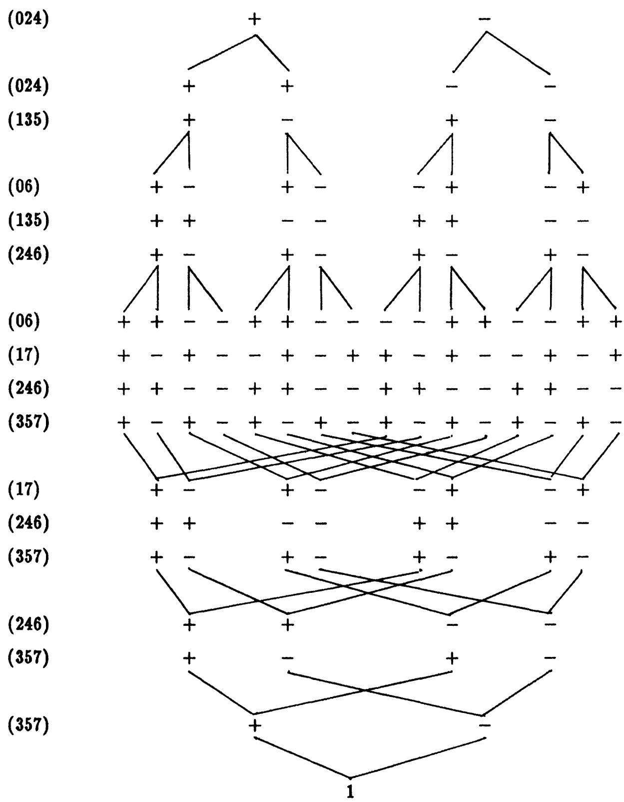


Figure 2. The orthogonality of minimal projections in $Q_4, Q_5, \ldots, Q_{11}$ for $\{0,0,1,0,1,0,\ldots\}$.

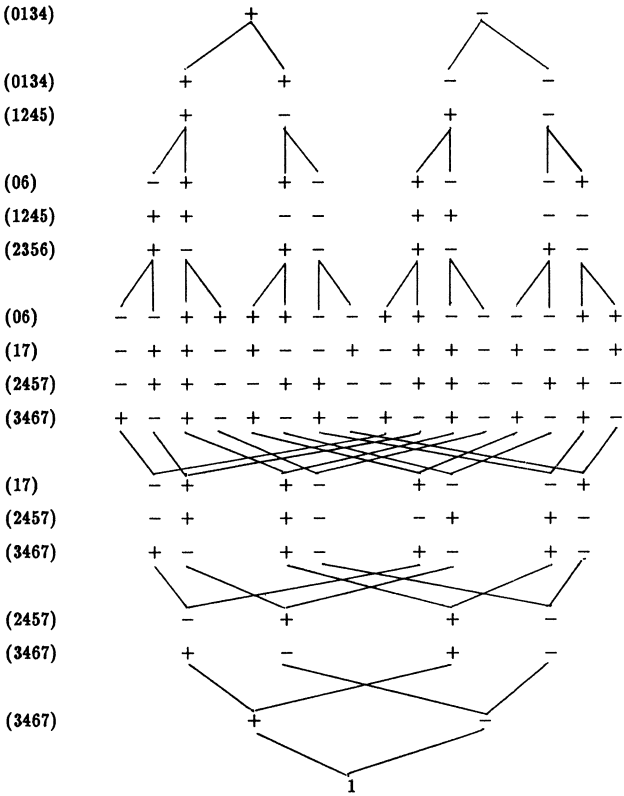


Figure 3. The orthogonality of minimal projections in $Q_4, Q_5, \ldots, Q_{11}$ for $\{0, 0, 1, 1, 1, 0, \ldots\}$.

## 3. Essentially periodic signature sequences.

In the following we shall treat the case of binary shifts $\sigma$ such that $C_k(\sigma)$ is trivial for all $k \geqq 0$.

DEFINITION 3.1. The sequence $a$ is called *essentially periodic* (or *ultimately periodic* if there exist integers $k(\geqq 0)$ and $p(> 0)$ such that, for any $n \geqq k$, $a(n + p) = a(n)$.

THEOREM 3.2. *Let $a$ be a non-periodic signature sequence and $\sigma_a$ be the associated shift of the hyperfinite* $\mathrm{II}_1$ *factor $R$. The sequence $a$ is essentially periodic if and only if there exists a positive integer $r$ such that $\sigma_a^r(R)' \cap R \neq \mathbf{C}I$.*

PROOF. Ar first we shall assume that there exists a positive integer $r$ such that $C_r(\sigma_a) \neq \mathbf{C}I$. Then there exists an element $x(\neq 1) \in G = \coprod_{i=0}^{\infty} G_i$, $G_i \cong Z_2$ such

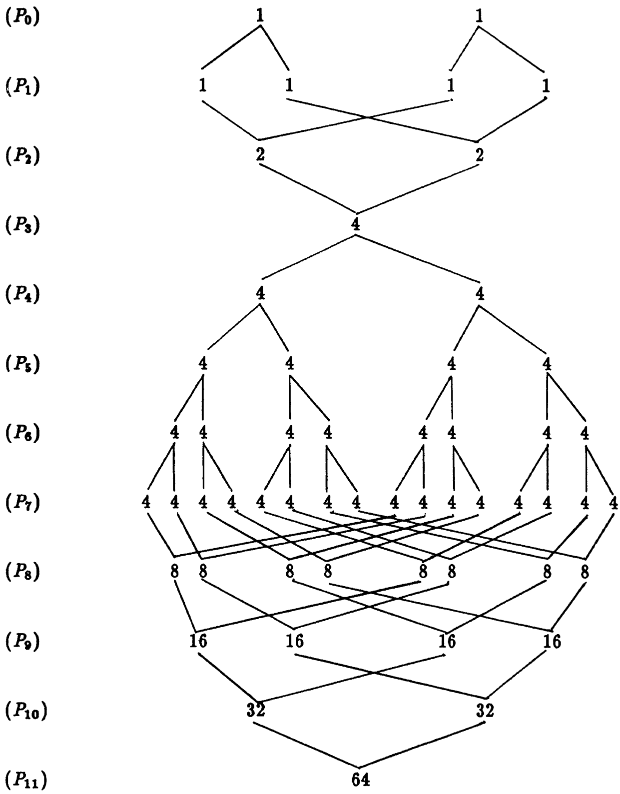


Figure 4. The Bratteli diagram of $P_0, P_1, \ldots, P_{11}$ for $\{0,0,1,0,1,0,\ldots\}$ and $\{0,0,1,1,1,0,\ldots\}$.

that

$$(*) \qquad m_a(x, e_n) = m_a(e_n, x) \quad \text{for } n \geqq r, \text{ where } e_n \in G,\ e_n(i) = 0\ (i \neq n).$$

For this $x$, $x$ is expressed as $x = \sum_{i=0}^{d} x(i)\, e_i$, where $d = \max\{i; x(i) \neq 0\}$. Then we have $\sum_{i=0}^{d} a(n-i)x(i) = 0$ for $n \geqq r$ by $(*)$. Put $l = \min\{i; x(i) \neq 0\}$. Let $D$ be the order of the sequence $\{x(l), x(l+1), \ldots, x(d)\}$. By Lemma 2.4, we have $a(m + D) = a(m)$ for $m \geqq r - d$. Thus $a$ is essentially periodic. Conversely, assume that $a$ is essentially periodic. So there exist integers $k\,(>0)$ and $p\,(>0)$ such that, for any $n \geqq k$, $a(n+p) = a(n)$. Then we have the following two cases.

(Case I). $\sum_{i=k}^{k+p-1} a(i) = 0$. Put $v = u_0 u_1 \ldots u_{p-1}$ ($\notin CI$). Then $v \in \sigma^{k+p-1}(R)' \cap R$.

(Case II). $\sum_{i=k}^{k+p-1} a(i) = 1$. Put $v = u_0 u_1 \ldots u_{2p-1}$ ($\notin CI$). Then $v \in \sigma^{k+2p-1}(R)' \cap R$, which it follows from the fact that $a(k+i) = a(k+p+i)$ for $0 \leqq i \leqq p-1$. So $\sum_{i=0}^{2p-1} a(k+i) = 0$. Thus we get the necessity of this theorem.

This theorem is also obtained by D. Bures and H. S. Yin [3] independently.

REMARK 3.3. There exist two signature sequences $a$ and $b$ such that $a$ has a finite support, $b$ does not have a finite support and $\sigma_a$ and $\sigma_b$ are outer conjugate.

For example, if we put $a = (0, 1, 0, 0, \ldots)$, $b = (0, 1, 1, 1, \ldots)$ and $w = \dfrac{1 - \sqrt{-1}u_0}{\sqrt{2}}$ where $u_0$ is a $\sigma_a$-generator then we have $\sigma_b = \operatorname{Ad} w \, \sigma_a$.

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