MATH. SCAND. 68 (1991), 131–147

MODELS OF REPRESENTATIONS OF SOME CLASSICAL SUPERGROUPS

D. LEITES and V. SERGANOVA

Abstract.

A *model of representations of a compact group* is a representation of this group which is the direct sum of all its irreducible representations, each of multiplicity one. For Lie supergroups it is unclear what is the good definitoon of a model: irreducible representations have parameters which run over a singular supervariety; in addition to that the representations, even finite-dimensional ones, are not, usually, completely reducible. The models we give (for the supergroups of series $\mathcal{GL}$ and $\mathcal{O}\mathfrak{sp}$) are straightforward generalizations of Gelfand-Zelevinsky's models for classical groups; they indicate which representations are in a sence more natural among other irreducible finite-dimensional representations. Generalization of this construction to the other series of classical Lie supergroups will be given elsewhere. Our results can be considered transversal to the description of irreducible representations of Lie superalgebras à la Borel-Weil-Bott – ... due, mainly, to I. Penkov – a realization of the representations in cohomology of invertible sheaves on flag supervarieties [P].

Since the concepts of supermanifold and Lie supergroup are so new in Mathematics we append the main text with basics so far poorly, especially as far as representations of Lie supergroups are concerned, presented in the litterature.

1. Let W be the superspace of the standard (identity) representation of the Lie supergroup $\mathcal{G} = \mathcal{O}\mathfrak{sp}(2m + 1|2n)$ with the parity function on W fixed so that $\mathcal{O}\mathfrak{sp}$ preserves an even nondegenerate *symmetric* bilinear form $\langle .,. \rangle$.

Fix maximal isotropic subspaces V, $V' \subset W$ and a vector $z_0 \in W_0$ such that $V + V' = W$ and $\langle z_0, z_0 \rangle = 1$. Denote by $\mathfrak{g}_0 = \mathfrak{gl}(m|n)$.

Fix a maximal toral subalgebra $\mathfrak{h} \subset \mathfrak{g}_0 \subset \mathrm{Lie}(\mathcal{G}) = \mathfrak{g}$ and a standard basis $\langle \varepsilon_1, \ldots, \varepsilon_m, \delta_1, \ldots, \delta_n \rangle$ of $\mathfrak{h}$ as in [OV], fix a Borel subalgebra $\mathfrak{b}^+$ and let $\mathfrak{b}^-$ be its opposite.

Lemma ([K], [L1]). a) *Any finite-dimensional irreducible representation of* $\mathcal{G}$ *has a unique* $\mathfrak{b}^+$*-highest vector of weight* $\lambda = \Sigma a_i \varepsilon_i + \Sigma b_j \delta_j$, *where* $a_1 \geqq \ldots \geqq a_m \geqq l$ *with* $l \in \{, \ldots, n\}$, $b_1 \geqq \ldots \geqq b_n \geqq 0$.

There are two irreducible representations with $\mathfrak{b}^+$*-highest vector* v *of weight* λ: *with an even* v *and with an odd* v.

b) ([S]) *The irreducible finite-dimensional representation of* $\mathfrak{gl}(m|n)$ *with weight*

Received March 28, 1990; in revised form April 10, 1990.

λ is a direct summand in the tensor algebra $T(V)$; i.e. *the representation of* $\mathfrak{gl}(m|n)$ *corresponding to λ is a polynomial one.*

2. Let C be a supercommutative superalgebra, $\mathcal{M} = \mathcal{G}/\mathcal{G}_0$. Then the set of C-points (for definition see [L2, #30, 31]) of $\mathcal{M}$ is $\mathcal{M}(C) = \{(V, V'', z)$: V, V'' are maximal isotropic direct modules of the C-module W, $z \in W_0$ and $\langle z, z\rangle = 1\}$.

There is a natural $\mathcal{G}$-module structure on a sheaf $O_{\mathcal{M}}$ of regular algebraic functions on $\mathcal{M}$.

THEOREM. *$O_{\mathcal{M}}$ is a model of representations of $\mathcal{G}$.*

REMARK. Both $\mathcal{G}$ and $\mathcal{M}$ have two connected components. Denote by $\mathcal{G}^0 = \mathcal{S}\mathcal{O}\mathit{sp}(2m + 1|2n)$ the connected component of the unit and by $\mathcal{M}^0$ either of the connected components of $\mathcal{M}$. Then, clearly, $\mathcal{M}^0 = \mathcal{G}^0/\mathcal{G}_0$ and a statement equivalent to that of Theorem is

THEOREM^0. *$O_{\mathcal{M}^0}$ is a model of representations of $\mathcal{G}_0$.*

3. EXAMPLE. $\mathcal{G} = \mathcal{O}\mathit{sp}(1|2)$, $\dim W = (1, 2)$. Fix a basis $\langle f, e_1, e_2\rangle = W$ such that $\langle f, f\rangle = 1$, $\langle e_1, e_2\rangle = 1$. Then.

$\mathcal{M}^0(C) = \{(X, Y, z)$: X, Y are direct submodules of $W \otimes C$ and $z \in W(C) = (W \otimes C)_0$ such that $\langle z, z\rangle = 1\}$.

In the above basis of W, on an open subsuperdomain $\mathcal{U}$ of $\mathcal{M}^0$ containing a triple (X, Y, z) such that X is transversal to, say, e_1 and Y is transversal to e_2, we can take for local coordinates the patrameters x, y, ξ, η in the following expressions:

$$X = C(e_1 + e_2),\ Y = C(e_2 + ye_1),\ z = (1 - \zeta\eta)f + \zeta e_1 + \eta e_2.$$

Let us express in these coordinates vector fields on $\mathcal{U}$ that correspond to the $\mathcal{G}$-action in $O_{\mathcal{M}}$. Let us express a basis of $\mathfrak{osp}(1|2)$ in the nonstandard format:

$$X^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \nabla^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\nabla^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, X^+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the corresponding vector fields are:

$$\begin{aligned} X^- &= \xi\partial/\partial\eta + \partial/\partial x - y^2\partial/\partial y \\ \nabla^- &= \partial/\partial\eta - \xi\eta\partial/\partial\eta + (\eta - \xi x)\partial/\partial x + (\eta y^2 - \xi y)\partial/\partial y \\ H &= \eta\partial/\partial\eta - \xi\partial/\partial\xi + 2x\partial/\partial x - 2y\partial/\partial y \end{aligned}$$

$$\nabla^+ = -\partial/\partial\xi + \xi\eta\partial/\partial\xi + (\eta x - \xi x^2)\partial/\partial x + (\xi - \eta y)\partial/dy$$

$$X^+ = \eta\partial/\partial\xi + \partial/\partial y - x^2\partial/\partial x$$

The $\mathfrak{b}^+$-highest vectors are:

$$\eta(y + x^{-1})^{-m} \text{ and } (1 - m\xi\eta)(y + x^{-1})^{-m} \text{ for any } m \in \mathbf{Z}_+.$$

4. Proof of Theorem. Thanks to Remark 2, it suffices to prove Theorem^0. Let $\mathscr{F}_{m,n}$ be the linear supermanifold associated with the superspace $E^2(\mathrm{id}^*) \oplus \mathrm{id}^*$, where id is the identity $\mathfrak{gl}(m|n)$-module. Naturally, $\mathscr{F}_{m,n} = \mathscr{G}/\mathscr{P}^+$, where the parabolic subsupergroup corresponds to the standard $\mathbf{Z}$-grading of $\mathfrak{osp}(2m+1|n)$ of the form

$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$	$\mathfrak{g}_2$
$E^2(\mathrm{id}^*)$	id^*	$\mathfrak{gl}(m\|n)$	id	$E^2(\mathrm{id})$

4.1. Theorem. *$O_{\mathscr{F}} = O_{\mathscr{F}_{m,n}}$ is a model of polynomial representations of $\mathscr{GL}(m|n)$.*

Proof. Thanks to Sergeev's results [S], since $O_{\mathscr{F}} = S^{\bullet}(E^2(V) \oplus V) \subset T(V)$ for $V = \mathrm{id}^*$, then $O_{\mathscr{F}}$ is a completely reducible $\mathfrak{gl}(m|n)$-module.

Let us show that for any polynomial weight $\lambda \in \mathfrak{h}^*$ there exists a vector highest with respect to $\mathfrak{b}_0^+ = \mathfrak{b}^+ \cap \mathfrak{g}_0$.

Indeed, let $\lambda = \Sigma\,\alpha_i\varepsilon_i + \Sigma\,\beta_j\delta_j$. In V, select a basis $\langle e_1, \ldots, e_m, f_1, \ldots, f_n\rangle$ of eigenvectors with respect to $\mathfrak{h}$:

$$h(e_i) = \varepsilon_i(h)e_i,\ h(f_j) = \delta_j(h)f_j \text{ for any } h \in \mathfrak{h},\ 1 \leqq i \leqq m,\ 1 \leqq j \leqq n$$

Set

$$\omega_1 = e_1, \omega_2 = e_1 \wedge e_2, \ldots, \omega_m = e_1 \wedge \ldots \wedge e_m,$$

$$\omega_{m+1} = e_1 \wedge \ldots \wedge e_m \otimes f_1, \omega_{m+2} = e_1 \wedge \ldots \wedge e_m \otimes f_1^2, \ldots$$

Then, clearly,

$$\omega_i \in E^i(V) \subset S^{[i/2]}(E^2(V)) \otimes V^{\otimes(i-2[i/2])}$$

and therefore $\omega_i \in O_{\mathscr{F}}$ and ω_i is the highest vector at that.

Let $(a_1^*, \ldots, a_k^*)$ be the dual partition to the partition $(a_1, \ldots, a_m)$, $\lambda_i = a_i^* + b_i$. Then $\lambda_1 \geqq \ldots \geqq \lambda_i \geqq \ldots \geqq \lambda_k > 0$.

Set $v_\lambda = w_{\lambda_1} \otimes \ldots \otimes w_{\lambda_k}$.

4.2. Exercise. Prove the simplicity of the spectrum.

4.3. Lemma. *There is a regular morphism $M^0 \to \mathscr{F}_{m,n}$ given by the formula*

$$p(X, Y, z) = (\omega_{X,Y}, \phi_z) \quad \text{[shortly denoted in what follows by } (\omega, \phi)],$$

where

$$\omega(v_1, v_2) = \langle \mathrm{pr}_{Y,X}(v_1), v_2\rangle + \langle v_1, z\rangle \langle v_2, z\rangle/2$$

and $\mathrm{pr}_{Y,X}$ *is the projection of* W *onto* Y *along* $X + Cz$, $\phi_z(v) = \langle v, z\rangle$.

PROOF. Let us verify that ω is superskewsymmetric. Let $v_1 = ax + by + cz$, $v_2 = a'b' + b'y' + c'z$, where $x, x' \in X$, $y, y' \in Y$, $a, a', b, b', c, c' \in C$.
Then
$\langle \mathrm{pr}_{Y,X} v_1, v_2\rangle = ba'\langle y, x'\rangle(-1)^{p(y)p(a')}$; $\langle \mathrm{pr}_{Y,X} v_2, v_1\rangle = b'a\langle y', x\rangle(-1)^{p(y')p(a)}$

$\omega(v_2, v_1) = ba'\langle y, x'\rangle(-1)^{p(y)p(a')} + cc'/2$; $\omega(v_1, v_2) = b'a\langle y', x\rangle(-1)^{p(y')p(a)} + cc'/2$

$$\begin{aligned}\langle v_1, v_2\rangle &= ab'\langle x, y'\rangle(-1)^{p(x)p(b')} + ba'\langle y, x'\rangle(-1)^{p(y)p(a')} + cc' = \\ &= ab'\langle x, y'\rangle(-1)^{p(x)p(b')} + ba'\langle y, x'\rangle(-1)^{p(y)p(a')} + cc'/2 \\ &\qquad + (-1)^{p(v_1)p(v_2)}c'c/2 \\ &= ba'\langle y, x'\rangle(-1)^{p(y)p(a')} + cc'/2 + \\ &\quad b'a\langle y', x\rangle(-1)^{p(x)p(b') + p(b')p(a) + p(y')p(x)} + (-1)^{p(v_1)p(v_2)}c'c/2\end{aligned}$$

Since

$$p(v_1)p(v_2) = (p(y') + p(b))(p(a) + p(x)),$$

then

$$\begin{aligned}&b'a\langle y', x\rangle(-1)^{p(x)p(b') + p(b')p(a) + p(y')p(x)} + (-1)^{p(v_1)p(v_2)}c'c/2 \\ &= (-1)^{p(v_1)p(v_2)}((-1)^{p(y')p(a)}b'a\langle y', x\rangle + c'c/2 = (-1)^{p(v_1)p(v_2)}\omega(v_2, v_1)\end{aligned}$$

implying

$$\langle v_1, v_2\rangle = \omega(v_1, v_2) + (-1)^{p(v_1)p(v_2)}\omega(v_2, v_1) = 0.$$

4.4. LEMMA. *The superdomain* $\mathcal{S} = \{(X, Y, z): V \cap Y = 0\}$ *is an open dense subsuperdomain in* $\mathcal{M}^0$ *and* $\mathcal{S} = \mathcal{U} \times \mathcal{F}_m$, *where* $\mathcal{U}$ *is the normal unipotent subsupergroup in* $\mathcal{P} = \mathrm{Stab}(V)$.

PROOF. Select V, V' and z_0 as in n. 1. Clearly, V' is canonically isomorphic to V^* and we can (and will) identify $\omega \in E^2(V^*)$ with an operator $\omega: V \to V'$ and ϕ with a vector from V'.

Determine a morphism $s: \mathcal{F}_{m,n} \to \mathcal{M}$ by the formula

$$s(\omega, \phi) = (X, Y, z), \text{ where } Y = V', z = z_0 + f$$

and

$$X = \langle v - \omega(v) - \langle \phi, v\rangle\phi/2 - \langle \phi, v\rangle z_0 \text{ for all } v \in V\rangle.$$

It is subject to a direct verification that $p \cdot s = \mathrm{id}$ and that s is an embedding. Since $\mathscr{P}(V) = V$, then $\mathscr{S}$ id $\mathscr{P}$-invariant.

If $u \in \mathscr{U}$ and $u(V') = V'$ then, as is easy to see, $u = 1$.

For any maximal isotropic $V'' \subset W$ such that $V'' \cap V = 0$ there exists $u \in \mathscr{U}$ such that $V' = uV''$. Therefore the $\mathscr{U}$-action on $\mathscr{S}$ is free; hence, $\mathscr{S} \cong \mathscr{U} \times \mathscr{S}(\mathscr{F}) \cong \mathrm{U} \times \mathrm{F}$.

4.5. Let V_λ be the irreducible $\mathfrak{g}_0$-module with highest weight λ and W_λ the irreducible $\mathfrak{g}$-module with highest weight λ; $\mathscr{P}' = \mathrm{Stab}(V')$, $\mathfrak{p} = \mathrm{Lie}(\mathscr{P})$, $\mathfrak{p}' = \mathrm{Lie}(\mathscr{P}')$, u and u' the maximal nilpotent ideals in $\mathfrak{p}$ and $\mathfrak{p}'$ respectively; let $\mathscr{U}$ and $\mathscr{U}'$ be their respective Lie supergroups; $U(\cdot)$ the functor of universal envelope.

LEMMA. $O_{\mathscr{S}} \cong \mathrm{Hom}_{U(\mathfrak{p}')}(U(\mathfrak{g}), O_{\mathscr{F}})$

Proof. Let $\langle r_1, \ldots, r_k \rangle$ be a basis of u. Any $y \in \mathscr{S}$ is of the form

$$y = e^{u_1 r_1} \ldots e^{u_i r_i} x, \text{ where } x \in s(\mathscr{F}).$$

Let $g \in \mathfrak{g}$. Then

$$e^{gt} y = e^{u_1 [r_1, g]} e^{gt} e^{u_2 r_2} \ldots e^{u_i r_i} x = \ldots$$

implying $O_{\mathscr{S}} \cong (U(u))^* \otimes O_{\mathscr{F}} \cong \mathrm{Hom}_{U(\mathfrak{p}')}(U(\mathfrak{g}), O_{\mathscr{F}}))$.

4.6. Denote

$$I_\lambda(\mathfrak{p}') = \mathrm{Hom}_{U(\mathfrak{p}')}(U(\mathfrak{g}), V_\lambda); I_\lambda(\mathfrak{g}) = \mathrm{Hom}_{U(\mathfrak{p})}(U(\mathfrak{g}), V_\lambda)$$

LEMMA. $O_{\mathscr{S}} \cong \bigoplus_{\text{polynomial } \lambda} I_\lambda(\mathfrak{p}')$

Proof follows immediately from Theorem 4.1 and Lemma 4.5.

4.7. LEMMA. 1) $O_{\mathscr{M}} \cong \bigoplus c_\lambda$, *where* $c_\lambda(\mathfrak{p}')$

2) $O_{\mathscr{M}} \cong \bigoplus c_\lambda$, *where* $c'_\lambda \subset I_{\lambda^*}(\mathfrak{p})$ *and* λ^* *is the highest weight of the representation dual to* V_λ.

Proof. 1) follows from 4.6; 2) follows from 1) under the replacements $\mathfrak{p}' \to \mathfrak{p}$, $\mathfrak{u}' \to \mathfrak{u}$.

4.8. LEMMA. $O_{\mathscr{M}} \cong \bigoplus_\lambda W_\lambda$.

Proof. Let us show that $c_\lambda = W_\lambda$. By 4.7 we have $c_\lambda = \bigoplus_{1 \leq i \leq k} c_{\mu^i}$. But since the c_λ are indecomposable, $k = 1$ and $c_\lambda = c'_\mu$ for some μ. This implies irreducibility of c_λ. Hence, $c_\lambda = W_\lambda$.

Appendix. Supergroups and their representations.

In this Appendix we will also describe some classical supergroups whose models we will describe in a continuation of this paper.

A.1. *Superspaces and supermanifolds.*

A) *Superspaces.* A space V (over a field k) endowed with a $Z/2$-grading, i.e. a decomposition $V = V_{\bar{0}} \oplus V_{\bar{1}}$, is called a *superspace* and the $Z/2$-grading is called *parrity* and is denoted by p. (We bar the elements of $Z/2$ to distinguish them from the elements of Z). The non-zero elements of $V_{\bar{0}}$ and $V_{\bar{1}}$ are called *homogeneous* (*even* and *odd*, respectively) elements of V and we write $p(v) = i$, $i \in \{0, 1\}$, if and only if $v \neq 0$ and $v \in V_i$. A *subsuperspace* is a $Z/2$-graded subspace W of the superspace V such that $W_i = W \cap V_i$.

Let V and W be superspaces. The superspace structure in the spaces $V \oplus W$, $V \otimes W$ and $\mathrm{Hom}(V, W)$ is naturally introduced, e.g. $(V \otimes W)_i = \oplus_{p+q=i} V_p \otimes W_q$ etc. The even homomorphisms of superspaces are called *morphisms*. Put $\pi(V)$ for the superspace defined by the formula $(\pi(V))_i = V_{i+\bar{1}}$; its elements will be denoted by $\pi(v)$, where $v \in V$.

A *superalgebra* is a superspace A with a morphism $m: A \otimes A \to A$. An algebra homomorphism $\varphi: A \to B$, where A and B are superalgebras, is called a *superalgebra* homomorphism if $p((\varphi) = \bar{0}$.

CONVENTIONS. 1) *Sign Rule: We put* $(-1)^{\bar{0}} = 1$, $(-1)^{\bar{1}} = -1$; *the formulas which at first glance are only defined on homogeneous elements are actually defined everywhere by linearity; if something of parity p moves past something of parity q the sign* $(-1)^{pq}$ *accrues.*

2) *In what follows, classifying bilinear forms, etc. we will only confine ourselves to homogeneous objects since the study of nonhomogeneous objects takes us beyond the limits of the investigated part of science.*

The first convention makes it possible to superize without hesitation notions like commutator, Leibniz rule, Lie algebra, (co)homology, etc., e.g. the *supercommutator* is the map

$$[\cdot,\cdot]: a, b \mapsto ab - (-1)^{p(a)p(b)} ba.$$

Usually the term *superalgebra* is used for an associative superalgebra with unit. Let us give several examples of associative superalgebras. The superalgebra A is called *supercommutative* if $[a, b] = 0$ for any $a, b \in A$. An example of a commutative superalgebra is the *Grassmann superalgebra* $\bigwedge_C(n)$ in indeterminates $\xi_1, \ldots, \xi_n$, where $p(\xi_i) = \bar{1}$ for $1 \leq i \leq n$, over a commutative algebra C (we assume $p(c) = 0$ for $c \in C$).

The *tensor algebra* $T(V)$ of the superspace V is naturally defined, $T(V) = \oplus_{n \geq 0} T^n(V)$, where $T^0(V) = k$ and $T^n(V) = V \otimes \ldots \otimes V$ (n factors) for

$n > 0$. The *symmetric algebra* of the superspace V is $S(V) = T(V)/I$, where I is the two-sided ideal generated by $v_1 \otimes v_2 - (-1)^{p(v_1)p(v_2)} v_2 \otimes v_1$ for $v_1, v_2 \in V$. The *exterior algebra* of the superspace V is $E(V) = S(\pi(V))$. Evidently, both the exterior and symmetric algebras of the superspace V are commutative superalgebras. It is worthwhile to mention that if $V_{\bar{0}} \neq 0$, $V_{\bar{1}} \neq 0$ then both $E(V)$ and $S(V)$ are infinite-dimensional.

A *Lie superalgebra* is a superalgebra $\mathfrak{g}$ (defined over a field or, more generally, a supercommutative superalgebra k) with the multiplication called *bracket* and usually denoted by $[.,.]$ or $\{.,.\}$ which satisfies the following conditions: $[X, X] = 0$ and $[Y, [Y, Y]] = 0$ for any $X \in (C \otimes \mathfrak{g})_{\bar{0}}$ and $Y \in (C \otimes \mathfrak{g})_{\bar{1}}$ and any. supercommutative superalgebra C (we assume that the bracket in $C \otimes \mathfrak{g}$ is defined via Sign Rule).

The condition on the bracket may be rewritten in an equivalent and more familiar form via Sign Rule (as superskewcommutativity and super Jacobi identity). In section A.2 below we give a more adequate definition of a Lie superalgebra which is a must in applications, like ours.

From a (super)algebra A construct a new (super)algebra A_L with the same (super)space and the multiplication $(a, b) \mapsto [a, b]$. Another method to get a Lie superalgebra from an associative superalgebra A is to consider $\mathfrak{der} A$, the derivation algebra of A, defined via Sign Rule.

From a Lie superalgebra $\mathfrak{g}$ we construct the associative superalgebra $U(\mathfrak{g}) = T(\mathfrak{g})/I$, where I is the two-sided ideal generated by the elements $x \otimes y - (-1)^{p(x)p(y)} y \otimes x \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$, called the *universal enveloping algebra* of the Lie superalgebra $\mathfrak{g}$. The *Poincaré-Birkhoff-Witt theorem* extends to Lie superalgebras with the same proof (beware Sign Rule) and reads as follows:

if $\{X_i\}$ *is a basis in* $\mathfrak{g}_{\bar{0}}$ *and* $\{Y_j\}$ *is a basis in* $\mathfrak{g}_{\bar{1}}$ *then the monomials* $X_{i_1}^{n_1} \ldots X_{i_r}^{n_r} Y_{j_1}^{\varepsilon_1} \ldots Y_{j_s}^{\varepsilon_s}$, *where* $n_i \in \mathbb{Z}^+$ *and* $\varepsilon_j = 0, 1$, *constitute a basis in* $U(\mathfrak{g})$.

A superspace M is called a *left module* over a superalgebra A (or left *A-module*) if there is given an even map act. $A \otimes M \to M$ such that $(ab)m = a(bm)$ and $1m = m$ if A is an associative superalgebra with unit (or $[a, b]m = a(bm) - (-1)^{p(a)p(b)} b(am)$ if A is a Lie superalgebra), where $a, b \in A$ and $m \in M$. The definition of a *right A-module* is similar. A module M over a commutative superalgebra C is supposed to be two-sided and the left module structure is obtained from the right one and vice versa according to the formula $cm = (-1)^{p(m)p(c)} mc$, where $m \in M$, $c \in C$: such modules will be called C-modules. There are two ways to apply the functor π to C-modules; to get $\pi(M)$ and $(M)\pi$, so to say; the two-sided module structures on $\pi(M)$ and $(M)\pi$ are given via Sign Rule. (Actually, there are *two* canonical ways to do this, see [L1]; the meaning of such an abundance is obscure.)

Sometimes instead of the map act a morphism $\rho: A \to \operatorname{End} M$ is defined if A is an associative superalgebra (or $\rho: A \to (\mathrm{End}\, M)_L$) if A is a Lie superalgebra): ρ is called a *representation* of A in M.

The simplest (in a sense) modules are those which are *irreducible* of general type (or *irreducible of G-type*); these do not contain invariant subspaces different from 0 and the whole module; and their "odd" counterparts, *irreducible modules of Q-type*, which do contain an invariant subspace that, however, is not a subsuperspace. Consequently, *Schur's lemma* states that over $\mathbf{C}$ the centralizer of a set of irreducible operators is either $\mathbf{C}$ or $\mathbf{C} \otimes \mathbf{C}^s = Q(1; \mathbf{C})$, see the definition of the superalgebras Q below.

The next in terms of complexity are *indecomposable* modules which cannot be represented as direct sum of invariant submodules.

A C-module is called *free* if it is isomorphic to a module of the form $C \oplus \ldots C \oplus \pi(C) \oplus \ldots \oplus \pi C$ (C occurs r times, $\pi(C)$ occurs s times). The *rank* of a free C-module M is the element $rkM = r + s\varepsilon$ from the ring $\mathbf{Z}[\varepsilon]/(\varepsilon^2 - 1)$ (over a field we usually write just $\dim M = (r, s)$ or $r|s$ and call this number the *dimension* of M).

The module $M^* = \mathrm{Hom}_C(M, C)$ is called *dual* to a C-module M. If $(.,.)$ is the pairing of modules M^* and M then to each operator $F \in \mathrm{Hom}_C(M, N)$, where M and N are C-modules, there corresponds the dual operator $F^* \in \mathrm{Hom}_C(N^*, M^*)$ defined by the formula

$$(F(m), n^*) = (-1)^{p(F)p(m)}(m, F^*(n^*)) \text{ for any } m \in M,\ n^* \in N^*.$$

A *supermatrix* is a matrix with entries from a superspace (sic!) and a parity assigned to each row and column. Usually, the even rows and columns are written first followed by the odd ones giving rise to a block expression of matrices in the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$; the elements of matrices usually belong to a supercommutative superalgebra. Such an expression of matrices is called the *standard format.*

Put $\mathrm{Mat}(n|m; C)$ for the set of all $(n, m) \times (n, m)$ matrices in the standard format with entries from a supercommutative superalgebra C.

To each operator $F \in \mathrm{End}_C M$ assign (in a fixed *standard* (the even vectors first) basis $\{m_i\}$ of a free module M of rank $n|m$ over a supercommutative superalgebra C) the matrix ${}^mF = (F_{ji})$ putting $Fm_i = \Sigma\, m_j F_{ji}$. Thus we obtain a one-to-one correspondence between $\mathrm{End}_C(M)$ and $\mathrm{Mat}(n|m; C)$. Now it is evident that the space $\mathrm{Mat}(n|m; C)$ is endowed with the natural superspace structure.

(The parity of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ equals $\bar{0}$ (resp. $\bar{1}$) if and only if $p(A_{ij}) = p(D_{rs}) = \bar{0}$ (resp. $\bar{1}$), $p(B_{is}) = p(C_{rj}) = \bar{1}$ (resp. $\bar{0}$)) and with an associative superalgebra structure.)

Put

$$Q(n; C) = \left\{ X \in \mathrm{Mat}(n|n; C): \left[X, \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \right] = \right\}.$$

The Lie superalgebras of series Q_L, or rather their traceless projectivizations, discovered by Gell-Mann, Mitchell and Radicatti, are examples of what V. Kac dubbed as "strange" series, Pe and Q, which stands, perhaps, for *queer* and *peculiar*. This one is a "queer" analogue of the matrix algebra $\mathrm{Mat}(n; k)$. The elements of $Q(n; C)$ preserve the complex structure given by an odd operator, cf. the definition of $\mathbf{C}^s$ and Schur's lemma.

Analogues of the trace tr: $\mathrm{gl}(n) \to \mathrm{gl}(1)$ are the *supertrace* str: $\mathrm{gl}(n|m) \to \mathrm{gl}(1|0) = \mathrm{gl}(1)$ and defined on C-points only, see sec. 1.3 belo) the *queertrace qtr:* $\mathrm{gl}(n) \xrightarrow{\sim} \mathrm{q}(1)$, where $\mathrm{gl} = \mathrm{Mat}_L$ and $\mathrm{q} = Q_L$, by the formulas, cf. [L1]:

$$\mathrm{str}\begin{pmatrix} A & B \\ D & E \end{pmatrix} = \mathrm{tr}\, A - (-1)^{p(X)} \mathrm{tr}\, E \text{ for } X = \begin{pmatrix} A & B \\ D & E \end{pmatrix}$$

$$\mathrm{qtr}\begin{pmatrix} A & B \\ B & A \end{pmatrix} = (\mathrm{tr}\, B)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Both str and qtr are (under definitions from 1.3) Lie superalgebra morphisms.

The even invertible elements from $\mathrm{Mat}(n|m; C)$ constitute the *general linear group* $\mathrm{GL}(n|m; C)$ Put $\mathrm{GQ}(nn; C) = Q(n; C) \cap \mathrm{GL}(n|n; C)$.

On the group $\mathrm{GL}(n|m; C)$ an analogue of the determinant is defined; it is called the *Berezinian* (in honour of F.A. Berezin who discovered it):

$$\mathrm{Ber}\begin{pmatrix} A & B \\ D & E \end{pmatrix} = \det(A - BE^{-1}D) \det E^{-1}$$

For the matrices from $\mathrm{GL}(n|m; C)$ the identity $\mathrm{Ber}\, XY = \mathrm{Ber}\, X\, \mathrm{Ber}\, Y$ holds, i.e. $\mathrm{Ber}: \mathrm{GL}(n|m; C) \to \mathrm{GL}(1|0; C)$ is a group homomorphism.

As is well known, the determinant is connected with the trace by the formula $\det X = \exp \mathrm{tr} \log X$ that holds when both parts of the formula are defined. We also have $\mathrm{Ber}\, X = \exp \mathrm{str} \log X$ whenever the right hand side is defined. (This formula extends the domain of Ber onto nonhomogeneous matrices).

On the group $\mathrm{GQ}(n; C)$ the Berezinian is identically equal to 1. However, on this "queer" analogue of GL the "queer" determinant is defined by the formula

$$\mathrm{qet}\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \exp \mathrm{qtr} \log \begin{pmatrix} 1_n & A^{-1}B \\ A^{-1}B & 1_n \end{pmatrix}$$

satisfying $\mathrm{qet}\, XY = \mathrm{qet}\, X \cdot \mathrm{qet}\, Y$ for $X, Y \in \mathrm{GQ}(n; C)$. Put $\mathrm{SL}(n|m; C = \{X \in \mathrm{GL}(n|m; C)\colon \mathrm{Ber}\, X = 1\}$, $SQ(n; C) = \{X \in \mathrm{GQ}(n; C)\colon \mathrm{qet}\, X = 1\}$. These are *special linear groups*, and, as we will see, the functors $G \mapsto \mathrm{GQ}(n; C)$, etc. are represented by supergroups.

A *bilinear form* is an additive in each variable map $B: M \times N \to C$ such that $B(mc, n) = B(m, cn)$, $B(m, nc) = B(m, n)c$, where $m \in M, n \in N, c \in C$ and $p(B(m, n) - p(\mathrm{m}) - \mathrm{p(n)})$. The superspace of bilinear forms is denoted by $\mathrm{Bil}_C(M, N)$ or $\mathrm{Bil}_C(M)$ if $M = N$. The *upsetting of forms* uf: $\mathrm{Bil}_C(M, N) \to \mathrm{Bil}_C(N, M)$, is defined by the formula

$$B^{\mathrm{uf}}(n, m) = (-1)^{p(n)(p(B)+p(m))+p(B)p(m)}B(m, n).$$

(The adequate way to express bilinear forms in supercase would be $\langle m|B|n\rangle$.) A form $\mathbf{B} \in \mathrm{Bil}_C(M)$ is called (*skew*) *symmetric* if $B^{\mathrm{uf}} = (-)B$.

Given bases $\{m_i\}$ and $\{n_j\}$ of C-modules and N and a bilinear form $B: M \otimes N \to C$, we assign to B the matrix

$$({}^{\mathrm{mf}}B)_{ij} = (-1)^{p(m_i)p(B)}B(m_i, n_j).$$

If $X \in \mathrm{GL}_C(M)$, $Y \in \mathrm{GL}_C(N)$ then with respect to the bases $\{Xm_i\}$ and $\{Yn_j\}$ an operator $F: M \to N$ and a bilinear form form B are given by the matrices

$$({}^{\mathrm{m}}F)' = ({}^{\mathrm{m}}X)^{-1}({}^{\mathrm{m}}F)({}^{\mathrm{m}}Y), \; ({}^{\mathrm{mf}}B)' = ({}^{\mathrm{m}}X)^{\mathrm{st}}({}^{\mathrm{mf}}B)({}^{\mathrm{m}}Y),$$

where ${}^{\mathrm{m}}X$ and ${}^{\mathrm{m}}Y$ are the matrices of the operators X and Y with respect to the bases $\{m_i\}$ and $\{n_j\}$ (and similarly with ${}^{\mathrm{m}}F$ and ${}^{\mathrm{n}}G$) and where *st* denotes the *supratransposition* defined by the formula

$$X^{\mathrm{st}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{\mathrm{st}} = \begin{cases} \begin{pmatrix} A^{\mathrm{t}} & C^{\mathrm{t}} \\ -B^{\mathrm{t}} & D^{\mathrm{t}} \end{pmatrix} & \text{if } p(X) = \bar{0} \\ \begin{pmatrix} A^{\mathrm{t}} & -C^{\mathrm{t}} \\ B^{\mathrm{t}} & D^{\mathrm{t}} \end{pmatrix} & \text{if } p(X) = \bar{1}. \end{cases}$$

REMARK. The order of the supertransposition is equal to 4.

LEMMA. *A non-degenerate homogeneous symmetric bilinear form B over* $\mathbf{C}$ *can be reduced to the canonical form with the matrix* $\mathrm{diag}(1_n, J_{2m})$ *or, if we wish to consider a split form* (for which the maximal torus in the Lie superalgebra that preserves B is situated on the main diagonal), to the form

$$\mathrm{diag}(\mathrm{ant}(1_n), J_{2m}) \text{ if } p(B) = \bar{0}$$

or to the form

$$J_{2n} \text{ if } p(B) = \bar{1}.$$

Respectively, the skewsymmetric bilinear form can be reduced to the canonical form with the matrix

$$\mathrm{diag}(J_{2m}, 1_n) \text{ or } \mathrm{diag}(J_{2m}, \sigma_n)) \text{ if } p(B) = \bar{0}$$

or to the form

$$P_{2n} \text{ if } p(B) = \bar{1}.$$

The *orthosympletic* group is the group $\mathrm{Osp}(n|2m; C)$ of automorphisms of the bilinear form with the even canonical matrix and the *peculiar*, or as A. Weil suggested *periplectic*, group $\mathrm{Pe}(n; C)$ is the group of automorphisms of the odd canonical form. The *special peculiar* (*periplectic*) group is $\mathrm{SPe}(n; C) = \mathrm{Pe}(n; C) \cap \mathrm{SL}(n|n; C)$. (As noted by A. Sergeev, the square root of the Berezinian, $\mathrm{Ser}(X) = \sqrt{\mathrm{Ber}(X)}$ is multiplicative on $\mathrm{SPe}(n)$, see [L2, n. 30].)

If $G(C) \subset \mathrm{GL}(n|m;\ C)$ is a subgroup containing the subgroup $\mathrm{Scal}(C)$ of scalar matrices, then put $\mathrm{PG}(C)$ for the group $G(C)/\mathrm{Scal}(C)$. It is called the *projective group of type* G (unitary, special, etc.), e.g. $\mathrm{PGL}(n|n)$, $\mathrm{PSQ}(n)$.

B) *Supermanifolds.* Recall that a *sheaf of groups* (*algebras*, *superalgebras*, *modules*, etc.) on a topological space X is a law F that to any open set $U \subset X$ assigns a group (algebra, superalgebra, module, etc.) $\mathscr{F}(U)$ and to each inclusion $U \subset V \subset X$ of open subsets $\mathscr{F}$ assignes a restriction morphism of groups (algebras, superalgebras, modules, etc.) $r_U^V: \mathscr{F}(V) \to \mathscr{F}(U)$ (we often write $s|_U$ instead of $r_U^V(s)$, where $s \in \mathscr{F}(V)$, and $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$ so that the following conditions of presheaf (psh) and sheaf (sh) are satisfied:

(psh) $\mathscr{F}(\emptyset) = 0$; $r_U^U = \mathrm{id}$; if $U \subset V \subset W$ then $r_U^W = r_U^V r_V^W$;

(sh) Let $U = \bigcup_\alpha U_\alpha$ be an open covering of an open set $U \subset X$. If $s \in \mathscr{F}(U)$ is an element such that $s|_{U_\alpha} = 0$ for all α then $s = 0$ and if the set $\{s_\alpha \in \mathscr{F}(U_\alpha)\}$ is such that $s_\alpha|_{\cap_\alpha U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all α, β then there exists $s \in \mathscr{F}(U)$ such that $s|_{U_\alpha} = s_\alpha$.

Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves of groups (algebras, superalgebras, modules, etc.) on X. A *sheaf morphism* $\varphi: \mathscr{F} \to \mathscr{G}$ is a collection of morphisms of groups (algebras, superalgebras, modules etc.) $\varphi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ for any open set U such that if $U \subset V$ are open sets then the diagram

$$\begin{array}{ccc} \mathscr{F}(V) & \xrightarrow{\varphi(V)} & \mathscr{G}(V) \\ \downarrow r_U^V & & \downarrow r_U^V \\ \mathscr{F}(U) & \xrightarrow{\varphi(U)} & \mathscr{G}(U) \end{array}$$

commutes. An *isomorphism* of sheaves is a morphism with two-sided inverse.

If $f: X \to Y$ is a continuous map of topological spaces and $\mathscr{F}$ is a sheaf on X, then the *direct image* or *pushforward* of $\mathscr{F}$ is the sheaf $f * (\mathscr{F})$ on Y defined by the formula $(f * (\mathscr{F})(U) = \mathscr{F}(f^{-1}(U))$ for any open $U \subset Y$. Definitions of a sheaf of modules over a sheaf of (super) algebras, of tensor operations over sheaves of modules and also of the restriction of the sheaf, $\mathscr{F}|_U$, onto the subspace $U \subset X$ are straightforward and are left to the reader.

EXAMPLES. The sheaf of functions on a manifold, the sheaf of sections of a vector bundle over a manifold.

Recall also that a *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A *morphism* of ringed spaces is a pair $(f, f^*): (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consisting of a continuous map $f: X \to Y$ and a sheaf morphism $f^*: \mathcal{O}_Y \to f * \mathcal{O}_X$.

If $\mathcal{O}_{\mathcal{M}}$ is a sheaf of commutative superalgebras over the structure sheaf of a smooth manifold M and for sufficiently small open domains U the superalgebras $\Gamma(U, \mathcal{O}_{\mathcal{M}})$ are free over $\Gamma(U, \mathcal{O}_{\mathcal{M}})$, then $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ is called a *smooth supermanifold*. The manifold M is called the *underlying manifold* or the *base* of the supermanifold $\mathcal{M}$.

Supermanifolds can be also defined by means of charts and atlases. In what follows, defining supergroups, we will give one more, equivalent, definition of supermanifolds, in terms of the *representing functor*, also called in algebraic geometry the *functor of points*.

IMPORTANT REMARK. There is a one-to-one correspondence between objects from the category of smooth supermanifolds and objects from the category of smooth vector bundles (this follows easily from definitions and existence of a partition of unity), but in the category of supermanifolds there are many more morphisms (and that was what physicists were strifing for in the first place).

REMARK. There are more analytic supervarieties (even non-singular ones) than vector bundles. The reason for this is that the grading in the steaf $\mathscr{L}_{E(M)}$ by powers of the steaf of ideals Odd generated by odd elements can (in the category of analytic supermanifolds) correspond to several filtrations and parameters describing such deformations can be odd, cf. Vaintrob's calculations in [L2], n. 24.

EXAMPLE. Let pr: $M \to B$ be a smooth vector bundle with fiber V and $E(M)$ the exterior algebra of M (with fiber $E(V)$); let $v_1, \ldots, v_m$ be a basis in V and $x_1, \ldots, x_n$ local coordinates on B; then the formula (where the summation over the repeated indices is assumed)

$$x_i \mapsto a_i(x) + \sum_{k \geqq 1} a_i^{j_1 \ldots j_{2k}}(x) \cdot v_{j_1} \ldots v_{j_{2k}}, \quad 1 \leqq i \leqq n$$

$$v_j \mapsto b_j^l(x) \cdot v_l + \sum_{k \geqq 1} b_j^{j_1 \ldots j_{2k+1}}(x) \cdot v_{j_1} \ldots v_{j_{2k+1}}, \quad 1 \leqq j \leqq m$$

defines an endomorphism of the supermanifold $\mathfrak{B} = (B, \mathscr{L}_{E(M)})$, where $\mathscr{L}_{E(M)}$ is the sheaf of sections of the bundle $E(M)$, and the terms that are not in a box define an endomorphism of the bundle $E(M)$.

The dimension of a smooth connected supermanifold $\mathscr{B} = (B, \mathcal{O}_{\mathscr{B}})$ is the pair (n, m), where $n = \dim B$ and $m = \dim V$ for V such that $\mathcal{O}_{\mathscr{B}} = \mathscr{L}_{E(M)}$, where M is a smooth bundle over B with fiber V.

The *linear supermanifold* of dimension (n, m) over the field $k (= \mathbf{C}$ for us) is $\mathscr{K}^{n,m} = (k^n, \mathcal{O}_{n,m})$ where $\mathcal{O}_{n,m} = \mathcal{O}_{k^n} \otimes \Lambda_k(m)$. There is a correspondence which to a linear supermanifold ${}^sV = (V_{\bar{0}}, \mathcal{O}_{V_{\bar{0}}} \otimes \Lambda[V_{\bar{1}}^*])$ assignes the linear superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ and the former is uniquely recovered from the latter.

A *superdomain of dimension* (n, m) is a subsupermanifold $\mathscr{U} \subset \mathscr{K}^{n,m}$ whose underlying manifold is a domain U in k^n and $\mathcal{O}_U = \mathcal{O}_{n,m}|_U$. The sections of the sheaf $\mathcal{O}_U$ are called *functions on* $\mathscr{U}$; if $\mathscr{U}$ is a smooth supermanifold then instead of $\Gamma(U, \mathcal{O}_{\mathscr{U}})$ we write $C^\infty(\mathscr{U})$.

An element $f \in C^\infty(\mathscr{U})$ in a sufficiently small neighbourhood $\mathscr{V} \subset \mathscr{U}$ (i.e. the undelying V of $\mathscr{U}$ is a sufficiently small neighbourhood) is uniquely expressable in the form $f(x) = \Sigma f_\alpha(u)\xi^\alpha$, where $f_\alpha \in C^\infty(V)$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_i = 0, 1$ and $\xi^0 = 1$.

Let $\mathscr{U}$ be a smooth supermanifold, $u_1, \ldots, u_n$ coordinates on U and $\xi_1, \ldots, \xi_m$ generators of $\Lambda(m)$, then the set $x = (u, \xi)$ and *also all its images* under the action of the automorphism group of the superalgebra $C^\infty(\mathscr{U})$ are called *coordinate systems* of the supermanifold $\mathscr{U}$.

Clearly, the collection of coordinate systems is in 1–1 correspondence with the elements of the group of diffeomorphisms of $\mathscr{U}$. How to describe the coordinate systems that correspond to the *supergroup* of diffeomorphisms of $\mathscr{U}$ will become clear from A.2. Meanwhile let us give the final answer: a *coordinate system* on a supermanifold is a set of (homogeneous) generators of the superalgebra of local functions with values in a background supercommutative superalgebra C and the passage to another such set is performed *over* C by the formulas:

$$x_i \mapsto a_i(x) + \sum_{k \geqq 1} a_i^{j_1 \ldots j_k}(x) \cdot v_{j_1} \ldots v_{j_k}, \quad 1 \leqq i \leqq n$$

$$v_j \mapsto b_j(x) + \sum_{k \geqq 1} b_j^{j_1 \ldots j_k}(x) \cdot v_{j_1} \ldots v_{j_k}, \quad 1 \leqq j \leqq m$$

where the coefficients (belonging to C) of x's and v's respectively in the power series expansion in v's is of the same (opposite) parity as that of the power.

The *partial derivatives* $\partial/\partial x_i$, derivations of the superalgebra $C^\infty(\mathscr{U})$, are defined by the formula $\partial/\partial x_i(x_j) = \delta_{ij}$, together with the Leibniz and Sign rules. The sections of the sheaf of the $\mathcal{O}_U$-module *Vect* $\mathscr{U}$ of derivations of the sheaf $\mathcal{O}_{\mathscr{U}}$ are called *vector fields*.

A *closed subsuperdomain* in a smooth supermanifold $\mathscr{M}^{n,m}$ is a pair $(F, \partial\mathscr{F})$, where F is a closed domain with smooth boundary ∂F and $\partial\mathscr{F}$ is a subsupermanifold in $\mathscr{F}$ of codimension $(1, 0)$ whose manifold coincides with ∂F. (*Attention*: To an open superdomain many closed superdomains may correspond!).

Bundles (vector, principal, etc.) in the category of supermanifolds are defined like in that of manifolds,

VERY IMPORTANT REMARK. It is very difficult, psychologically, to become used to the practice to speak about vector bundles, representations of Lie supergroups and their orbits (in fibers of a bundle, etc.) having in mind that the fiber is *not a superspace but a linear supermanifold*, the space of the representation is tacidly replaced by the corresponding linear supermanifold (see section A.2 on Lie supergroups). Tensor operations over superspace generalize in an obvious way to vector bundles over supermanifolds, for example the fiberwise change of parity transforms a vector bundle $\mathscr{V} \to \mathscr{E} \to \mathscr{M}$ (for brevity we often write just $\mathscr{E}$) into the bundle $\pi\mathscr{V} \to \pi\mathscr{E} \to \mathscr{M}$.

It is possible to interpret differential forms as fiberwise polynomial functions on the bundle $T\mathscr{M}$. For some unknown reason, unlike differential (skewsymmetric) forms, fiberwise polynomial functions on $T\mathscr{M}$ do not possess any interesting properties unless they are homogeneous of degree 2, i.e. bilinear forms on $T\mathscr{M}$. If a symmetric form is nondegenerate it is called a *metric*.

A.2. Lie supergroups and homogeneous superspaces in terms of the point functor. In the 50's A. Grothendieck invented way to describe spaces ringed by sheaves of algebras with nilpotents. Such algebras are very natural even in the freshperson's course of calculus: when one considers the Taylor series expansion up to the n-th power and ignores the $(n + 1)$-th one. Grothendieck showed that instead of considering *one* set-theoretical model of the space (which suffices for smooth manifolds) we ought to consider, simultaneously, *many* models.

Let $\mathscr{M} = (\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ be a supermanifold. To any supermanifold $\mathscr{X} = (X, \mathscr{O}_X)$ assign the set $P_{\mathscr{M}}(\mathscr{X}) = \operatorname{Mor}(\mathscr{X}, \mathscr{M})$. Clearly, the sets $P_{\mathscr{M}}(\mathscr{X})$ for all $\mathscr{X}$ define $\mathscr{M}$ (for a nice manifold it suffices to consider only one $\mathscr{X}$: a one-point set).

It is more convenient though to reformulate the problem. If C is a sufficiently large ring of global functions on $\mathscr{X}$ (i.e. $\mathscr{X}$ is rather thick, not just a mere point) then $P_{\mathscr{M}}(\mathscr{X})$ can be recovered from C, and some functorial in C properties are satisfied.

Now, the other way around. Let for each supercommutative superalgebra C be given the set $P_{\mathscr{M}}((C)$, called the set of *C-points* of $\mathscr{M}$, and for each morphism of superalgebras $\alpha: C \to C'$ there corresponds a map of sets $\alpha^{\mathscr{M}}: P_{\mathscr{M}}(C') \to P_{\mathscr{M}}(C)$ such that $\mathrm{id}^{\mathscr{M}} = \mathrm{id}$ and $(\alpha\beta)^{\mathscr{M}} = \beta^{\mathscr{M}}\alpha^{\mathscr{M}}$. In other words, *$\mathscr{M}$ represents a functor* from the category of supercommutative superalgebras into that of sets. To a morphism of supermanifolds $\varphi: \mathscr{M} \to \mathscr{N}$ there corresponds a morphism of functors i.e. for each C a map of sets $\varphi(C): P_{\mathscr{M}}(C) \to P_{\mathscr{N}}(C)$ such that $\alpha^{\mathscr{M}} \circ \varphi(C) = \varphi(C') \circ \alpha^{\mathscr{N}}$ for all α..

Now we can give another, functorial, definition of a Lie superalgebra and the definition of a Lie supergroup to match.

If each set $P_{\mathscr{G}}(C)$ is a group (a Lie algebra) and all morphisms $\alpha^{\mathscr{G}}$ are homomorphisms of groups (Lie algebras) then the supermanifold $\mathscr{G}$ is called a Lie *supergroup* (*superalgebra*). An *action a* of the supergroup $\mathscr{G}$ on a supermanifold $\mathscr{M}$ is a set of actions $a(C): P_{\mathscr{G}}(C) \times P_{\mathscr{M}}(C) \to P_{\mathscr{M}}(C)$ consistent with transformations $\alpha: C \to C'$.

Now, let $P(\cdot)$ be an arbitrary functor assigning to a commutative superalgebra C a set $P(C)$ so that for any supermanifold $\mathscr{M}$ maps $\varphi(C): P(C) \to P_{\mathscr{M}}(C)$, $\varphi'(C): P_{\mathscr{M}}(C) \to P(C)$ are defined and to a morphism $\alpha: C \to C'$ there corresponds a map $\alpha^P: P(C') \to P(C)$ satisfying $\alpha^P \circ \varphi(C) = \varphi(C') \circ \alpha^{\mathscr{M}}$ and $\alpha^{\mathscr{M}} \circ \varphi'(C) = \varphi(C) \circ \alpha^P$. The functor $P(\cdot)$ is not necessarily representable in the form $P_{\mathscr{M}}(\cdot)$ for some supermanifold $\mathscr{M}$ and there is no general way to find out if it is representable. Each case is to be considered ad hoc.

EXAMPLES. 1) The linear supermanifold ${}^sV = \mathscr{V}^{n,m} = (V_{\bar{0}}, \mathscr{O}_{V_0} \otimes \Lambda(m))$ of dimension $(n|m)$, where $\dim V_{\bar{0}} = m$, $\dim V_{\bar{1}} = m$ for $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Put $P_{\mathscr{V}}(C) = (V \otimes C)_{\bar{0}}$. Note, that the supermanifold $\mathscr{V}$ possesses a natural commutative Lie supergroup structure.

2) A Lie superalgebra $\mathfrak{g}$ is usually understood as an object from the category of linear superspaces. To speak about representations of $\mathfrak{g}$ or in $\mathfrak{g}$, assign to $\mathfrak{g}$ the functor $C \mapsto P_{\mathfrak{g}(C)} = (\mathfrak{g} \otimes C)_{\bar{0}}$. The associated linear supermanifold $\mathscr{G} = {}^s\mathfrak{g}$ is endowed with the Lie algebra structure in the category of supermanifolds and the action of the supergroup $\mathscr{K}^{1,0} = (K, \mathscr{O}_K)$ on $\mathscr{G}$ is defined.

Exercise: what is a representation of a Lie superalgebra? What does this new definition add to the naive definition given above? Hint: cf. 4), 5) below and definition of qet.

3) The *general linear Lie supergroup* $\mathscr{GL}(p|q)$. Put $P_{\mathscr{GL}(p|q)}(C) = \mathrm{GL}(p|q; C)$. More generally, given a linear supermanifold ${}^sV$ of dimension (p, q), the Lie supergroup $\mathscr{GL}(V)$ defines the functor of points

$$P_{\mathscr{GL}(V)}(C) = \mathrm{GL}(V; C) = \mathrm{GL}_C(V \otimes_k C)_{\bar{0}}$$

Having chosen a basis in V we may identify $\mathscr{GL}(V)$ with $\mathscr{GL}(\dim V)$.

4) A representation ρ of a Lie supergroup $\mathscr{G}$ in a superspace V is a Lie supergroup morphism $\rho: \mathscr{G} \to \mathscr{GL}(V)$ defining a $\mathscr{G}$-action in the linear supermanifold ${}^sV$ associated with V. In terms of C-points to define a representation ρ is the same as to define for any commutative superalgebra C a group homomorphism (functorial in C):

$$\rho(C): P_{\mathscr{G}}(C) \to P_{\mathscr{GL}}(C) = \mathrm{GL}(V; C)$$

For example, the C- or R-points of qtr and qet are 1-dimensional trivial representations, but if we take into account odd parameters we see that qtr and qet are nontrivial representations of the Lie superalgebras and supergroups on which they are defined.

In general, having found the C-points of a representation of G in V we can describe its odd parameters after calculating $(H^1(\mathfrak{g}; \mathrm{End}(V)))_{\bar{1}}$. To describe the *supervariety* of parameters one has to calculate the Massey powers of the elements from $(H^1(\mathfrak{g}; \mathrm{End}(V)))$.

5) The *coadjoint action* of the supergroup $\mathcal{GL}(p|q)$ is the action of $\mathcal{GL}(p|q)$ on the superspace $\mathfrak{gl}(p|q)^*$, i.e. the set of actions of the groups $GL(p|q; C)$ in the spaces $(\mathfrak{gl}(p|q) \otimes C)_{\bar{0}})^*$ defined for all commutative superalgebras C in a way compatible with homomorphhisms $C \to C'$.

6) The homogeneous superspace $\mathcal{G}/\mathcal{H}$ is defined by the functor $C \mapsto P_{\mathcal{G}}(C)/P_{\mathcal{H}}(C)$. If $H \subset G$ is a Lie subgroup then this functor is represented by a supermanifold.

REMARK. Defining a supermanifold $\mathcal{M}$ in terms of its C-points, $P_{\mathcal{M}}(C)$, we can confine ourselves to Grassmann superalgebras C and, moreover, to one sufficiently large superalgebra $\Lambda(N)$. For another supermanifold such a defining superalgebra may be another one, therefore it is convenient not to fix N, but just consider it "very large" or ∞. More precisely, what does it mean that the $\Lambda(N))$-points define a supermanifold? The answer is given by the following statement, cf. [L2], n. 31.

LEMMA. 1) *Let* $\varphi, \psi \colon \mathcal{M} \to \mathcal{N}$ *be morphisms of supermanifolds,* $\varphi(n)$, $\psi(n)$: $P_{\mathcal{M}}(\Lambda(n)) \to P_{\mathcal{N}}(\Lambda(n))$ *the corresponding maps of sets. If* $\dim \mathcal{M} = (p|q)$ *and* $n \geqq q$ *then* $\varphi(n) = \psi(n)$ *implies* $\varphi = \psi$.

2) *Let* $\mathcal{M}, \mathcal{N}$ *be supermanifolds,* $\dim \mathcal{M} = (p|q)$. *Let a set of maps* $\tilde{\alpha}(n)\colon P_{\mathcal{M}}(\Lambda(n)) \to P_{\mathcal{N}}(\Lambda(n))$ *be defined for* $n \geqq q$. *Let for a homomorphism* $\varphi\colon \Lambda(n) \to \Lambda(n')$ *given for* $n, n' \geqq q$ *there correspond maps* $\varphi^{\mathcal{M}}\colon P_{\mathcal{M}}(\Lambda(n')) \to P_{\mathcal{M}}(\Lambda(n))$ *and* $\varphi^{\mathcal{N}}\colon P_{\mathcal{N}}(\Lambda(n')) \to P_{\mathcal{N}}(\Lambda(n))$ *such that* $\varphi^{\mathcal{M}} \circ \tilde{\alpha}(n) = \tilde{\alpha}(n') \circ \varphi^{\mathcal{N}}$.

Then there exists a supermanifold morphism $\alpha\colon \mathcal{M} \to \mathcal{N}$ *such that* $\alpha(n) = \tilde{\alpha}(n)$.

Examples 1)–4) illustrate functors represented by supermanifolds. On the contrary, actions of supergroups (on supermanifolds) supply with examples of objects that we are tempted to consider as supermanifolds but which are not. Indeed, they are not ringed spaces, and the functor corresponding to them is not represented by any supermanifold. Consider for instance $GL(n)$-orbits in the standard (identity) n-dimensional representation. There are 2 orbits: the origin and its complement. Now, assume that the space of the representation is purely odd. Then, there are, evidently, also 2 orbits, but the one, which is not the origin, is not a supermanifold and, besides, it has no points over a field (or a commutative ring without odd elements) at all!

Such examples appear all the time. It is absolutely not clear though how to describe the category of such objects or whether it is possible to extend differential or algebraic geometry to this category. (There is a sea of papers where such an extension is wrongly declared to be done. To test whether this is so substitute for C above a plain *commutative* algebra without odd part.)

REFERENCES

[GZ1] I. M. Gelfand, A. V. Zelevinsky, *Representation models for classical groups and their hidden symmetries*, Functional Anal. Appl. 18, #3 (1984), 14–31 (Russian).

[GZ2] I. M. Gelfand, A. V. Zelevinsky, *Representation models for classical groups and their higher symmetries*, Astérisque, hors séries (1985), 117–128.

[OV] A. L. Onishchik, E. B. Vinberg, *Seminar on algebraic groups and Lie groups*, Springer, Berlin 1990.

[P] I. Penkov, *Borel-Weil theory for superalgebras*, in: Modern probl. of math., 1988, 71–124 (Russian) = Engl. transl. in: Sov. J. Math. (JOSMAR), 51, # 1, 1990; an expanded version: "*Classical Lie supergroups and Lie superalgebras and their representations*" is to appear in Asterisque, 1991; its preprinted version is: Prépubl. de l'Inst. Fourier, n. 117, 1988, 150 p.

[S] A. Sergeev, *Invariant functions on Lie superalgebras*, in: [L2, #32].

[K] V. G. Kac, *Lie superalgebras*, Adv. in Math. 26 (1977), 8–96.

[L1] D. Leites, *Lie superalgebras*, Sov. J. Math. (JOSMAR), 30 (6) (1984), 2500–2567.

[L2] D. Leites (ed), *Seminar on supermanifolds*, ##1–35, Reports of Dept. of Math. of Stockholm Univ., 1987–90.

DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
BOX 6701
S-11385 STOCKHOLM
SWEDEN